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AN OPTIMAL SEQUENCE OF LANDFILLS IN MUNICIPAL SOLID WASTE MANAGEMENT

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ABSTRACT

Given that landfills are depletable and replaceable resources, the right approach, when dealing with landfill management, is that of designing an optimal sequence of landfills rather than every single landfill separately. In this paper we use Optimal Control models, with mixed elements of both continuous and discrete time problems, to determine an optimal sequence of landfills, as regarding their capacity and lifetime. The resulting optimization problems involve dividing a time horizon of planning into subintervals the length of which has to be decided. In each of the subintervals some costs, the amount of which depends on the value of the decision variables, have to be borne. The obtained results are useful for other economic problems such as private an public investments, consumption decisions on durable goods, etc.

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1 Introduction

The aim of this paper is to analyse how the optimal capacity and the swithching time of a sequence of landfills has to be decided, considering both the construction and management costs.

As noted in Ready and Ready (1995), landfills are depletable and replaceable resources. Unlike other natural resources, whose depletion is irreversible, once a landfill is full it can be replaced at some cost, by constructing a new one. The new landfill will also be depleted and so on. As a consequence, the capacity of a landfill should not be decided from a static point of view, just by considering the costs associated with the present landfill, but also the costs linked to the following ones. Therefore, instead of optimally designing a landfill, the appropriate approach is that of designing an optimal sequence of landfills. In Ready and Ready (1995), Huhtala (1997) and Gaudet, Moreaux and Salant (1998) the sequential aspect of landfills is recognized. However, in these papers, the capacity is a given and therefore the problem of obtaining the optimal capacity is not considered. In Huhtala (1997) the substitution of a landfill by a new one is studied and the capacity of the new landfill is a decision variable; however the capacity of the first landfill is given and moreover, the decision capacity is not studied in depth.

The higher is the capacity of the landfill to be constructed, the higher is the construction cost. If the objective of the planner is just to minimise the present building costs for this landfill, the obvious solution is to construct a landfill with the smallest capacity. The consequence will be a short lifetime of the constructed capacity and the construction of a new landfill will have to be undertaken very soon. There is a conflict between present and future costs and so we are faced with a typical economic dynamic problem, where the present and future decisions are not independent, but have to be jointly taken. Specifically, the planning time horizon has to be divided into several subintervals the length of which has to be decided, and in each of the subintervals some costs, the amount of which depends on the values given to the decision variables, are realised. In this paper, the described problem is formalized and the solution is discussed under different assumptions. The models we state are Optimal Control problems, with mixed elements of both continuous and discrete time problems. The obtained results are useful for other economic problems such as private and public investments, consumption decisions on durable goods, etc.

The rest of the paper has the following structure: In section 2 the basic problem, under the assumption of constant waste generation, is stated and its main characteristics are discussed. In section 3 a generalization is studied, assuming that the instant generation of waste follows a given evolution through time. In section 4 the joint problem of optimal capacity of landfills and the optimal landfilled and recycled amount of waste is analysed. The paper finishes with the conclusions (section 5) and the proof of the mathematical results in an appendix (section 6).

2 Problem with Constant Generation of Waste

A social planner has to take the following actions in order to manage, with the smallest possible cost, the waste produced in a time horizon of length τ:

1. At instant $t = 0$, to construct a landfill, with arbitrary capacity Y_0, with a set up cost which depends on Y_0, according to the increasing, convex and $C^{(2)}$ cost function $C(Y_0)$.

2. While the first landfill is being used, he has to pay the instantaneous waste management cost, given by the linear function $h_0(Q(t)) = \phi_0 Q(t)$, where ϕ_0 is the unit management cost, basically representing the collection, transportation and processing costs of one unit of waste, and $Q(t)$ is the amount of waste produced at instant t, which we assume is exogenous. In this section, we assume that $Q(t) = Q$ is constant. This assumption is relaxed in section 3.

3. When the capacity of the first landfill is exhausted, which happens at time T_1, implicitly determined by the condition $\int_0^{T_1} Q(t)\,dt = Y_0$, the planner has to close it and to construct a new one, in another place, with capacity Y_1, which will last until time T_2 given by $\int_{T_1}^{T_2} Q(t)\,dt = Y_1$.

4. Between T_1 and T_2, he has also to pay the management costs of the waste produced in this period. These costs are given by the function $h_1(Q(t)) = \phi_1 Q(t)$, where the unit cost ϕ_1, in general, is different from ϕ_0, due to the different transport costs, land types, etc.

And so on, until the last landfill, denoted by $K-1$, being K a decision variable. In general, a landfill

constructed at T_i with a capacity Y_i lasts until T_{i+1}, implicitly defined by the equation $\int_{T_i}^{T_{i+1}} Q(t)\,dt = Y_i$, and the instantaneous management costs associated with such a landfill are given by $h_i(Q(t)) = \phi_i Q$.

From a mathematical point of view, the described problem (P) has a particular structure which incorporates some continuous time and some discrete time elements. On the one hand, the time variable t is continuous, waste is generated in continuous time and the management costs $h_i(Q(t))$ are produced in continuous time. The variables T_i, which refer to time can take any real value, as corresponds to a continuous time optimal control model. On the other hand, the construction costs happen at a finite number of times, as happens in discrete time optimal control problems.

Nevertheless, assuming that the capacities of all landfills are depleted under the solution, the problem can be expressed as one in discrete time, in the following way: given the assumption $Q(t) = Q$, we obtain $T_1 = \frac{Y_0}{Q}, \ldots, T_{i+1} = T_i + \frac{Y_i}{Q}, \ldots, \tau = T_K = T_{K-1} + \frac{Y_{K-1}}{Q}$, so that, the planner has to find a number of landfills K, and a sequence of capacities $\{Y_0, Y_1, \ldots, Y_{K-1}\}$, in order to minimise the function

$$\sum_{i=0}^{K-1} e^{-\delta T_i} C(Y_i) + \sum_{i=0}^{K-1} \left[\int_{T_i}^{T_{i+1}} e^{-\delta t} h_i(Q(t))\,dt \right] = \sum_{i=0}^{K-1} e^{-\delta T_i} \left[C(Y_i) + \int_{T_i}^{T_{i+1}} e^{-\delta(t-T_i)} \phi_i Q\,dt \right] \qquad \text{(P)}$$

subject to the following constraints

$$\begin{aligned} T_0 &= 0,\ T_K = \tau, \\ T_{i+1} &= T_i + \frac{Y_i}{Q}, \qquad i = 0, 1, 2, \ldots, K-1, \\ \underline{Y} &\le Y_i \le \overline{Y}, \end{aligned} \qquad (1)$$

where $\underline{Y}$ and $\overline{Y}$ represent the minimum and maximum capacity constraints and δ is the discount rate. Note that (P) can be regarded as a discrete time optimal control problem, where the "discrete time" is not given by the chronological time t, but by the different landfills $i = 0, 1, \ldots, K-1$, and (1) is the state equation.

