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Documento de Trabajo

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No.9603

Febrero 1996

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A SOLUTION METHOD FOR A CLASS OF *LEARNING BY DOING* MODELS

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ABSTRACT

We obtain in the closed-form the optimal policy for a class of *learning by doing* models, in which a monopolist operating in a market with linear demand and finite time horizon, faces a lower bound in the cost reduction that can be achived through production. By using Dynamic Programming principles we show that the existence of a lower bound in the unit production cost implies that the optimal decision for output is a function which is indexed by initial unit cost. There is an optimal set of threshold values beyond wich the parameters of the production rule change. Some examples with specific parameter values are provided.

RESUMEN

En este artículo se obtiene la solución analítica para una familia de problemas de *Learning by Doing*, en los que un monopolista opera en un mercado con demanda lineal y horizonte temporal finito, teniendo un límite inferior en la reducción de costes, vía producción. El método de solución, basado en la Programación Dinámica, permite obtener la solución óptima, así como una partición del intervalo de posibles valores del coste inicial, de modo que la regla de decisión óptima cambia dependiendo del subintervalo al que pertenezca el coste inicial. Se presentan ejemplos con valores específicos para los parámetros.

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Alvarez has been supported by DGICYT grant PS94-0027.

We want to thank E. Domínguez and C. Mazón for their suggestions. Any error is our responsability.

Π.Ο.Γ.: α-93-201793-Y
Ν.Ε.: 5306522984

I Introduction

A phenomenon widely observed in industries which are in an early stage of their productive life is that they reduce their cost as a result of accumulating experience, that is, they reduce their cost with their output. This is known in the economic literature as *learning by doing*, and it was studied for the first time by Arrow (1962). Other authors studied the relationship between industry structure and *learning by doing* (Fudenberg and Tirole(1983), Stokey (1986), Dasgupta and Stiglitz (1988), and Parente (1994)). They analyzed the differences in the learning process if the industry structure is a monopoly or a nationalized industry. In general, the results available in the literature give properties of the optimal policy, but they do not present the optimal policy in the closed-form.

In this paper we obtain in closed-form the optimal policy for a class of *learning by doing* models. We consider a monopoly, without possible competition. The demand function is linear. The problem is deterministic, dynamic, with a finite time horizon and it is formulated in discrete time. The state variable is the unitary cost (c(t)) and the control variable is the quantity to be produced (q(t)). The objective function is to maximize the discounted benefit flow and the state equation is: $c(t+1)=\max\{\tau,c(t)-\beta q(t)\}$, with $c(0)>\tau$ given.

The problem is solved by using dynamic programming (Bellman(1957)). The key of the method is to identify the role of τ, the lower bound of the unit cost, as a binding constraint. We obtain a partition of the set of possible values for c(0). Depending on the value of c(0) in the partition, an equivalent problem is formulated for which we obtain the optimal solution. It is shown that there is an optimal set of threshold values of c(0) beyond which the parameters of the production rule change. The method to obtain the closed-form optimal policy is formally presented in five theorems and two corollaries.

In Section II we formulate the problem, and in Section III we present the solution. Section IV gives some examples with specific parameter values. Finally, in Section V we present the conclusions and some ideas for further research.

II The model

We consider the problem of a monopolist, without possible competition, who maximizes the discounted profit flow along T periods, where T is known. The discount parameter is λ. The first period is 0, so the last is T-1. The monopolist faces a demand which is constant over time, and such that, the price in period t, p(t), is a linear function of the produced output in that period, q(t). Specifically, the inverse demand function is:

$$p(t) = a - bq(t) \quad t = 0,\ldots,T-1 \tag{1}$$

where a,b>0.

In the period t, the firm chooses q(t), the output to be produced at a unitary cost c(t); that output is sold at price p(t). There are no fixed costs. In next period, t+1, the unitary cost is c(t+1). The change in the unitary cost from period t to t+1 is given by:
$c(t+1)=\max\{\tau,c(t)-\beta q(t)\}$. This equation is used by Dasgupta and Stiglitz (1988). In this equation, the unitary cost is a linear function on the output produced in the last period, while the cost remains above a certain value τ, and if the cost takes that value, remains in it forever. Furthermore, β determines the abilitity of present output to reduce future cost.

Given **(1)**, the profit of the monopolist in period t is: (a-bq(t)-c(t))q(t). Since the monopolist maximizes the profit flow discounted by λ, subject to the evolution of the costs given above, the problem can be expressed, in mathematical terms, as follows:

Problem I

$$\underset{q(0),\ldots,q(T-1)}{MAX} \quad \{ \sum_{t=0}^{T-1} \lambda^t (a-bq(t)-c(t))q(t) \} \tag{2}$$

subject to:

$$c(t+1) = \max\{\tau, c(t)-\beta q(t)\} \quad t = 0,\ldots,T-2 \tag{3}$$

$q(t)\geq 0$ for every $t\in\{0,\ldots,T-1\}$.

c(0) is given. The parameters a,b,λ,β,τ,T are known.

Other additional assumptioms are:

$$c(0) > \tau \tag{4}$$

$$a > c(0) \tag{5}$$

$$\tau > 0 \; ; \; \beta > 0 \; ; \; b > 0 \; ; \; \lambda \in (0,1) \tag{6}$$

The assumption given in **(4)** ensures that in **problem I** there can be cost reduction. The assumption given in **(5)** is normally used in the economic literature, along with other conditions which are shown later (related to the concavity of the objective function) to ensure that the output is positive. We must note that in a static problem without *learning by doing*, the condition **(5)** is necessary for a positive output. However it can be shown that in **problem I** the output can be positive although **(5)** does not hold. In **(6)** there are other assumptions which are widely used in economics: the lower bound on the costs must be positive ($\tau>0$), to raise the output reduces future costs ($\beta>0$), the demand function is decreasing (b>0) and the discount factor is between 0 and 1.

III Solution method for problem I

The problem stated above is a dynamic optimization problem, with a finite time horizon and in discrete time, and it is solved by using the well-known methodology proposed by Bellman. The state variable is unit cost and the control variable is output. The interest of the solution method that we present is that it finds the analytical solution for Bellman's functional equation associated with **problem** I. The key idea is to identify the role of τ as a binding constraint. For example, given all the parameters, it is possible that, under the optimal policy, τ is not reached after T periods, if so, τ does not represent a constraint, in the sense that if the bound for the cost reduction did not exist, the cost would not take a lower value after T periods. A different case is that in which τ is reached in the last period, if so, τ may represent a constraint, because if it did not exist, the cost could take a lower value. A third case occurs if τ is reached in the period T-2, and in this case τ represents a stronger constraint than in the previous examples. We can continue and so, we can consider all possible cases. Of course only one will occur, that is, or the cost reaches τ in a specific period, or it never reaches τ. The solution method identifies in which period the cost reaches τ (if τ is reached), for the first time, by succesive resolution of linear quadratic problems. Once we have identified that period, the problem is already analytically solved, since for every case to be considered, the remaining problem is linear quadratic, if we can be certain that the output is positive.

