A dimension-based representation in multicriteria decision making

D. GÓMEZ, J.MONTERO, J. YÁÑEZ
Complutense University
Madrid (Spain)
email: dagomez@estad.ucm.es, javier_montero@mat.ucm.es

Abstract

Dimension Theory allows the representation of any finite set of alternatives in a real space, provided that the associated preference relation defines a partial order set. Such a representation can be very useful whenever criteria are not known, are therefore we can not even address the problem of evaluating their respective weights. In this paper we propose that the importance of underlying criteria can be approached taking into account those possible representations associated to the dimension of the binary preference relations between criteria.

Keywords: Multicriteria Decision Analysis, Valued Preference Relations, Dimension Theory.

1 Introduction

Assigning weights to criteria is a key issue in multicriteria decision making, since they use to be associated to the importance of those criteria explaining decision making. In fact, most approaches in the literature assume that decision maker is always able to evaluate importance of all existing criteria by means of a real value. But sometimes it is enough to know the relative position of each criteria in the real line (see, e.g., [8, 11] and [3].

Real-life applications show that, quite often, decision makers can not decide which one between two criteria is more important, since they are incomparable for such a decision maker. In most of the weighting methods this situation is unacceptable: decision makers should always be able establish a linear order on the set of criteria.

In this paper we want to stress the need of methods allowing to model those inconsistencies between criteria, if they really appear in decision maker mind. Our approach is based upon a main argument: a good representation of alternatives should give some hint about the structure of each decision making problem, and should therefore help decision maker to understand the problem and determine the importance or weights of criteria.

The paper is organized as follow: in section 2 we present a short review of classical dimension theory, pointing out its limitations. In section 3 we present a new dimension concept allowing the representation of arbitrary binary preference relations, and we extend this concept to the valued preference relation context. Then, in section 4 we propose a new method to determine criteria weights by means of the information in this way obtained. The paper ends with a final comments section stressing the relevance of this approach.

2 Classical dimension theory

Dimension concept has been widely developed in the context of crisp binary relations $R \subset X \times X$, i.e, mappings

$$\mu^R : X \times X \to \{0, 1\}$$

where $X = \{x_1, x_2, \ldots, x_n\}$ represents a finite set of alternatives and $\mu^R(x_i, x_j) = 1$ whenever $x_i R x_j$ and $\mu^R(x_i, x_j) = 0$ otherwise.

Dimension theory was initially developed by Dushnik-Miller [2], and subsequently applied to partial orders, i.e., crisp binary relations such that the following conditions hold: non reflexivity ($\mu^R(x_i, x_i) = 0 \quad \forall x_i \in X$), antisymmetry ($\mu^R(x_i, x_j) = 1 \Rightarrow \mu^R(x_j, x_i) = 0$), and transitivity ($\mu^R(x_i, x_j) = \mu^R(x_j, x_k) = 1 \Rightarrow \mu^R(x_i, x_k) = 1$).

Based on a result due to Szpilrajn [12] proving that every partial order can be represented as an intersection of linear orders, the dimension of a partial order R is defined by Dushnik-Miller [2] as the minimum number of linear orders (complete partial orders) whose intersection is R. Being R a partial order set (*poset*) with dimension d, each element $x_i \in X$ can be represented in the real space $(x_i^1, \ldots, x_i^d) \in \Re^d$ in such a way that $x_i R x_j$ if and only if

$$x_i^k > x_j^k \quad \forall k \in \{1, \ldots d\} \quad \forall x_i, x_j \in X$$

Of course, dimension is unique, but the associated representation is not unique (see Trotter [13]).

2.1 Dimension function

Given X a finite set of alternatives, a valued preference relation in X is a fuzzy subset of the cartesian product $X \times X$, being characterized by its membership function

$$\mu : X \times X \rightarrow [0, 1]$$

in such a way that $\mu(x_i, x_j)$ represents the degree to which alternative x_i is preferred to alternative x_j. We shall assume that such a preference intensity is referred to a *strict* preference, so by definition $\mu(x_i, x_i) = 0 \quad \forall x_i \in X$.