This problem is conceptually similar to that of exploiting a sequence of deposits of a natural resource, as studied in Herfindahl (1967), Hartwick (1978), Weitzman (1976) or Hartwick, Kemp and Long (1986), where the role of extraction cost is played by the management costs in our problem. Anyway, there are two important difference: first, in our case, the rhythm of depletion of the landfill, analogous to the rhythm of natural resource extraction, can not be decided because it is given by the exogenous generation of waste. Second, the initial landfill capacity (analogous to the initial resource stock) is not given in our problem, as in natural resource extraction models, but it is a decision variable.

The classic result by Herfindahl (1967) for various natural resource deposits, which states that the extraction has to be done in an increasing order of marginal extraction costs, apply here. In our case, if there is no other difference among landfill places, it is optimal to deplete the landfills in an increasing order of their marginal management costs[1].

Because K is a decision variable, (P) is a free time horizon problem. The easiest way to solve it consists of finding the solution for all possible values of K, and choosing that which provides the minimum total cost. Specifically, K can take any integer value such that $K \in \{K_{\min}, K_{\min}+1, \ldots, K_{\max}-1, K_{\max}\}$, where

$$K_{\min} = \begin{cases} \dfrac{\tau Q}{\overline{Y}} & \text{if } \dfrac{\tau Q}{\overline{Y}} \text{ is an integer,} \\ Int\left(\dfrac{\tau Q}{\overline{Y}} + 1\right) & \text{otherwise,} \end{cases} \quad ; \quad K_{\max} = Int\left(\frac{\tau Q}{\underline{Y}}\right).$$

[1] Let us assume that the solution of the problem (P) is a sequence of capacities $\{Y_i^*\} = \{Y_0^*, Y_1^*, \ldots, Y_k^*, Y_{k+1}^*, Y_{k+2}^* \ldots\}$, where $\phi_k > \phi_{k+1}$. The discounted cost of the solution $\{Y_i^*\}$ can be reduced just by changing the order of the landfills k y $k+1$.

$Int(\xi)$ denoting the integer part of ξ. Henceforth, $K_{\max}-K_{\min}+1$ discrete time optimal control problems have to be solved. Let $\hat{C}_K$ be the optimal discounted cost which can be obtained constructing K landfills. The optimal value of K is given by $K^* = \underset{\{K=1,\dots,\bar{K}\}}{\arg\min}\ \hat{C}_K$.

Now, let us regard the solution for each possible value of K. The most interesting case, as for its economic interpretation, is the one where the minimum and maximum capacity constraints are not binding. For that reason, in what follows we concentrate on the interior solutions. The capacities of an optimal sequence of K landfills, in an interior solution, have to meet what we call the Optimal Capacity Condition, which is stated in the following proposition:

Proposition 1 *Given an arbitrary value of K, in an interior solution of problem (P), for two arbitrary consecutive landfills, k and $k+1$, the following Optimal Capacity Condition holds:*

$$\begin{aligned} C'(Y_k) &= e^{-\frac{\delta}{Q}Y_k}\left[C'(Y_{k+1}) + \tfrac{\delta}{Q}C(Y_{k+1}) + \Delta\phi_k\right] \\ &= e^{-\delta(T_{k+1}-T_k)}\left[C'(Y_{k+1}) + \tfrac{\delta}{Q}C(Y_{k+1}) + \Delta\phi_k\right], \qquad k = 0,1,\dots,K-2 \end{aligned} \tag{2}$$

where $\Delta\phi_k = \phi_{k+1} - \phi_k$ is the unit management cost increment from landfill k to landfill $k+1$■

Proof: see section 6■

Condition (2) is a nonlinear first order difference equation which represents the relation between the optimal capacity of two consecutive landfills, k and $k+1$. In order to economically interpret this condition, think of a situation in which $\Delta\phi_k = 0\ \forall k$ and $\delta = 0$, that is, the unit cost of management is identical for all the landfills and there is no time discount, so that all the costs have the same weight in the objective function. Then (2) takes the form

$$C'(Y_k) = C'(Y_{k+1}), \tag{3}$$

which can be taken as a non-arbitrage condition: if $C'(Y_k) < (>)\, C'(Y_{k+1})$, total cost could be reduced by reducing Y_k (Y_{k+1}) and increasing Y_{k+1} (Y_k). Condition (3) establishes the impossibility of reducing total cost by transferring some capacity from one landfill to another one. With a strictly positive discount rate and different unit management costs, the relevant equation is (2), which is still a non- arbitrage condition, but now the marginal effect of transferring capacity from one landfill to another has two additional components: the delay (or advance) of the future construction costs and the difference between the management costs borne on one or other landfill. The greater is the expected transportation cost increment for the next landfill, $\Delta\phi_k$, the greater is the value of the right hand side of (2). In order to maintain the equality, the left hand side, that is, the marginal construction cost of Y_k, has to be greater too. Assuming that C is a convex function, and therefore $C'(Y_k)$ is increasing with Y_k, it follows that the greater is $\Delta\phi_k$, the greater is the optimal capacity of landfill k. This conclusion is reasonable from an economic point of view: if future landfills are subject to large management costs increments, it is optimal to increase the capacity of the present landfill in order to extend its lifetime and to delay future management costs associated with the next landfills.