Next, we present formally the solution. The following definitions will be used.

Definition 1 For every $t \in \{0,\dots,T-1\}$, $\pi(t)=\{q(t),\dots,q(T-1)\}$ where $q(j)\geq 0$ for every $j\in\{t,\dots,T-1\}$ is a feasible sub-policy that begins in period t. Furthermore, if t=0, then $\pi(0)$ is called feasible policy.

Definition 2 For every $t\in\{0,\dots,T-1\}$, we denote by S(t) the set of all the feasible subpolicies which begin in period t.

Definition 3 For every $t\in\{0,\dots,T-1\}$, given a feasible sub-policy $\pi(t)$ and c(t) we define:

$$J(c(t),\pi(t),t) = \sum_{j=t}^{T-1} \lambda^j (a-bq(j)-c(j))q(j)$$

Where, for every $j\in\{t+1,\dots,T-1\}$, c(j) is given by **(3)**. Furthermore, for any integer s which verifies $t\leq s\leq T-1$ we write either $J(c(t),\pi(t),t)$ or $J(c(t),q(t),\dots q(s-1),\pi(s),t)$.

For any period t, the definitions given above characterize the set of all the possible decisions the monopolist can make, S(t), and for any of them, the discounted profit that they produce. The definitions that follow are used for the optimal decisions.

Definition 4 For every $t\in\{0,\dots,T-1\}$, given c(t), we say that $\pi^*(t)=\{q^*(t),\dots,q^*(T-1)\}$ (with $\pi^*(t)\in S(t)$) is an optimal sub-policy (or policy if t=0), if: $J(c(t),\pi(t),t)\leq J(c(t),\pi^*(t),t)$ for every $\pi(t)\in S(t)$.

Definition 5 For every $t\in\{0,\dots,T-1\}$ we define the value function as: $J^*(c(t),t) = J(c(t),\pi^*(t),t)$.

Theorem 1 states when it is optimal not to reach τ after T periods. As we have indicated previously in an intuitive way, this is expected to occur, given all the parameters, when there is *enough* difference between c(0) and τ. So the theorem formalizes what is *enough*. Next notation will be used later in the theorem.

Let:

$$K(T,T) = 0\ ;\ R(T-1,T-2) = \tau \qquad (7)$$

$$\phi(t,T) = \frac{1+2\lambda\beta K(t+1,T)}{2b-2\lambda\beta^2 K(t+1,T)} \quad t = 0,\dots,T-1 \qquad (8)$$

$$R(t,T-2) = \frac{R(t+1,T-2)+\beta\phi(t,T)a}{1+\beta\phi(t,T)} \quad t = 0,\dots,T-2 \qquad (9)$$

$$K(t,T) = \lambda K(t+1,T)+\frac{1}{2}(1+2\lambda\beta K(t+1,T))\phi(t,T) \quad t = 0,\dots,T-1 \qquad (10)$$

Theorem 1

If:

$$c(0) > R(0,T-2) \qquad (11)$$

$$b > \lambda\beta^2 K(t,T) \quad t = 1,\dots,T-1 \qquad (12)$$

where R(0,T-2) and K(t,T) for every $t\in\{1,\dots,T-1\}$ are defined in **(7)** to **(10)**, then:

i) $q^*(t)=\phi(t,T)(a-c(t))$; $J^*(c(t),t)=K(t,T)(a-c(t))^2$; $t=0,\dots,T-1$; where $\phi(t,T)$ for every $t\in\{0,\dots,T-1\}$ is defined in **(8)**.

ii) In **problem I**, under $\pi^*(0)$, $c(T-1)>\tau$ holds.

■

The proofs of all theorems and corollaries are included in appendix I.

The next corollary, and the following theorem and corollary, state when, under the optimal policy, τ is reached in period $s\in\{2,\dots,T-1\}$. Intuitively, if there must be enough difference between c(0) and τ (given all the other parameters) not to reach τ in T periods, then, to reach τ in a period different from period 1, c(0) must be between two values. Firstly, c(0) must be sufficiently greater than τ so as not to reach τ before. On the other hand, c(0) must be close enough to τ so as not to reach (for the first time) τ later. Formally, we can establish a partition in the set of all possible values for c(0) (the open interval (τ,a)) such that, depending on the value of c(0) in the partition, we know when (if so) the cost reaches τ for the first time. Furthermore, we show that, depending on the value of c(0) in the partition, the parameters of the production rule change.

Corollary 1

If **(12)** holds and furthermore **(11)** does not verify then, under $\pi^*(0)$, we have $c(T-1)=\tau$.

■

The following notation is used in the next theorem.

Let $s \in \{2,\ldots,T-1\}$; let:

$$K(t,s) = 0 \;;\; t = s,\ldots,T \qquad R(s-1,s-2) = Q(s,s-1) = \tau \tag{13}$$

$$K_0(t,s) = \frac{1}{4b}(a-\tau)^2\frac{1-\lambda^{T-t}}{1-\lambda} \;;\quad t = s,\ldots,T-1 \tag{14}$$

$$\phi(t,s) = \frac{1+2\lambda\beta K(t+1,s)}{2b-2\lambda\beta^2 K(t+1,s)} \;;\quad t = 0,\ldots,T-1 \tag{15}$$

$$R(t,s-2) = \frac{R(t+1,s-2)+\beta\phi(t,s)a}{1+\beta\phi(t,s)} \;;\quad t = 0,\ldots,s-2 \tag{16}$$

$$Q(t,s-1) = \frac{Q(t+1,s-1)+\beta\phi(t,s)a}{1+\beta\phi(t,s)} \;;\quad t = 0,\ldots,s-1 \tag{17}$$

$$K(t,s) = \lambda K(t+1,s)+\frac{1}{2}(1+2\lambda\beta K(t+1,s))\phi(t,s) \;;\quad t = 0,\ldots,s-1 \tag{18}$$

$$K_0(t,s) = \lambda K_0(t+1,s) \;;\quad t = 0,\ldots,s-1 \tag{19}$$

Theorem 2

Let $s \in \{2,\ldots,T-1\}$. If:

$$R(0,s-2) < c(0) \le Q(0,s-1) \tag{20}$$

$$b > \lambda\beta^2 K(t,s) \quad t = 1,\ldots,s-1 \tag{21}$$

where R(0,s-2), Q(0,s-1) and K(t,s) for every $t \in \{1,\ldots,s-1\}$ are defined in **(13)** to **(18)**, and furthermore $c(s)=\tau$ under $\pi^*(0)$; then:

i) $q^*(t)=\phi(t,s)(a-c(t))$; $J^*(c(t),t)=K_0(t,s)+K(t,s)(a-c(t))^2$; $t=0,\ldots,T-1$; where $\phi(t,s)$ and $K_0(t,s)$ for every $t \in \{0,\ldots,T-1\}$ are defined in **(15)** and **(19)**.

ii) In **problem I**, the unitary cost, under $\pi^*(0)$, reaches τ, for the first time, in period s.