Once a value $\alpha \in (0, 1]$ has been fixed, the α-cut of a valued preference relation μ is defined as the crisp binary relation R^α in X such that

$$x_i R^\alpha x_j \iff \mu(x_i, x_j) \geq \alpha$$

Then, meanwhile R^α is a partial order set (antisymmetric and transitive), it defines a crisp partial order, so its dimension $d(\alpha)$ is defined. A *dimension function* has been in this way defined

$$d : [0, 1] \rightarrow N$$

where $d(\alpha) = dim(R^\alpha)$ whenever such a dimension is well defined. In this way, dimension function translates classical dimension into a valued preference context.

Crisp dimension of all α-cuts, as proposed in [5], seems a useful hint for decision makers in order to understand the decision making problem. In fact, they are taken into account in [1] in order to obtain operative bounds.

However, a first approach requires antisymmetry and transitivity for each α-cut. In case our valued preference relation is max-min transitive, i.e.,

$$\mu(x_i, x_j) \geq \min\{\mu(x_i, x_k), \mu(x_k, x_j)\}$$

for all $x_i, x_j, x_k \in X$, then R^α is a partial order set whenever antisymmetry holds, i.e., meanwhile those α-cuts do not show 2-order cycles. In particular (see [6]), R^α is antisymmetric for all $\alpha > \alpha_2$, being

$$\alpha_2 = \max_{x_i \neq x_j} \min\{\mu(x_i, x_j), \mu(x_j, x_i)\}$$

Therefore, since μ is max-min transitive if and only if every α-cut R^α is transitive (see [7] but also [1]), if μ is max-min transitive, then we can consider the dimension of R^α for every $\alpha > \alpha_2$.

So, some kind of general representation for any arbitrary α-cut is desirable, even if it is non symmetric or non transitive. Any useful representation should allow a dimension function being defined in the whole unit interval, but in some way showing every inconsistency. Hence, we should be searching for explanatory representations of arbitrary crisp preference relations. As pointed out in [5, 6], there is an absolute need to understand and explain decision maker inconsistencies: accepted inconsistencies are some times extremely informative. These considerations suggests a generalization of Dushnik-Miller representation theorem, as shown in the next section.

3 Dimension theory for arbitrary preference relations

As already pointed out, representation of arbitrary binary preference relation is itself an objective, perhaps more interesting than the associated dimension value. The following result shows that any strict preference relation can be represented in terms of unions and intersections of linear orders (see [5, 6] but also [4]): meanwhile *incomparability* is explained by the intersection operator, *inconsistencies* (i.e., symmetry and non transitivity) will be associated to the union operator, at least at a first stage.

Theorem 3.1 *Let $X = \{x_1, ..., x_n\}$ be a finite set of alternatives, and let us consider*

$$\mathcal{C} = \{L/L \text{ linear order on } X\}$$

Then for every non-reflexive crisp binary relation R on X there exist a family of linear orders $\{L_{st}\}_{s,t} \subset \mathcal{C}$ such that

$$R = \bigcup_s \bigcap_t L_{st}$$

Proof: see [5].

The classical concept of dimension is in this way generalized into a more general framework, now not being restricted to partial order sets.

Definition 3.1 *Let us consider X a finite set of alternatives. The* generalized dimension *of a crisp binary relation R is the minimum number of different linear orders, L_{st}, such that*

$$R = \bigcup_s \bigcap_t L_{st}$$

It should be pointed out that, in general, generalized dimension of a partial order may not be equal to its classical dimension value.