2.1 Example

Let us assume that the building cost function has the following cuadratic form:

$$C(Y) = a + bY + \frac{c}{2}Y^2, \qquad a,b,c > 0.$$

and the unit management cost increases at a rate ε , from landfill i to landfill $i+1$, according to the following equation:

$$\phi_{k+1} = (1+\varepsilon)\,\phi_k.$$

The numerical solution is obtained, using the Matlab optimization toolbox for the following parameter values:

$$\begin{array}{lll} a = 100000, & \tau = 50, & \phi_0 = 1 \\ b = 1, & \bar{Y} = 5000, & \varepsilon = 0.1 \\ c = 0.5, & \underline{Y} = 333.3, & \delta = 0.02. \\ Q = 100, & & \end{array} \tag{4}$$

from which, we know that $K_{\min} = 1$ and $K_{\max} = 15$. The solution is shown in figure 2.1

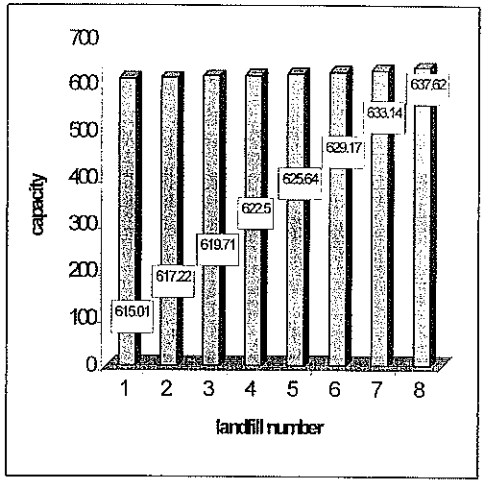

Figure 2.1. Solution for example 2.1

As shown in the labels, in this solution, landfill capacities are slightly increasing.

We now perform sensitivity analysis to describe the effects of changes in parameter values. Figure 2.2 shows the effects of parameters a, b, c, δ, Q and τ on the optimal number of landfills, K^*, and the average capacity of landfills in the solution, that is, $\tilde{Y} = \frac{1}{K}\sum_{i=0}^{K-1} Y_i$, which can be used to measure the level of the sequence $Y_0, Y_1, \dots, Y_{K-1}$.

Note the economic meaning of these results: increasing parameter a, which measures fixed construction costs, makes it optimal to build fewer landfills with a bigger capacity. The opposite occurs when increasing the variable building cost parameters b and c: it is optimal to build more landfills with a smaller capacity. Increasing the rate of discount δ leads to an increase in the weight given in the objective function to short term costs versus long term costs. As a consequence, it is better to build more landfills with smaller capacity to delay costs.

Changing parameters a, b, c or δ does not change the overall quantity of waste produced throughout the planning period, given by τQ, so that, in the solution, although the individual capacities Y_i change, the sum of capacities $\sum_i Y_i$ does not. Increasing Q or τ, enlarges the overall waste generated in the period $[0,\tau]$ and, hence, makes a bigger total capacity necessary. Note that "small" increases of τ or Q, lead to increases in the average individual capacity and keep K^* unchanged, up to a point that the increase of τQ is large enough to make building a new landfill profitable, allowing a decrease in average capacity. As a result, K^*, as a function of Q and τ, has a stair shape and $\tilde{Y}$, as a function of Q and τ, has a sawtooth shape.