■

Corollary 2

Let $s \in \{2,\ldots,T-1\}$; if **(21)** holds, $c(0) \le R(0,s-2)$ and furthermore $c(s)=\tau$ under $\pi^*(0)$, then, under $\pi^*(0)$, we have $c(s-1)=\tau$.

■

If the hyphotesis of the last corollary hold for $s>2$, **theorem 2** defines a optimal solution if $c(s-1)=\tau$ and $c(s-2)>\tau$. If the hyphotesis of the last corollary hold for $s=2$ then, under $\pi^*(0)$, we have $c(1)=\tau$, in this case, the next theorem defines a optimal solution. The following notation is used in the theorem bellow.

Let:

$$K(t,1) = 0 \;;\; t = 1,\ldots,T \tag{22}$$

$$K_0(t,1) = \frac{1}{4b}(a-\tau)^2\frac{1-\lambda^{T-t}}{1-\lambda} \;;\; t = 1,\ldots,T-1 \tag{23}$$

$$\phi(t,1) = \frac{1}{2b} \;;\; t = 0,\ldots,T-1 \tag{24}$$

$$Q(0,0) = \frac{\tau+\beta\phi(0,1)a}{1+\beta\phi(0,1)} \tag{25}$$

$$K(0,1) = \frac{1}{4b} \tag{26}$$

$$K_0(0,1) = \lambda K_0(1,1) \tag{27}$$

Theorem 3

If:

$$c(0) \le Q(0,0) \tag{28}$$

where Q(0,0) is defined in **(24)** and **(25)**, and furthermore $c(2)=\tau$ under $\pi^*(0)$ and $b>0$; then:

i) $q^*(t)=\phi(t,1)(a-c(t))$; $J^*(c(t),t)= K_0(t,1)+K(t,1)(a-c(t))^2$; $t=0,\ldots,T-1$; where $\phi(t,1)$ and $K_0(t,1)$

for every $t\in\{0,\ldots,T-1\}$ are defined in **(23)**, **(24)** and **(27)**.

ii) In **problem I**, the unitary cost, under $\pi^*(0)$, reaches τ in the period 1 for the first time. ■

Next, we prove that, since the monopolist solves a strictly concave problem in every period, then, there cannot exist a value of c(0) such that it satisfies the conditions to reach τ, for the first time, in two different periods. This is shown formally in the next theorem. Before, we must note that, in view of **theorems 1, 2** and **3**, we can interpret R(j,t) as the minimum value that c(j) can take that, under the optimal policy for **problem I**, $c(t+1)>\tau$ holds (for $j\le t$, furthermore $R(t+1,t)=\tau$). It is also possible to make a similar interpretation for Q(j,t): it can be interpreted as the maximum value that c(j) can take that, under the optimal policy of the **problem I** with c(0)=c(j) and T=t+1, $c(t+1)=\tau$ holds (for $j\le t$, furthermore $Q(t+1,t)=\tau$).

Theorem 4

If **(12)** and **(21)** hold, then:

$$Q(0,t) \ < \ R(0,t) \quad t = 0,\ldots,T-2 \tag{29}$$

where R(0,t) and Q(0,t) for every $t\in\{0,\ldots,T-2\}$ are defined in **(7)** to **(10)**, **(13)** to **(18)**, **(24)** and **(25)**. ■

In the view of the last theorem, for example, we can have $Q(0,T-2)<c(0)\le R(0,T-2)$. The theorems presented up to this point do not define the optimal policy if, in a general way for $s\in\{0,\ldots,T-2\}$, $Q(0,s)<c(0)\le R(0,s)$ holds. The next theorem states the optimal policy for this case, and so the **problem I** is solved for any $c(0)\in(\tau,a)$.

The idea is very simple. Let, for example: $Q(0,T-2)<c(0)\le R(0,T-2)$. In this case c(0) is too low not to reach τ (under the optimal policy) in T periods ($c(0)\le R(0,T-2)$), so $c(T-1)=\tau$ holds. But if we have $c(T-1)=\tau$, then, for c(T-2) given, the problem to be solved is to maximize present benefit subject to the fact that the cost must be τ in T-1. By making s=T-1 in **theorem 2** we have: if $c(0)\le Q(0,T-2)$, then, for c(T-2) given, the output that maximizes the present benefit already satisfies the constraint that cost in T-1 must be τ. So, if $Q(0,T-2)<c(0)$ holds, then, for c(T-2) given, under concavity of the present profit function, the best q(T-2) to be taken is not the one that maximizes that function (it does not satisfy the constraint $c(T-1)=\tau$ since $Q(0,T-2)<c(0)$), but the nearest to the maximum which satisfies the constraint, that is, the point which exactly satisfies the constraint, which is: $\beta^{-1}(c(T-2)-\tau)$. We calculate for this value of q(T-2) the present profit in T-2, and the remaining problem from T-2 to

0 is linear quadratic, so we can solve Bellman's equation.

Next notation is used in the theorem.

Let $s\in\{1,\ldots,T-1\}$; let:

$$p_0(s-1) = -\frac{\tau}{\beta}\ ;\ p_1(s-1) = -\frac{1}{\beta} \tag{30}$$

$$K_{0,E}(s-1) = -\frac{\tau}{\beta}\left(a+\frac{b\tau}{\beta}\right)+\lambda K_0(s,s) \tag{31}$$

$$K_{1,E}(s-1) = \frac{1}{\beta}\left(a+\tau+2\frac{b\tau}{\beta}\right) \tag{32}$$

$$K_{2,E}(s-1) = -\frac{1}{\beta}\left(1+\frac{b}{\beta}\right) \tag{33}$$

and, if s>1, then, for every $t\in\{0,\ldots,s-2\}$:

$$p_0(t) = \frac{a-\lambda\beta K_{1,E}(t+1)}{2b-2\lambda\beta^2K_{2,E}(t+1)}\ ;\ p_1(t) = \frac{1+2\lambda\beta K_{2,E}(t+1)}{2b-2\lambda\beta^2K_{2,E}(t+1)} \tag{34}$$

$$K_{0,E}(t) = \lambda K_{0,E}(t+1)+\frac{1}{2}(a-\lambda\beta K_{1,E}(t+1))p_0(t) \tag{35}$$

$$K_{1,E}(t) = \lambda K_{1,E}(t+1)-(1+2\lambda\beta K_{2,E}(t+1))p_0(t) \tag{36}$$

$$K_{2,E}(t) = \lambda K_{2,E}(t+1)+\frac{1}{2}(1+2\lambda\beta K_{2,E}(t+1))p_1(t) \tag{37}$$