3.1 Dimension Function of arbitrary valued preference relations

Based upon the above generalized representation of crisp preferences we can therefore assure the existence of a *generalized* dimension function:

Definition 3.2 *Given a valued strict preference relation*

$$\mu : X \times X \to [0, 1]$$

its generalized dimension function *is given by the mapping*

$$\begin{array}{rccl} D: & (0,1] & \to & N \\ & \alpha & \to & D(\alpha) = Dim(R^\alpha) \end{array}$$

where $Dim(R^\alpha)$ is the generalized dimension of R^α.

This approach will then lead to a *generalized dimension function* showing the generalized dimension for every α-cut, no matter if our valued preference relation μ is max-min transitive or not. In general, being X a finite set of alternatives, the interval $(0, 1]$ is divided in two subsets, depending on the existence of union operators in the above generalized representation, suggesting the existence or non existence of inconsistencies for each $R^\alpha, \alpha \in (0, 1]$.

4 Determination of weights

If the search for good representation models is an objective, dimension theory seems a natural alternative. In this section we will show how we can use this representation in order to determine the importance of each criteria, avoiding the strict constraints assumed in standard approaches (see [10, 11, 8, 3]), both in the crisp and valued context (see [4]).

Given $\mu_C : C \times C :\longrightarrow: [0, 1]$ a valued preference relation between the criteria, the generalized dimension function, and the associated representations, for each value $\alpha \in [0, 1]$ will be useful to determine the importance of each criteria in a decision making problem.

4.1 Partial order criteria sets

Let $\mu_C : C \times C :\longrightarrow \{0, 1\}$ be the binary preference relation between criteria. Let us suppose first that μ_C is a partial order set. Let $R = \bigcap_{k=1,d} L_k$ be a representation of R, being accepted (understood) by the decision maker. Notice that most classical approaches assume $dim(R) = 1$, so criteria can be represented in the real line.

Definition 4.1 *Let C be the set of criteria and let L be a lineal order on C, then we will say that $F_L : C \longrightarrow [0, 1]$ is a fair allocation rule for the pair (C, L) if and only if:*

- *If $c_i L c_j$ then $F_L(c_i) < F_L(c_j)$.*

- $\sum_{c\in C} F_L(c) = 1.$

An example of fair allocation rule could be $F(c_j) = W_j$ where

$$W_j = \frac{\frac{1}{r_j}}{\sum_{i=1}^{n} \frac{1}{r_i}}$$

or

$$W_j = \frac{(n - r_j + 1)}{\sum_{i=1}^{n} (n - r_i + 1)}$$

being r_j the position of the j-th criteria in the above ordering.

It is easy to see that Saaty, Simos and Figueira-Roy models [10, 11, 3] are *fair allocation rules* in the sense of the above definition, although some of these methods require additional information.

Once we have a *fair allocation rule* F_L for each lineal order L, we only need to aggregate the weights $(F_L(c_1), F_L(c_2), \ldots F_L(c_{|C|}))$ in such a way the sum of final weights is one.

So, given a representation of the Binary preference relation $R = \bigcap_{k=1}^{d} L_k$, a family of *fair* allocation rules for this representation $\{F_{L_k}\}$ and given $\phi : [0,1]^d \longrightarrow [0,1]$ an aggregation operator, the final weights of the criteria $(W_1, W_2, \ldots W_{|C|})$ can be obtained as

$$W_i = \phi\left(F_{L_1}(c_i), \ldots, F_{L_d}(c_i)\right) \ \forall i = 1, \ldots, |C|$$

Notice that aggregation operators can not be chosen arbitrarily, since the final sum of weights must be one.

Definition 4.2 *Let $\phi : [0,1]^d \longrightarrow [0,1]$ be an aggregation operator, we will say that this operator is doubly efficient respect to the sum if*

$$\sum_{i=1}^{|C|} \phi(w_i^1, w_i^2, \ldots, w_i^d) = 1$$

when $\sum_{j=1}^{|C|} w_i^j = 1 \quad \forall i = 1, 2, \ldots, d$

Proposition 4.1 *If ϕ is a additive aggregation operator i.e.*

$$\phi_n(a_1, \ldots, a_d) = \sum_{k=1}^{d} c_k a_k \qquad (1)$$

then ϕ is doubly efficient respect to the sum.