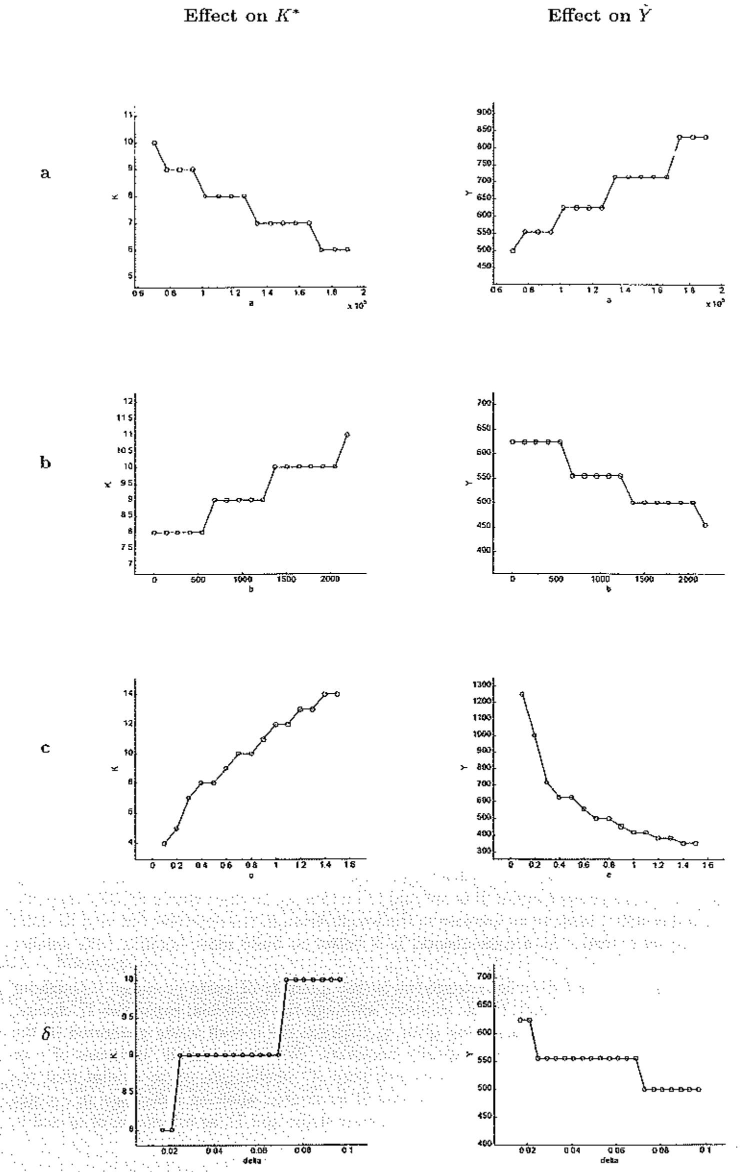

Figure 2.2a. Effect of parameters on K^* and $\tilde{Y}$

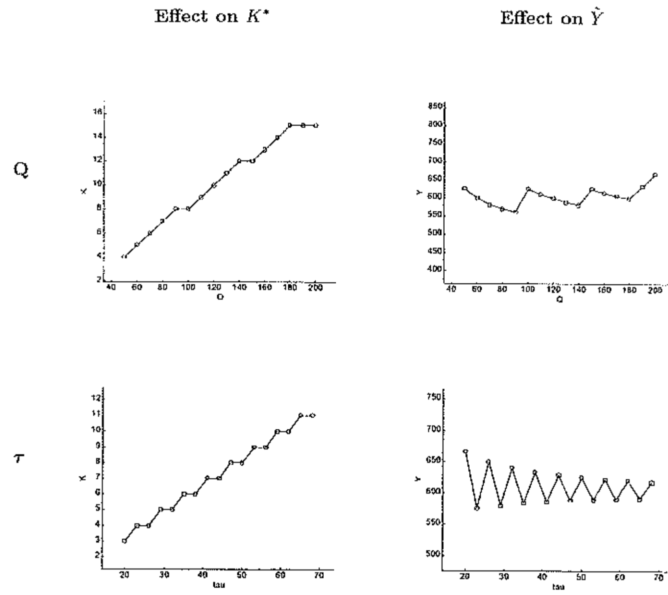

Figure 2.2b. Effect of parameters on K^* and $\tilde{Y}$

As for parameter ε, its effect is shown in figures 2.3 and 2.4. First, note that ε does not affect the whole quantity of waste τQ, and so, varying ε does not change the sum of capacities, although it does imply a change in particular capacities. Increasing ε makes the difference in unit management costs larger from landfill to landfill. As a consequence, as ε increases, the optimal solution implies a sequence of more sharply decreasing capacities, that is, the capacity of the initial landfills is bigger and bigger and the capacity of the latter landfills is smaller and smaller (as shown in figure 2.3).

When the increase in ε is big enough, it is optimal to decrease the number of landfills K^* to save management costs. (as shown in figure 2.4).

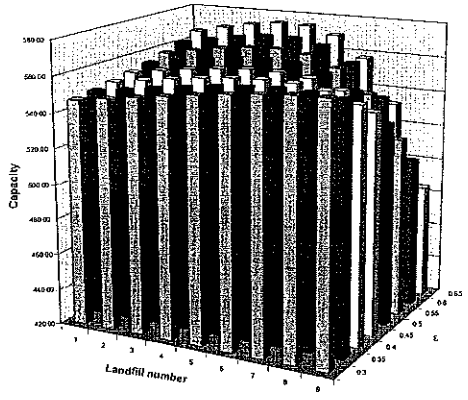

Figure 2.3. Solution for different values of ε.

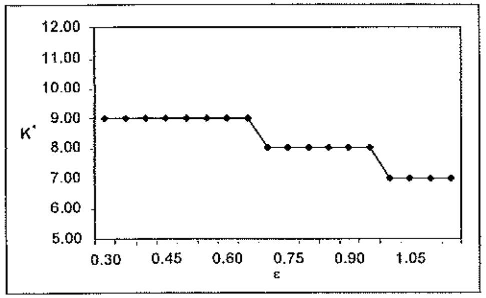

Figure 2.4. Optimal value of K^* for different values of ε.

3 A Variable Quantity of Waste

In this section, we explore an extension to the basic case, assuming that the planner expects that the quantity of landfilled waste will not be constant along the planning horizon. This belief may come from several circumstances, such as a foresight for economic growth or technological change, that will imply a change of production and consumption patterns, some forthcoming environmental regulation concerning packaging, recycling promotion, etc.

Let $Q(t)$ be the amount of waste generated by the population at time t, and assume that such amount evolves according to the following differential equation:

$$\dot{Q}(t) = G(Q(t), t), \tag{5}$$

where the concrete expression of function $G(t)$ depends on the expectations about the future evolution of waste. The problem is to that to find a number of landfills K and a sequence of capacities $\{Y_0, Y_1, ..., Y_{K-1}\}$ which minimize the discounted addition of management and construction costs given by

$$\sum_{i=0}^{K-1} \left[e^{-\delta T_i} C(Y_i) + \int_{T_i}^{T_{i+1}} e^{-\delta t} \phi_i Q(t)\, dt \right]$$

subject to the constraints

$$\begin{aligned} \dot{Q}(t) &= G(Q(t), t), \\ \int_{T_i}^{T_{i+1}} Q(t)\, dt &= Y_i, \quad i = 0, 1, 2, \ldots, K-1 \\ T_0 &= 0,\ T_K = \tau,\ Q(0) = Q_0, \end{aligned} \tag{6}$$

where Q_0 is known and represents the instantaneous generation of waste at time $t = 0$.

Problem (6), like the one studied in section 2, contains some continuous time and some discrete time elements. The evolution of $Q(t)$ and the constraint of landfill i capacity are formulated in continuous time[2], but the objective function does not have the typical form of an optimal control problem in continuous time, because it consists of a sum, as occurs in discrete time Optimal Control problems. Next, a form of approaching the problem(6), employing usual dynamic optimization techniques is proposed. The method has the following steps:

1. Solve the differential equation $\dot{Q}(t) = G(Q(t), t)$, with initial condition $Q(0) = Q_0$, obtaining the expression for $Q(t)$ as a function of time.

2. Substitute the expression obtained in step **1.** in the equation $Y_i = \int_{T_i}^{T_{i+1}} Q(t)\, dt$ and solve this definite integral, as a function of the limit values of integration T_i, T_{i+1}. The result can be written as

$$Y_i = F(T_i, T_{i+1}).$$

where function $F(T_i, T_{i+1})$ measures the total amount of waste generated between any two times T_i and T_{i+1}. On the other hand, the total landfill i discounted management cost is given by

$$MC(T_i, T_{i+1}) = \int_{T_i}^{T_{i+1}} e^{-\delta t} \phi_i Q(t)\, dt$$

3. From here on, there are two possibilities:

3.1. Substitute $Y_i = F(T_i, T_{i+1})$ in the objective function. The resulting problem is that of finding the sequence of construction times $\{T_1, T_2, ..., T_{K-1}\}$ which minimizes

$$\sum_{i=0}^{K-1} H(T_i, T_{i+1}) = \sum_{i=0}^{K-1} \left\{ e^{-\delta T_i} C(F(T_i, T_{i+1})) + MC(T_i, T_{i+1}) \right\} \tag{7}$$

with the initial condition $T_0 = 0$ and the final condition $T_K = \tau$.

Taking $i = 0, 1, 2, ..., K-1$ as the time index, (7) is a discrete time Calculus of Variations problem, being T_i the state variable, and $\sum_{i=0}^{K-1} H(T_i, T_{i+1})$ the objective function. Let us observe that, due to the different periods length, the term $e^{-\delta T_i}$ can not be interpreted as a discount, but as a part of the objective function. In order to solve this problem, the Euler equation[3] has to be applied:

$$H_2(T_i, T_{i+1}) + H_1(T_{i+1}, T_{i+2}) = 0,$$

[2] In Optimal Control theory, the constraints of the type $\int_{T_i}^{T_{i+1}} Q(t)\, dt = Y_i$, are called isoperimetric constraints.