Theorem 5

Let $s\in\{1,\ldots,T-1\}$; if **(12)** and **(21)** hold and furthermore:

$$Q(0,s-1) \ < \ c(0) \ \le \ R(0,s-1) \tag{38}$$

$$b \ > \ \lambda\beta^2K_{2,E}(t)\ ;\ \ t = 1,\ldots,s-2 \quad (\text{for } s>1) \tag{39}$$

where $K_{2,E}(t)$, Q(0,s-1) and R(0,s-1) for every $t \in \{1,...,s-2\}$ are defined in **(7)** to **(10)**, **(13)** to **(18)**, **(24)**, **(25)**, **(33)** and **(37)**; then:

i) For $t \geq s$: $q^*(t)=\phi(s,s)(a-c(t))$; and $J^*(c(t),t)=K_0(t,s)$. Furthermore, for $t<s$: $q^*(t)=p_0(t)-p_1(t)c(t)$; and $J^*(c(t),t)=K_{0,E}(t)+K_{1,E}(t)c(t)+K_{2,E}(t)c(t)^2$. Where $\phi(s,s)$ and $K_0(t,s)$ for $t<s$ are defined in **(13)**, **(14)** and **(15)**.

ii) In **problem I**, the unitary cost reaches τ, under $\pi^*(0)$, for the first time, in the period s.

■

This theorem completes the solution method. It is important to observe that in view of the last theorem, we have established a partition in (τ,a), such that, depending on the specific value of c(0) the parameters of the production rule change.

IV Some examples

In this section, we illustrate the solution method with some examples with specific parameter values.

We fix, arbitrarily, T=60, λ=.9 and b=10 for all the problems solved (one problem for every set of parameters). We consider two possible values for a, which are: 10 (small in terms of b) and 40 (large in terms of b). Furthermore we consider two different values for the pair $(c(0),\tau)$: a) (8,1), that is, a large difference from the initial to the final cost, and b) (5,4), which represents a smaller difference. Finally we consider two different values for β: .5 and .1.

The results for every combination of the parameters given above are presented in table 1. Each row represents a different problem. In appendix II we plot a graph with the optimal policy, and in the first column of the table we put the number of the graph in appendix II corresponding to this problem. Columns two to five contain values of the parameter for each problem, and the last two columns have the value of the discounted benefit flow ($J^*(c(0),0)$) and the first period of cost τ (a blank appears if τ is not reached), under the optimal policy.

graph	a	c(0)	τ	β	$J^*(c(0),0)$	first time τ
1	10	8	1	.5	2.05	48
2	10	5	4	.5	8.03	8
3	10	8	1	.1	1.10	
4	10	5	4	.1	6.85	37
5	40	8	1	.5	334.14	9
6	40	5	4	.5	321.46	3
7	40	8	1	.1	280.70	40
8	40	5	4	.1	318.32	7

Table 1

The results presented in table 1 are reasonable. First, for every combination $(c(0),\tau,\beta)$ we have: the higher a, the higher $J^*(c(0),0)$ and the sooner the cost reaches τ (compare rows 1 to 5, 2 to 6, 3 to 7 and 4 to 8). Second, for every combination $(a,c(0),\tau)$ we have: the higher β, the higher $J^*(c(0),0)$ and the sooner the cost reaches τ (compare rows 1 to 3, 2 to 4, 5 to 7 and 6 to 8). Third, for every combination (a,β) we have: the shorter distance between c(0) and τ, the sooner the cost reaches τ (compare rows 1 to 2, 3 to 4, 5 to 6 and 7 to 8). Finally, also for every combination (a,β), in three cases we have that, the shorter distance between c(0) and τ, the higher $J^*(c(0),0)$ (compare rows 1 to 2, 3 to 4 and 7 to 8) and in one case the reverse holds (compare rows 5 to 6).

V Conclusions and further research

We obtain in closed-form the optimal policy for a class of models with *learning by doing*. The models consider an industry with a single agent, and are deterministic, with a finite time horizon and in discrete time. The demand function is linear. The cost evolution equation is taken from Dasgupta and Stiglitz (1988), in this equation the unitary cost of the next period is reduced linearly with the present output, but if the cost reaches a lower bound (τ), it remains in that value until the end of the decision problem.

The interest of the method presented is that, for the models described here, it finds the closed-form optimal policy. The key idea is to identify the role of τ as binding constraint. To do that, we construct a partition in the set of all possible values for the initial cost, such that, depending on the value of c(0) in the partition, an equivalent problem is formulated. We show that the optimal decision is a linear function indexed by initial cost, that is, depending on the value of the initial cost, the parameters of the decision rule change. The solution method is formally presented, and some examples with specific parameter values are also given.

Many extensions are being studied now. We outline here some of them. The solution method is developed for the case of a monopolist, but it is equally valid for the case of a nationalized industry. So, an extension to this paper is to apply this solution method to a nationalized industry and to study the differences between both structures in the view of the closed-form optimal policy. Furthermore, it is possible to extend this solution method for some simple stochastic problems, and to find the closed-form optimal policy for those problems. So, for some stochastic models, it is also possible to compare the behaviour of a monopoly to the behaviour of a nationalized industry, and to compare their behaviour in the stochastic and the deterministic case.

REFERENCES

ARROW, K. J. (1962), "The Economic Implications of learning by Doing", Review of Economic Studies, 29, 155-173.

BELLMAN, R. (1957). "Dynamic Programming", Princeton University Press, New Jersey.

DASGUPTA, P. y STIGLITZ, J. (1988), "Learning-by-Doing, Market Structure and Industrial and Trade Policies", Oxford Economic Papers, 40, 246-268.

FUDENBERG, D. y TIROLE, J. (1983), "Learning-by-Doing and Market Perfomance", Bell Journal of Economics, 14, 522-530.

PARENTE, S. L. (1994), "Technology Adoption, Learning-by-Doing, and Economic Growth", Journal of Economic Theory, 63, 346-369.

STOCKEY, N. (1986), "The Dynamics of Industry-wide Learning", en W. P. Heller, R. M. Starr y D. A. Starrett, Eds, "Equilibrium Analysis: Essays in Honour of Kenneth J. Arrow ", Vol II, Cambrigde University Press.

APPENDIX I

In this appendix we present the proofs of all theorems and corollaries. Previously we present a lemma, which will be used later in the proofs.