Proof: As $\phi(1, \ldots, 1) = 1$ then we have that $\sum_{i=1,n} a_i = 1$. Then

$\sum_{i=1}^{|C|} \phi(w_i^1, w_i^2, \ldots, w_i^d) = \sum_{i=1}^{|C|} \sum_{r=1}^{d} a_r w_i^r =$

$\sum_{r=1}^{d} a_r \sum_{i=1}^{|C|} w_i^r = \sum_{r=1}^{d} a_r = 1.$

So we can use any additive aggregation operator.

Example 4.1 *Let $C_1, \ldots C_3$ be the criteria of a Decision Making problem, and let $R = \mu_C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$, i.e. $c_1 < c_3$ (the importance of c_1 is less than c_3), $c_2 < c_3$ and $c_1 || c_2$ (we know that the importance of these criteria are incomparable). Observe that this possibility is not allowed in standard methods. In this case, $dim(R) = 2$ because $R = [c_1, c_2, c_3] \bigcap [c_2, c_1, c_3]$. If decision maker accepts and understands this representation, and for example if $F_{L_k}(c_j) = w_j$, where $w_j = \frac{\frac{1}{r_j}}{\sum_{i=1}^{n} \frac{1}{r_i}}$, then the weights for L_1 are $(0.2, 0.3, 0.5)$, and for L_2 are $(0.3, 0.2, 0.5)$. If ϕ is the median aggregate operator $W = (0.25, 0.25, 0.5)$*

4.2 Valued preference relation

Let $\mu_C : C \times C : \longrightarrow [0,1]$ be a valued preference relation between the criteria of the decision problem. In this sense $\mu_C(c_i, c_j)$ represents the degree to which the decision maker believes that c_i is more important than c_j. In order to generalize the classical representation into valued representation we need to impose that valued preference relation is max-min transitive and also $\mu_C(c, c) = 0$. For each $\alpha \in (0,1]$ a representation of criteria can be obtained and we can determine the importance (measuring as weights) of each criteria. So we have a weighting function $W(\alpha)$ depending of the aptitude of the decision maker it is important to notice that this function shows different decision maker attitudes. In fact, different people with the same valued preference relation, if forced to be crisp, can face different problems depending on their exigency level.

If we consider that any aptitude of the decision maker is possible, we can aggregate this information, for example as $\int_0^1 W(\alpha)d\alpha$. Of course, if we know that the probability distribution of the aptitude (define by means of their density function $a(\alpha)$), we can aggregate $W(\alpha)$ as $\int_0^1 W(\alpha)\ a(\alpha)d\alpha$.

We want to emphasize the importance of the weights function as an additional representation tool to dimension function, in order to understand better the problem.

4.3 Generalized dimension function

The above sections allow us to obtain the importance of each criteria when decision maker preferences between criteria are max-min transitive, but of course this is not always possible. In order to determine the weights in a more general case, Pachon *et al.* (2003) introduce a new dimension concept that allow us to represent any binary relation, as already shown in a section 3.

Given R a binary preference relation represented as $R = \bigcup_{r=1}^{k} \bigcap_{s=1}^{d_r} L_{rs}$ we can aggregate this information taking into account a *fair* allocation rule for each L_{rs}. Now we need to aggregate the information in two steps in order to obtain the final importance of each criteria. Let us denote $w_j^{rs} = F_{L_{r,s}}(c_j)$ the weigh of the j-th criteria in the lineal order L_{rs}. First, we need to aggregate the information contained in $\cap_{s=1,d_r} L_{rs}$, obtaining $w_j^r = \phi_r(w_j^{r,1}, w_j^{r,2}, \ldots, w_j^{r,d_r})$. The final weight W_j is the aggregation of w_j^r $\forall r = 1, \ldots, k$, so $W_j = \varphi(w_j^1, w_j^2, \ldots, w_j^k)$.