[3] See, for example, Stockey and Lucas (1989).

where H_j represents the partial derivative of H with respect to its j-th argument. According to the definition of H given in (7), the Euler becomes

$$\begin{aligned} 0 &= e^{-\delta T_i} C'(F(T_i, T_{i+1})) . F_2(T_i, T_{i+1}) + MC_2(T_i, T_{i+1}) \\ &+ e^{-\delta T_{i+1}} C'(F(T_{i+1}, T_{i+2})) . F_1(T_{i+1}, T_{i+2}) - \delta e^{-\delta T_{i+1}} C(F(T_{i+1}, T_{i+2})) + MC_1(T_{i+1}, T_{i+2}) \end{aligned}$$

3.2. If it is possible to solve $Y_i = F(T_i, T_{i+1})$ for T_{i+1}, an expression like $T_{i+1} = \Phi(T_i, Y_i)$ is obtained, giving the exhaustion time of a capacity-Y_i- and-initial-time T_i landfill and, by construction, $\Phi_1, \Phi_2 > 0$, the later the landfill i begins to be used (time T_i) and the greater its capacity (Y_i) is, the later it is exhausted (time T_{i+1}). Using Φ, we have a discrete time optimal control model, being T_i the state variable and Y_i the control variable. A solution to the problem is a sequence of capacities $\{Y_0, Y_1, ..., Y_{K-1}\}$ and the associated sequence of switching times $\{T_0, T_1, ..., T_{K-1}\}$ which minimise the objective functional

$$J = \sum_{i=0}^{K-1} \{e^{-\delta T_i} C(Y_i) + MC(T_i, \Phi(T_i, Y_i))\}$$

subject to the state equation $T_{i+1} = \Phi(T_i, Y_i)$, the initial condition $T_0 = 0$ and the final condition $T_K = \tau$. Let us define the Lagrangian function

$$\mathcal{L} = \sum_{i=0}^{K-1} e^{-\delta T_i} C(Y_i) + MC(T_i, \Phi(T_i, Y_i)) + \lambda_0 [\Phi(0, Y_0) - T_1] + ... + \lambda_{K-1} [\Phi(T_{K-1}, Y_{K-1}) - \tau].$$

λ_i measuring the total discounted cost reduction that happens when landfill i lifetime is marginally prolonged, so that it can be called the (opposite of) "shadow price of time", referring to the lifetime of landfill i. The first order conditions are

$$\frac{\partial \mathcal{L}}{\partial T_i} = -\delta e^{-\delta T_i} C(Y_i) + MC_1 + MC_2 \Phi_1 - \lambda_{i-1} + \lambda_i \Phi_1(T_i, Y_i) = 0, \quad i = 1, 2, \ldots, K-1 \tag{8}$$

$$\frac{\partial \mathcal{L}}{\partial Y_i} = e^{-\delta T_i} C'(Y_i) + MC_2 \Phi_2 + \lambda_i \Phi_2(T_i, Y_i) = 0, \quad i = 0, 1, \ldots K-1, \tag{9}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_i} = 0 \rightarrow T_{i+1} = \Phi(T_i, Y_i), \quad i = 0, 1, \ldots, K-1. \tag{10}$$

with $T_0 = 0$ and $T_K = \tau$. From (9) we obtain that the costate variable λ_i is negative

$$\lambda_i = -\frac{e^{-\delta T_i} C'(Y_i) + MC_2 \Phi_2}{\Phi_2(T_i, Y_i)} < 0 \qquad \forall i$$

and the evolution of this shadow price follows a difference equation, obtained from (8):

$$\lambda_i = \frac{\lambda_{i-1} + \delta e^{-\delta T_i} C(Y_i) - MC_1 - MC_2 \phi_1}{\Phi_1(T_i, Y_i)}.$$

Condition (9) is the Optimal Capacity Condition for landfill i and it states the equality between the marginal cost and the marginal profit of increasing the capacity of landfill i. The marginal cost is given by $e^{-\delta T_i} C'(Y_i)$, that is, the (discounted) derivative of the building cost. The marginal profit is the "value of time gained", that is, the discounted saving produced by using landfill i for more time. This saving is obtained by multiplying Φ_2, the marginal increase of the landfill i lifetime due to an increment in Y_i, $\left(\text{that is, } \frac{\partial T_{i+1}}{\partial Y_i}\right)$ by λ_i (that increment shadow price).

3.1 Example

Assume that the construction cost function is quadratic:

$$C(Y) = a + bY + \frac{c}{2} Y^2$$

and the instantaneous generation of waste follows the differential equation

$$\dot{Q}(t) = \alpha Q(t), \tag{11}$$

with the initial condition $Q(0) = Q_0$. From 11, we have $\frac{\dot{Q}(t)}{Q(t)} = \alpha$, so that, waste generation increases or decreases at a constant rate equal to α. The concrete value of α depends on the expectations about the future evolution of the waste. If the sole reason for the expected increment of the waste is economic growth, a proxy variable for α may be the GNP or the Industrial Production growth rate. Solving equation (11) we obtain

$$Q(t) = Q_0 e^{\alpha \cdot t},$$

therefore, the relation between Y_i, T_i and T_{i+1} is given by the equation

$$Y_i = \int_{T_i}^{T_{i+1}} (Q_0 e^{\alpha \cdot t})\, dt = \frac{Q_o}{\alpha} \left[e^{\alpha T_{i+1}} - e^{\alpha T_i}\right]. \tag{12}$$

Assume, moreover, that waste management cost is identical for all the landfills, in such a way that this component can be taken as a constant and, the problem solved taking into account just the construction costs. Substituting (12) in the objective function, the problem can be formulated as the following Calculus of Variations problem, with initial condition $T_0 = 0$ and the final condition $T_K = \tau$:

$$\min_{\{T_1, T_2, ..., T_{K-1}\}} \sum_{i=0}^{K-1} \left\{ e^{-\delta T_i} \left[a + b \frac{Q_o}{\alpha} \left[e^{\alpha T_{i+1}} - e^{\alpha T_i}\right] + \frac{c\, Q_o^2}{2\, \alpha^2} \left[e^{\alpha T_{i+1}} - e^{\alpha T_i}\right]^2 \right] \right\}$$

Another possibility for approaching this problem consists of solving the equation (12) for T_{i+1}, obtaining the following optimal control problem in discrete time:

$$\min_{\{Y_0, Y_1, ..., Y_{K-1}\}} \sum_{i=0}^{K-1} \left\{ e^{-\delta T_i} \left[a + bY_i + \frac{c}{2} Y_i^2 \right] \right\}$$

subject to the state equation

$$T_{i+1} = \frac{1}{\alpha} \log\left(\frac{\alpha}{Q_0} Y_i + e^{\alpha T_i} \right), \tag{13}$$

the initial condition $T_0 = 0$ and the final condition $T_K = \tau$.