Lemma I

Given the problem:

$$\underset{q\geq 0}{Max}\ \{\ f(q)\ =\ (a-bq-c)q+\lambda K(a-c+\beta q)^2\} \qquad \textbf{(40)}$$

with $\lambda>0$, $\beta>0$, $K\geq 0$, and $a>c$.

If $b>\lambda\beta^2K$; then the optimal solution is:

$$q^*\ =\ \phi(a-c) \qquad \textbf{(41)}$$

where:

$$\phi\ =\ \frac{1+2\lambda\beta K}{2b-2\lambda\beta^2K} \qquad \textbf{(42)}$$

Furthermore:

$$f(q^*)\ =\ K'(a-c)^2 \qquad \textbf{(43)}$$

where:

$$K'\ =\ \lambda K+\frac{1}{2}(1+2\lambda\beta K)\phi \qquad \textbf{(44)}$$

■

Proof

The problem is a maximization problem subject to a inequality constraint. We denote by μ the Lagrange multiplier associated with the constraint $q\geq 0$, and by q^* the value which solves the problem. The necessary Kuhn-Tucker optimality conditions are: (i) $f'(q^*)+\mu=0$; (ii) $\mu\geq 0$; (iii) $q^*\geq 0$ and (iv) $\mu q^*=0$ (where the prime denotes derivative). If $\mu>0$ then it must be that $q^*=0$, and so $f'(0)+\mu=0$. From **(40)** we have: $f'(0)=(1+2\lambda\beta K)(a-c)$, and under the hypothesis of the lemma this is strictly positive, so it cannot be $f'(0)+\mu=0$ with $\mu>0$, so it must be $\mu=0$. By taking $\mu=0$, from (i) we have: $f'(q^*)=0$, and solving for q^* we have $q^*=\phi(a-c)$; where: $\phi=(1+2\lambda\beta K)(2b-2\lambda\beta^2K)^{-1}$. Furthermore: $f''(q)=-2b+2\lambda K\beta^2$, for every $q\in\mathbb{R}$. So, if $b>\lambda\beta^2K$, then $f''(q)<0$, for every $q\in\mathbb{R}$, f(q) is concave in q, the program is convex and hence q^* is a global maximum. Finally, by substituting q^* in f(q) we have: $f(q^*)=K'(a-c)^2$; where: $K'=\lambda K+½(1+2\lambda\beta K)\phi$.

■

Proof of theorem I

We define the auxiliary problem:

$$\underset{q(0),\ldots,q(T-1)}{MAX} \quad \{ \sum_{t=0}^{T-1} \lambda^t(a-bq(t)-c(t))q(t) \} \qquad (45)$$

subject to:

$$c(t+1) = c(t)-\beta q(t) \quad t = 0,\ldots,T-2 \qquad (46)$$

$q(t)\geq 0$ for every $t\in\{0,\ldots,T-1\}$

c(0) given

If the optimal policy of this auxiliary problem satisfies the next additional constraint:

$c(t)\geq\tau$, for every $t\in\{1,\ldots,T-1\}$

then, it it also the optimal policy for **problem I**.

The proof of the theorem has two steps. In the first step we solve the auxiliary problem and we prove that the optimal policy for this problem is the one that appears in the thesis i of the theorem. In the second step we prove that under the conditions given in the hypothesis and under the optimal policy of the auxiliary problem $c(T-1)>\tau$ holds.

First step. We denote by $J_a^*(c(t),t)$ the value function in the period t, and by $\pi_a^*(0)=\{q_a^*(0),\ldots,q_a^*(T-1)\}$ the optimal policy for the auxiliary problem. Bellman's equation associated with the auxiliary problem is:

$$J_a^*(c(t),t) = \underset{q(t)\geq 0}{MAX}\{(a-bq(t)-c(t))q(t)+\lambda J_a^*(c(t+1),t+1)\} \quad t = 0,\ldots,T-1 \qquad (47)$$

subject to **(46)**, and taking $J_a^*(c(T),T)=0$. Next we prove, by finite induction on t, that $q_a^*(t)=q^*(t)$ and $J_a^*(c(t),t)=J^*(c(t),t)$ for every $t\in\{0,\ldots,T-1\}$. In effect: let $t=T-1$, that is, c(T-1) given, then the problem to be solved is static, in fact is the problem of **lemma I** taking $c=c(T-1)$, $q=q(T-1)$ and $K=0$. From that lemma we obtain: $q_a^*(T-1)=\phi(T-1,T)(a-c(T-1))$. The sufficient condition for maximum is $b>0$, and if it holds, then **(5)** assures that the output is positive. From the lemma we also obtain $J_a^*(c(T-1),T-1)=K(T-1,T)(a-c(T-1))^2$. Let it now be the induction hypothesis for t+1: $q_a^*(t+1)=\phi(t+1,T)(a-c(t+1))$ and $J_a^*(c(t+1),t+1)=K(t+1,T)(a-c(t+1))^2$. Let c(t) given, we must solve the functional equation **(47)**, but this is the problem of **lemma I** taking: $c=c(t)$, $q=q(t)$ and $K=K(t+1,T)$. From that lemma we obtain: $q_a^*(t)=\phi(t,T)(a-c(t))$. The sufficient condition for maximum is $b>\lambda\beta^2K(t+1,T)$, and if it holds, then **(5)** and **(6)** assure that the output is positive. From the lemma we also have: $J_a^*(c(t),t)=K(t,T)(a-c(t))^2$. This concludes the first step.

Second step. We prove, under $\pi_a^*(0)$, that $c(0)>R(0,T-2)\Leftrightarrow c(T-1)>\tau$. We demonstrate that, under $\pi_a^*(0)$, the next chain of equivalences holds:
$c(T-1)>\tau\Leftrightarrow c(T-2)>R(T-2,T-2)\Leftrightarrow\ldots\Leftrightarrow c(t)>R(t,T-2)\Leftrightarrow\ldots\Leftrightarrow c(0)>R(0,T-2)$. Note that, for $t=T-2$, we have $c(T-1)>\tau\Leftrightarrow c(T-2)-\beta q_a^*(T-2)>\tau\Leftrightarrow c(T-2)>R(T-2,T-2)$, where the last implication is obtained by considering that $q_a^*(T-2)=\phi(T-2,T)(a-c(T-2))$. The induction hypothesis for t+1 is: $c(T-1)>\tau\Leftrightarrow c(t+1)>R(t+1,T-2)$. Now it must be proved for t. In effect, let be c(t) given: $c(T-1)>\tau\Leftrightarrow c(t+1)>R(t+1,T-2)\Leftrightarrow c(t)-\beta q_a^*(t)>R(t+1,T-2)\Leftrightarrow c(t)>R(t,T-2)$, where the last implication comes from the fact that $q_a^*(t)=\phi(t,T)(a-c(t))$. This demonstrates the chain of equivalences, and so concludes the second step.