Given $R = \cup_{r=1,k} \cap_{s=1,d_r} L_{rs}$ a representation of the binary preference relation, $\phi_r : [0,1]^{d_r} \longrightarrow [0,1]$ be a family of aggregation operator for $r = 1, \ldots k$. and $\varphi : [0,1]^k \longrightarrow [0,1]$ other aggregation operator, we need to impose

$$\sum_{j=1}^{|C|} \varphi\left(\phi_1\left(w_j^{11}, \ldots w_j^{1,d_1}\right), \ldots, \phi_k\left(w_j^{k1}, \ldots w_j^{k,d_k}\right)\right)$$

takes value 1, for all w_j^{rs} such that

$$\sum_{j=1}^{|C|} w_j^{rs} = 1 \quad \forall s = 1, \ldots, d_r \ ; \ \forall r = 1, \ldots, k$$

Proposition 4.2 *Given $R = \cup_{r=1,k} \cap_{s=1,d_r} L_{rs}$ a representation of the binary preference relation, $\phi_r : [0,1]^{d_r} \longrightarrow [0,1]$ a family of aggregation operators for $r = 1, \ldots k$, and $\varphi : [0,1]^k \longrightarrow [0,1]$ other aggregation operator, if φ and $\{\phi_r\}r = 1, k$ are aggregation operators being double efficient with respect to the sum, then*

$$\sum_{j=1}^{|c|} \varphi\left(\phi_1\left(w_j^{11}, \ldots w_j^{1,d_1}\right), \ldots, \phi_k\left(w_j^{k1}, \ldots w_j^{k,d_k}\right)\right)$$

is one, for all w_j^{rs} such that

$$\sum_{j=1}^{|C|} w_j^{rs} = 1 \quad \forall s = 1, \ldots, d_r \ ; \ \forall r = 1, \ldots, k$$

Proof: Let us denote by $w_j^r = \phi_r(w_j^{r1}, \ldots, w_j^{rd_r})$. As φ is a aggregation operator double efficient respect to the sum we only need to prove that $\sum_{j=1}^{|C|} w_j^r = 1$ $\forall r = 1, \ldots k$. Fixed $r \in \{1, \ldots k\}$ $\sum_{j=1}^{|C|} w_j^r = \sum_{j=1}^{|C|} \phi_r(w_j^{r1}, \ldots, w_j^{rd_r})$ where $\sum_{s=1,d_r} w_j^{rs} = 1, \forall j$, then as ϕ_r is doubly efficient $\sum_{j=1}^{|C|} \phi_r(w_j^{r1}, \ldots, w_j^{rd_r}) = 1$ so the result holds.

Proposition 4.3 *If φ and ϕ are additive aggregation rules then $\sum_{i=1}^{|C|} W_i = 1$*

Proof: direct from 4.1 and 4.2.

5 Final remarks

In this paper we propose an alternative method to determine the importance of criteria in a decision problem, based on a representation of criteria in a real space. This method will be specially useful when incomparability between criteria appears.

From the above comments it is clear that additive operator is not unique. Operators should therefore be chosen carefully, taking into account the idea of aggregation our decision maker has in mind.

It can be also observed from the definition of dimension that the associated representation is not unique. We should be looking forward a useful representation, understandable by the decision maker.

The above dimension approach to preference relations (valued or not valued) opens in our opinion an interesting approach in order to help decision maker to put in clear the importance of criteria, although alternative rationalities should be also tried (see [7]).

Anyway, this paper stresses the role of representation tools in order to get a good understanding of every decision making problem, as a previous requirement for decision making. In particular, we explore representation techniques based upon dimension theory, showing how this approach can give some interesting hint about the underlying criteria and their relative importance in a quite general context, but still being consistent with some key standard procedures.

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