Figure 3.1 shows the solution to the problem with the following parameter values:

$$\begin{array}{lll} a = 50000, & \delta = 0.04, & K_{\min} = 1, \\ b = 1, & Q_0 = 30, & K_{\max} = 15, \\ c = 0.4, & \tau = 60, & \alpha = 0.021. \end{array}$$

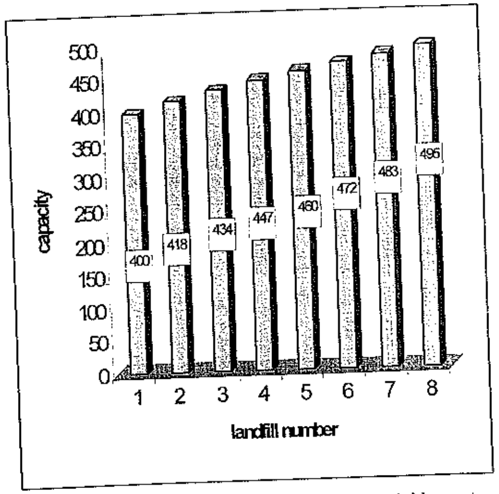

Figure 3.1. Solution to the problem with variable waste.

Figures 3.2.a. and 3.2.b. show the effect of changes in the parameter α on the optimal number of landfills, K^* and the average capacity of the landfills, $\bar{Y} = \frac{1}{K}\sum_{i=0}^{K-1} Y_i$. Note that the total volume of waste generated in period $[0, \tau]$ increases with α. To keep feasibility, either the number of landfills or their average capacity should increase. As can be seen in the graphics, for small increments of α, it is optimal to increase the average capacity (leaving K^* unchanged), whereas, for large increments of α it is optimal to increase K^* (perhaps decreasing $\bar{Y}$).

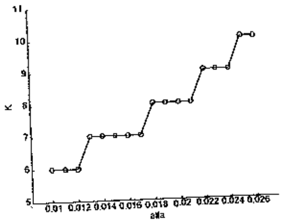

Figure 3.2.a. Effect of α on K^*

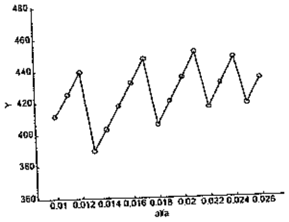

Figure 3.2.b. Effect of α on $\bar{Y}$

4 Landfilling and Recycling

A relevant matter, concerning waste management, is that of deciding which method, or combinations of methods, among the available ones (landfilling, incineration, recycling, composting, etc.) to use for the treatment of a given amount of waste. Selecting management technologies and building landfills are clearly related decisions. In previous sections, for the sake of simplicity, the second decision was assumed to be made, taking the answer to the first one as given. In this section, a model is presented in which building of landfills is performed taking into account the existence of a different technology aside from landfilling, so that optimal landfill capacity and management technologies are jointly decided. Recycling is selected as the alternative technology because it is being the object of a great and increasing interest nowadays for its economic and environmental advantages (see, for example, Weinstein and Zeckhauser (1974), Highfill and McAsey (1997) or Huhtala (1994, 1997, 1999)).

Assume a constant instantaneous waste quantity $Q(t) = Q$ is generated. From the total amount, a portion $R(t)$ is recycled and the rest $V(t)$ is landfilled. The following *mass balance* condition must hold:

$$V(t) + R(t) = Q, \qquad \forall\, t. \tag{14}$$

Disposing of any quantity of waste in landfill i has a unit cost of c_i, and the cost of recycling and amount $R(t)$ is given by $r_i(R(t))$, where r_i is a $C^{(2)}$ function holding $r_i' > 0$, $r_i'' > 0$[1]. The recycling cost functions are supposed to be increasing and convex to represent the different technical recycling complexity attached to different materials. For example, glass is more easily recyclable than paper, and paper more than plastic. In practice, it is reasonable to recycle first those materials which are technically easier (and hence, cheaper) to recycle. As a bigger amount of waste is to be recycled, more complex materials are affected and the attached cost increases faster and faster.

A landfill of capacity Y_i built at time T_i, will be depleted at T_{i+1}, given by

$$\int_{T_i}^{T_{i+1}} V(t)\, dt = Y_i.$$

Let $Y_i(t)$ denote the available capacity of landfill i at instant t. Assuming that landfill i is not available until T_i, the building moment, and that landfill capacity is always depleted, the time evolution of $Y_i(t)$ is given by

$$\dot{Y}_i(t) = \begin{cases} 0 & t < T_i, \\ V(t) = -Q + R(t) & T_i \le t \le T_{i+1}, \qquad i = 0, 1, \ldots \\ 0 & t > T_{i+1}, \end{cases} \tag{15}$$

with the boundary conditions $Y_i(T_i) = Y_i$ and $Y_i(T_{i+1}) = 0$, where Y_i is a decision variable. Between T_i and T_{i+1}, $Y_i(t)$ decreases as waste is disposed of and is totally exhausted at T_{i+1}.