■

Proof of corollary 1

If $c(T-1)>\tau$, then **problem I** is equivalent to the auxiliary problem defined in the demonstration of **theorem 1**, but the optimal policy of this auxiliary problem, which is defined under **(12)**, verifies $c(T-1)>\tau$ iff $c(0)>R(0,T-2)$, as we have seen in the second step of the proof of the **theorem 1**. So if $c(0)\leq R(0,T-2)$ then $c(T-1)=\tau$ holds, since for hypothesis it is $c(T-1)\geq\tau$.

■

Proof of theorem 2

Let the auxiliary problem:

$$\underset{q(0),\dots,q(T-1)}{MAX} \quad \{ \sum_{t=0}^{T-1} \lambda^t (a-bq(t)-c(t))q(t) \} \tag{48}$$

subject to:

$$c(t+1) = c(t) - \beta q(t) \quad t = 0,\dots,s-2 \tag{49}$$

$$c(t) = \tau \quad t = s,\dots,T-1. \tag{50}$$

$q(t) \geq 0$ for every $t \in \{0,\dots,T-1\}$.

$c(0)$ given

If, under $\pi^*(0)$, it holds $c(s)=\tau$, and if the optimal policy of this auxiliary problem satisfies the additional constraints:

$c(t) \geq \tau$, for every $t \in \{1,\dots,s-1\}$

$c(t)-\beta q(t) \leq \tau$, for every $t \in \{s-1,\dots,T-2\}$ (in fact, it is sufficient that $c(s-1)-\beta q(s-1) \leq \tau$)

then this optimal policy is also the optimal policy for **problem I**.

The proof is similar to the proof of **theorem 1** and it has two steps. In the first step we solve the auxiliary problem, and we show that the optimal policy for this problem is the one given in the thesis of the theorem. In the second step we prove that, under the hypothesis of the theorem, the optimal policy of the auxiliary problem satisfies the additional constraints given above.

First step. We denote by $J_a^*(c(t),t)$ the value function in the period t, and by $\pi_a^*(0)=\{q_a^*(0),\dots,q_a^*(T-1)\}$ the optimal policy for the auxiliary problem. The Bellman's equation associated with the auxiliary problem is:

$$J_a^*(c(t),t) = \underset{q(t)\geq 0}{MAX}\{(a-bq(t)-c(t))q(t)+\lambda J_a^*(c(t+1),t+1)\} \quad t = 0,\dots,T-1 \tag{51}$$

subject to **(49)** and **(50)**, and taking $J_a^*(c(T),T)=0$. Let $t=T-1$, that is $c(T-1)$ given, then the problem to be solved is static, indeed it is the problem of **lemma I** taking $c=c(T-1)$, $q=q(T-1)$ and $K=0$. From that lemma we have: $q_a^*(T-1)=\phi(T-1,s)(a-c(T-1))$. The sufficient condition for maximum is $b>0$, and if it holds, then **(5)** ensures that the output is positive. From the lemma we also have $J_a^*(c(T-1),T-1)=K_0(T-1,s)$. Now let the next induction hypothesis for $t \geq s$:

$J_a^*(c(t+1),t+1)=K_0(t+1,s)$ and $q_a^*(t+1)=\phi(t+1,s)(a-c(t+1))$. Let be $c(t)$ given, then the function to be maximized is again the function of **lemma 1** taking $c=c(t)$, $q=q(t)$ and $K=0$, plus the constant $\lambda K_0(t+1,s)$. From that lemma we have: $q_a^*(t)=\phi(t,s)(a-c(t))$, with $c(t)=\tau$. The sufficient condition for maximum holds if $b>0$. We also have, from **lemma 1**: $J_a^*(c(t),t)=K_0(t,s)$. Thus, by following the induction until $t=s$ we obtain: $J_a^*(c(s),s)=K_0(s,s)$ and $q_a^*(s)=\phi(s,s)(a-c(s))$. Let $t=s-1$, that is $c(s-1)$ given, then the function to be maximized obtained from Bellman's equation is the function of the **lemma 1** taking: $c=c(s-1)$, $q=q(s-1)$ y $K=0$, plus the constant $\lambda K_0(s,s)$. From **lemma 1** we have: $q_a^*(s-1)=\phi(s-1,s)(a-c(s-1))$. The sufficient condition for maximum is $b>0$. From **lemma 1** we also have: $J_a^*(c(s-1),s-1)=K_0(s-1,s)+K(s-1,s)(a-c(s-1))^2$. Let it now the induction hypothesis for $t \leq s-1$: $q_a^*(t+1)=\phi(t+1,s)(a-c(t+1))$ and $J_a^*(c(t+1),t+1)=K_0(t+1,s)+K(t+1,s)(a-c(t+1))^2$. Let $c(t)$ be given (with $t \leq s-1$), then we must solve Bellman's equation, and to do so, the function to be maximized is the function of the **lemma 1** taking $c=c(t)$, $q=q(t)$ and $K=K(t+1,s)$, plus the constant $\lambda K_0(t+1,s)$. From that lemma we obtain: $q_a^*(t)=\phi(t,s)(a-c(t))$. The sufficient condition for maximum is $b>\lambda\beta^2K(t+1,s)$. If the last inequality holds, then **(5)** and **(6)** ensure that output is positive. We also obtain from **lemma 1**: $J_a^*(c(t),t)=K_0(t,s)+K(t,s)(a-c(t))^2$.

Second step. Note first that: $c(s-1)-\beta q_a^*(s-1) \leq \tau \Leftrightarrow c(s-1) \leq Q(s-1,s-1)$. Next we prove, by induction on t, that, under $\pi_a^*(0)$, it verifies: $\tau < c(s-1) \leq Q(s-1,s-1) \Leftrightarrow R(s-2,s-2) < c(s-2) \leq Q(s-2,s-1) \Leftrightarrow \dots \Leftrightarrow R(t,s-2) < c(t) \leq Q(t,s-1) \Leftrightarrow \dots \Leftrightarrow R(0,s-2) < c(0) \leq Q(0,s-1)$. In effect, let $t=s-2$: $\tau < c(s-1) \leq Q(s-1,s-1) \Leftrightarrow \tau < c(s-2)-\beta q_a^*(s-2) \leq Q(s-1,s-1) \Leftrightarrow R(s-2,s-2) < c(s-2) \leq Q(s-2,s-1)$. Induction hypothesis for $t+1$: $\tau < c(s-1) \leq Q(s-1,s-1) \Leftrightarrow R(t+1,s-2) < c(t+1) \leq Q(t+1,s-1)$. Let $c(t)$ be given: $\tau < c(s-1) \leq Q(s-1,s-1) \Leftrightarrow R(t+1,s-2) < c(t+1) \leq Q(t+1,s-1) \Leftrightarrow R(t+1,s-2) < c(t)-\beta q_a^*(t) \leq Q(t+1,s-1) \Leftrightarrow R(t,s-2) < c(t) \leq Q(t,s-1)$. Thus the chain of equivalences we wanted to prove is proved, so in particular: $\tau < c(s-1) \leq Q(s-1,s-1) \Leftrightarrow R(0,s-2) < c(0) \leq Q(0,s-1)$; and this concludes the second step.