Given that the total number of landfills K is a decision variable, $K_{\max} - K_{\min} + 1$ optimal control problems have to be solved, one for each possible value of K, and that value providing the least cost is selected. The minimum and maximum feasible values for K are given by

$$K_{\min} = 1, \qquad K_{\max} = Int\left(\frac{\tau Q}{\underline{Y}}\right).$$

The maximum number of landfills $K_{\max}$ (which is the relevant number when no waste is recycled, $V(t) = Q\ \forall t$, and all landfills are minimum capacity, $Y_0 = Y_1 = \cdots = Y_{K_{\max}-1} = \underline{Y}$), has the same expression as in section 2. Nevertheless, the minimum number of landfills is determined in a different way. Let us suppose (rather realistically) that the constraint $R(t) \le Q$ is never binding, because of the high marginal cost of recycling the whole amount of waste, in such a way that a positive amount of waste is landfilled at every instant t, and henceforth, at least one landfill is necessary. But nothing prevents the landfilled amount from being small enough for one single landfill to meet the requirements of the whole period $[0, \tau]$. As a consequence, $K_{\min} = 1$. For each possible value of K, the social planner faces the following dynamic optimization problem:

$$\min_{\{Y_0, Y_1, \ldots, Y_{K-1}\}, [R(t)]_{t=0}^{\tau}} \sum_{i=0}^{K-1} \left[e^{-\delta T_i} C(Y_i) + \int_{T_i}^{T_{i+1}} e^{-\delta t} \left\{ c_i \left[Q - R(t) \right] + r_i\left(R(t)\right) \right\} dt \right]$$

[1] Income obtained from recycled products trading is not explicitely taken into account in the model. This shortcoming may be overcome by interpreting r_i as recycling cost minus recycling income.

subject to

$$
\begin{aligned}
T_0 &= 0, \quad T_K = \tau, \\
\dot{Y}_i(t) &= -[Q - R(t)] \quad T_i \le t \le T_{i+1}, \\
Y_i(T_i) &= Y_i, \\
Y_i(T_{i+1}) &= 0, \\
\underline{Y} &\le Y_i \le \overline{Y}, \\
0 &\le R(t) \le Q,
\end{aligned}
$$

where (14) has be used to omit the variable $V(t)$.

This problem fits in the category of multiple-stage optimal control problems, whose solution can be found by applying the results of Tomiyama (1985) and Tomiyama and Rossana (1989)[5]. The main idea implies managing the whole problem as made of a sequence of K optimal control problems, each related to a time interval $[T_i, T_{i+1})$, for $i = 0, 1, \ldots$ and solving them backwards, as shown below:

1. First, solve the sub-problem related to landfill $K-1$, deciding the capacity Y_{K-1} and the recycling path $[R(t)]_{T_{K-1}}^{\tau}$, taking T_{K-1} and τ as given, to minimize

$$J(T_{K-1}) = e^{-\delta T_{K-1}} C(Y_{K-1}) + \int_{T_{K-1}}^{\tau} e^{-\delta t} \{c_{K-1}[Q - R(t)] + r_{K-1}(R(t))\}\, dt$$

subject to

$$
\begin{aligned}
&\dot{Y}_{K-1}(t) = -[Q - R(t)] \quad T_{K-1} \le t \le \tau, \\
&Y_{K-1}(T_{K-1}) = Y_{K-1}, \\
&Y_{K-1}(\tau) = 0, \\
&\underline{Y} \le Y_{K-1} \le \overline{Y}, \\
&0 \le R(t) \le Q.
\end{aligned}
$$

Once the solution is obtained, given by Y_{K-1}^* and $[R^*(t)]_{T_{K-1}}^{\tau}$, it is substituted in the objective function, and we define the value function as

$$J^*(T_{K-1}) = \min_{Y_{K-1}, [R(t)]_{T_{K-1}}^{\tau}} J(T_{K-1}).$$

2. The next step is to solve the sub-problem related to landfill $K-2$, taking T_{K-2} as given and T_{K-1} as a decision variable, that is, deciding Y_{K-2}, T_{K-1} and $[R(t)]_{T_{K-2}}^{T_{K-1}}$ which minimizes

$$J(T_{K-2}) = e^{-\delta T_{K-2}} C(Y_{K-1}) + \int_{T_{K-2}}^{T_{K-1}} e^{-\delta t} \{c_{K-2}[Q - R(t)] + r_{K-2}(R(t))\}\, dt + J^*(T_{K-1}).$$

subject to

$$
\begin{aligned}
&\dot{Y}_{K-2}(t) = -[Q - R(t)] \quad T_{K-2} \le t \le T_{K-1}, \\
&Y_{K-2}(T_{K-2}) = Y_{K-2}, \\
&Y_{K-2}(T_{K-1}) = 0, \\
&\underline{Y} \le Y_{K-2} \le \overline{Y}, \\
&0 \le R(t) \le Q.
\end{aligned}
$$

3. The value function $J^*(T_{K-2})$, obtained in step 2, is used to solve the problem related to landfill $K-3$, and so on, up to landfill $k = 0$, delimited by $t \in [0, T_1)$.

[5] Both papers deal with two-stage problems, but the extension of their results to problems with more than two stages is straightforward.

For each $k = 0, 1, 2, \ldots, K-1$, we have a continuous time optimal control problem with a state variable, $Y_k(t)$, and a control variable, $R(t)$, taking T_k as given and T_{k+1} as a decision variable, except for the case $k = K-1$, in which $T_K = \tau$ is also given. For the $k-th$ interval, $[T_k, T_{k+1})$, the current-value Hamiltonian is defined as

$$\mathcal{H}_k = c_k[Q - R(t)] + r_k(R(t)) + \Psi_k(t)[Q - R(t)] \qquad t \in [T_k, T_{k+1})$$

and the Lagrangian is given by

$$\mathcal{L}_k = c_k[Q - R(t)] + r_k(R(t)) + \Psi_k(t)[Q - R(t)] + \xi_k R(t) \qquad t \in [T_k, T_{k+1}).$$

$\Psi_k(t)$ (with $-\Psi_k(t) \le 0$) being the costate variable related to the available capacity of landfill k at instant t, representing the effect on the objective function, of a marginal increase in $Y_k(t)$. ξ_k is the Kuhn-Tucker multiplier associated with the non-negativity constraint of $R(t)$.