■

Proof of corollary 2

If $c(s-1)>\tau$ and $c(s)=\tau$, then **problem I** is equivalent to the auxiliary problem defined in the proof of **theorem 2**. The optimal policy for that problem is defined under **(21)**. Under the optimal policy for that auxiliary problem $c(s-1)>\tau$ holds iff: $c(0)>R(0,s-2)$, as we have proved in the second step of the proof of **theorem 2**, but we have $c(0) \leq R(0,s-2)$, so it must be that $c(s-1)=\tau$, since by definition $c(s-1) \geq \tau$.

■

Proof of theorem 3

Let the auxiliary problem:

$$\underset{q(0),\ldots,q(T-1)}{MAX} \quad \{ \sum_{t=0}^{T-1} \lambda^t (a - bq(t) - c(t)) q(t) \} \tag{52}$$

subject to:

$$c(t) = \tau \quad t = 1,\ldots,T-1. \tag{53}$$

c(0) given

$q(t) \geq 0$ for every $t \in \{0,\ldots,T-1\}$.

If under $\pi^*(0)$ verifies $c(1)=\tau$ and the optimal solution for this auxiliary problem satisfies the additional constraint:

$c(0)-\beta q(0) \leq \tau$

then this optimal policy is also the optimal for **problem I**.

The proof has two steps. In the first step we solve the auxiliary problem, and we prove that the optimal policy for this problem is the one given in the thesis of the theorem. In the second step we prove that, under the hypothesis of the theorem, the optimal policy of the auxiliary problem satifies the additional constraints given above.

First step. We denote by $J_a^*(c(t),t)$ the value function in the period t, and by $\pi_a^*(0)=\{q_a^*(0),\ldots,q_a^*(T-1)\}$ the optimal policy for the auxiliary problem. Bellman's equation associated with the auxiliary problem is:

$$J_a^*(c(t),t) = \underset{q(t)\geq 0}{MAX}\{(a-bq(t)-c(t))q(t)+\lambda J_a^*(c(t+1),t+1)\} \quad t = 0,\ldots,T-1 \tag{54}$$

subject to **(53)**, and taking $J_a^*(c(T),T)=0$. Let $t=T-1$, that is $c(T-1)$ given, then the problem to be solved is static, indeed it is the problem of **lemma I** taking $c=c(T-1)$, $q=q(T-1)$ and $K=0$. From that lemma we have: $q_a^*(T-1)=\phi(T-1,1)(a-c(T-1))$, with $c(T-1)=\tau$. The sufficient condition for maximum is $b>0$, and if it holds, then **(5)** ensures that the output is positive. From the lemma we also obtain $J_a^*(c(T-1),T-1)=K_0(T-1,1)$. Next, let the next induction hypothesis on $t+1$, for $t>1$:

$J_a^*(c(t+1),t+1)=K_0(t+1,1)$ and $q_a^*(t+1)=\phi(t+1,1)(a-c(t+1))$. Let c(t) be given, the function to be maximized, obtained from **(51)**, is the function of **lemma 1** taking: $c=c(t)$, $q=q(t)$ and $K=0$; plus the constant $\lambda K_0(t+1,1)$. From the equations of that lemma we have: $q_a^*(t)=\phi(t,1)(a-c(t))$, with $c(t)=\tau$. The sufficient condition for maximum is $b>0$. From **lemma 1** we have $J_a^*(c(t),t)=K_0(t,1)$. Thus, by continuing the induction we obtain $J_a^*(c(1),1)=K_0(1,1)$ and $q_a^*(1)=\phi(1,1)(a-c(1))$. Let $t=0$, that is, c(0) is given, then, the function to be maximized, obtained from the Bellman's equation, is the function of the **lemma 1** taking $c=c(0)$, $q=q(0)$ and $K=0$; plus the constant $\lambda K_0(1,1)$. Hence, we obtain: $q_a^*(0)=\phi(0,1)(a-c(0))$. The sufficient condition for maximum is $b>0$. Furthermore, from the **lemma 1**, we have: $J_a^*(c(0),0)=K_0(0,1)+K(0,1)(a-c(0))^2$.

Second step. Note that, under $\pi_a^*(0)$, we have: $c(1)=\tau \Leftrightarrow c(0)-\beta q_a^*(0)\leq\tau \Leftrightarrow c(0)\leq Q(0,0)$.

■

Proof of theorem 4

Let $t\in\{0,\ldots,T-2\}$. Indeed we prove that:

$$Q(j,t) < R(j,t) \; ; \; j = 0,\ldots,t \tag{55}$$

We define: $h(x,y)=(x+\beta ya)(1+\beta y)^{-1}$; $n(x)=(1+2\lambda\beta x)(2b-2\lambda\beta^2x)^{-1}$; $z(x)=\lambda x+(1+2\lambda\beta x)^2(4b-4\lambda\beta^2x)^{-1}$.

We have $Q(t,t)=h(\tau,\phi(t,t+1))$ and on other hand $R(t,t)=h(\tau,\phi(t,t+2))$, where $\phi(t,t+1)=n(0)$ and $\phi(t,t+2)=n(K(t+1,t+2))$ with $K(t+1,t+2)=z(0)$. Hence: $\phi(t,t+1)<\phi(t,t+2)$, and so $h(\tau,\phi(t,t+1))<h(\tau,\phi(t,t+2))$. Thus **(55)** is true for $j=t$. Furthermore, for the period t, associated with $\phi(t,t+1)$ we have $K(t,t+1)=z(0)$; and associated with $\phi(t,t+2)$ we have $K(t,t+2)=z(K(t+1,t+2))$. With simple algebra, and assuming the concavity condition it can be proved that $K(t,t+2)>K(t,t+1)$.

To prove **(55)** for any j, it is enough to establish the last result as an induction hypothesis that holds for $j+1$ and to prove that it also holds for j. So, let **(55)** be true for $j+1$, and let also $K(j+1,t+2)>K(j+1,t+1)$. By definition: $\phi(j,t+1)=n(K(j+1,t+1))$ and $\phi(j,t+2)=n(K(j+1,t+2))$, but since $n'(x)>0$ for all x, it verifies $\phi(j,t+2)>\phi(j,t+1)$. On other hand, since **(55)** is true for $j+1$ we have $Q(j+1,t)<R(j+1,t)$. Thus, since: $Q(j,t)=h(Q(j+1,t),\phi(j,t+1))$ and $R(j,t)=h(R(j+1,t),\phi(j,t+2))$; we have $Q(j,t)<R(j,t)$. Furthermore, $z'(x)>0$ for all x, and so we have $K(j,t+2)>K(j,t+1)$.

■

Proof of theorem 5

The **corollaries 1** and **2**, indicate that if $c(0) \leq R(0,s-1)$ and **(12)** and **(21)** hold, then, under $\pi^*(0)$, $c(s)=\tau$ must occur. In this case, $\pi^*(0)$ is also the optimal policy of the auxiliary problem defined in the proof of **theorem 2** for $s>1$ or **theorem 3** for $s=1$ if, the optimal policies of those auxiliary problems satisfy the additional constraints:

$$c(t) \geq \tau \text{ for every } t \in \{1,\ldots,s-1\} \text{ (if } s>1)$$

$$c(s-1)-\beta q(s-1) \leq \tau$$

Since $c(0)>Q(0,s-1)$, then it also verifies $c(0)>R(0,s-2)$ (for $s>1$) so, under the optimal policy of the auxiliary problems mentioned before, we have $c(s-1)>\tau$, and so the first of the additional constraints given above holds. However, the second constraint does not hold for those optimal policies. Since, under $\pi^*(0)$, this constraint must hold, then $\pi^*(0)$ is obtained from solving the auxiliary problem defined in the proof of **theorem 2** if $s>1$, and the one defined in the proof of **theorem 3** if $s=1$, and impossing:

$c(s-1)-\beta q(s-1) \leq \tau$.

For $t \geq s$, the optimal policy of the auxiliary problems satisfies both auxiliary constraints, so it is also the optimal policy for **problem I**. Thus, we obtain, for $t \geq s$, $q^*(t)=\phi(s,s)(a-c(t))$, with $c(t)=\tau$ and $J^*(c(t),t)=K_0(t,s)$. For $t=s-1$, from Bellman's equation associated with the auxiliary problems, we obtain the next problem to be solved: $\text{Max}_{q(s-1)} \{(a-bq(s-1)-c(s-1))q(s-1)+\lambda J^*(c(s),s)\}$ subject to $q(s-1) \geq \beta^{-1}(c(s-1)-\tau)$. The optimum of this problem without taking into account the constraint is: $(2b)^{-1}(a-c(s-1))$. From the second step of the **theorems 2** and **3** we obtain: $(2b)^{-1}(a-c(s-1)) \geq \beta^{-1}(c(s-1)-\tau) \Leftrightarrow c(s-1) \leq Q(s-1,s-1) \Leftrightarrow c(0) \leq Q(0,s-1)$. Since the last inequality does not hold, and $(a-bq(s-1)-c(s-1))q(s-1)+\lambda J^*(c(s),s)$ is globally concave in $q(s-1)$, then $q^*(s-1)$ is the lowest value of $q(s-1)$ such that it verifies the constraint, that is:

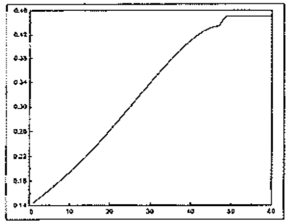
$q^*(s-1)=\beta^{-1}(c(s-1)-\tau)$, or $q^*(s-1)=p_0(s-1)-p_1(s-1)c(s-1)$. By substituting in the last objetive function we obtain: $J^*(c(s-1),s-1)=K_{0,E}(s-1)+K_{1,E}(s-1)c(s-1)+K_{2,E}(s-1)c(s-1)^2$. If $s>1$, let the next induction hypothesis for $t+1$, with $t<s-1$: $q^*(t+1)=p_0(t+1)-p_1(t+1)c(t+1)$ and $J^*(c(t+1),t+1)=K_{0,E}(t+1)+K_{1,E}(t+1)c(t+1)+K_{2,E}(t+1)c(t+1)^2$. Let $c(t)$ be given, the problem to be solved is: $\text{Max}_{q(t)\geq 0} \{(a-bq(t)-c(t))q(t)+\lambda K_{0,E}(t+1)+\lambda K_{1,E}(t+1)(c(t)-\beta q(t))+\lambda K_{2,E}(t+1)(c(t)-\beta q(t))^2\}$. The solution is

$q^*(t)=p_0(t)-p_1(t)c(t)$, and sufficient condition for maximum is $b>\lambda\beta^2 K_{2,E}(t)$. Furthermore, by substituting in the objetive function we obtain: $J^*(c(t),t)=K_{0,E}(t)+K_{1,E}(t)c(t)+K_{2,E}(t)c(t)^2$.

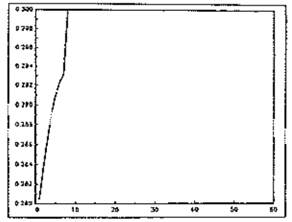
■

APPENDIX II

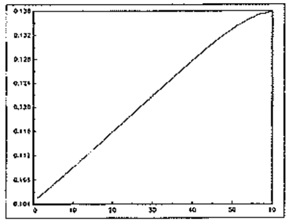
Horizontal axis represents time, and for every period we plot the optimal level of output.



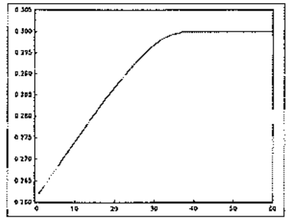
Graph 1



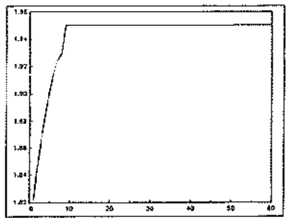
Graph 2



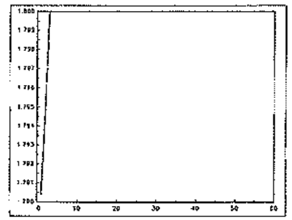
Graph 3



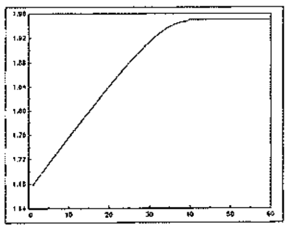
Graph 4



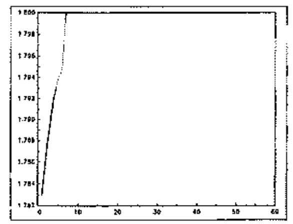
Graph 5



Graph 6



Graph 7



Graph 8