The first order conditions for each control problem, $k = 0, 1, \ldots$ are

$$
\left.
\begin{aligned}
&1.\ r_k'(R(t)) - \Psi_k(t) - c_k \ge 0 \quad (\text{with } "=" \text{ if } R(t) > 0) \\
&2.\ \dot{\Psi}_k(t) = \delta \Psi_k(t) \\
&3.\ C'(Y_k) - \Psi_k(T_k) + \lambda_k + \mu_k = 0 \\
&\quad 3'.\ \lambda_k [\overline{Y} - Y_k] = 0; \quad \mu_k [Y_k - \underline{Y}] = 0 \\
&\quad 3''.\ \lambda_k \ge 0; \quad \mu_k \le 0 \\
&4.\ e_k^{-\delta T_{k+1}} \mathcal{H}_k(T_{k+1}^-) = -\frac{\partial}{\partial T_{k+1}} J^*(T_{k+1})
\end{aligned}
\right\} t \in [T_k, T_{k+1}),
$$

λ_k and μ_k being the multipliers attached to maximum and minimum admissible capacity constraints for landfill k, and $\mathcal{H}_k(T_{k+1}^-)$ denoting $\lim_{t \to T_{k+1}^-} \mathcal{H}_k(t)$. These conditions can be interpreted as follows:

Equation 1 is the first order maximization condition of $\mathcal{H}_k$ subject to $R(t) \ge 0$, that is, $\frac{\partial \mathcal{L}_k}{\partial R} = 0$, which insures that total cost cannot be reduced by increasing or decreasing the recycled amount. If the marginal cost of recycling is greater than that of landfilling, that is to say, condition 1 holds with strict inequality, in the solution, no waste is recycled, $R(t) = 0$. In the case of an interior solution with $R(t) > 0$, the optimal quantity of recycled waste is determined according to

$$r_k'(R(t)) = c_k + \Psi_k, \tag{16}$$

which states that, for all landfills, and at every time, the marginal cost of recycling, given by $r_k'(R(t))$, must equal the marginal cost of landfilling, given by the unit cost c_k plus the shadow price of available capacity, in landfill k.

Condition 2 determines the optimal time evolution of the costate variable Ψ_k, that is to say, $\dot{\Psi}_k(t) = \delta \Psi_k(t) + \frac{\partial \mathcal{H}_k(\cdot)}{\partial Y_k(t)}$. For landfill capacity being a depletable resource, condition 2 is the classical Hotelling rule, which states that the shadow price of such resource grows at a rate that equals the temporal discount rate δ. Taking this result, and the fact that c_k is constant for each landfill, into account, we find that the right side of (16) is constantly increasing during the useful life of landfill k. Accordingly, to maintain the equality, the left side must also be increasing. Given the assumption $r_k'' > 0$, we conclude that, during the useful life of a given landfill, the recycled amount increases with time.

Equations 3, $3'$, $3''$ and 4 are the transversality conditions of the problem[6], which deserve some comments. Conditions 3, $3'$ and $3''$ are the transversality conditions for the initial state, Y_k, which is a decision variable with maximum and minimum threshold values. If no threshold conditions are binding, the optimal capacity of landfill k is determined by condition 3 alone, which takes the form

$$C'(Y_k) = \Psi_k(T_k). \tag{17}$$

Equation (17) is the Optimal Capacity Condition for landfill k, and it states the equality between marginal cost of Y_k, given by the increase in building cost, and its marginal gain, given by the shadow

[6] See Hestenes (1966) for a general treatment of transversality conditions.

price of the available landfill capacity at time T_k, $\Psi_k(T_k)$, that measures the effect of increasing Y_k on the total discounted costs from T_k on, coupling the saving in the management costs attached to landfill k and the discounted cost saving that may be obtained by delaying the building times of future landfills[7]. Equation 4 is the transversality condition for the optimal value of T_{k+1}, when a scrap value function exists that, from the viewpoint of period $[T_k, T_{k+1}]$, is $J^*(T_{k+1})$. For $k = K-1$, this equation is replaced by the final condition $T_K = \tau$. The left side of equation 4 represents the marginal cost of enlarging the useful life of landfill k, which is $\mathcal{H}_k$ evaluated at T_{k+1} and properly discounted. The right side of 4 represents the marginal gain obtained from enlarging the useful life, which is the effect of an increase in T_{k+1} on the scrap function $J^*(T_{k+1})$ and, by definition of $J^*(T_{k+1})$,

$$\begin{aligned}\frac{\partial}{\partial T_{k+1}} J^*(T_{k+1}) &= \frac{\partial}{\partial T_{k+1}}\left[e^{-\delta T_{k+1}} C\left(Y_{k+1}^*\right)\right] \\ &+ \frac{\partial}{\partial T_{k+1}}\left\{\int_{T_{k+1}}^{T_{k+2}} e^{-\delta t}\left\{c_i\left[Q - R^*(t)\right] + r_i\left(R^*(t)\right)\right\} dt\right\},\end{aligned}$$

where Y_{k+1}^* and $R^*(t)$ represent the optimal value of Y_{k+1} and $R(t)$.

Following Caputo and Wilen (1995) we know that

$$\frac{\partial}{\partial T_{k+1}}\left\{\int_{T_{k+1}}^{T_{k+2}} e^{-\delta t}\left\{c_i\left[Q - R(t)\right] + r_i\left(R(t)\right)\right\} dt\right\} = -e^{-\delta T_{k+1}}\mathcal{H}_{k+1}\left(T_{k+1}^+\right),$$

where $\mathcal{H}_{k+1}\left(T_{k+1}^+\right) = \lim_{t \to T_{k+1}^+} \mathcal{H}_{k+1}(t)$. Using this result, and given that

$$\frac{\partial}{\partial T_{k+1}}\left[e^{-\delta T_{k+1}} C\left(Y_{k+1}^*\right)\right] = -\delta e^{-\delta T_{k+1}} C\left(Y_{k+1}^*\right) + e^{-\delta T_{k+1}} C'\left(Y_{k+1}^*\right)\frac{\partial Y_{k+1}^*}{\partial T_{k+1}},$$

condition 4 can be expressed as

$$\mathcal{H}_k\left(T_{k+1}^-\right) = \delta C\left(Y_{k+1}^*\right) - C'\left(Y_{k+1}^*\right)\frac{\partial Y_{k+1}^*}{\partial T_{k+1}} + \mathcal{H}_{k+1}\left(T_{k+1}^+\right)$$

that is to say, at T_{k+1} a jump happens from the value of the $k-th$ Hamiltonian to the $(k+1)-th$ one. This conclusion is also obtained in Hartwick, Kemp and Long (1986), in the context of the exploitation of many deposits of an exhaustible resource, with the peculiarity that, in Hartwick et. al.'s paper, the jump is always the same size because all the deposits have the same initial capacity, while in this paper the jump size, given by $\frac{\partial}{\partial T_{k+1}}\left[e^{-\delta T_{k+1}} C\left(Y_{k+1}^*\right)\right]$, depends on the capacity of landfill $k+1$, Y_{k+1}^*, which is a decision variable.

4.1 Example

Assume that building costs are given by the linear function

$$C(Y) = a + bY$$

and the recycling costs, which are identical for all the landfills, have the form

$$r_0(R_t) = r_1(R_t) = \cdots = d \cdot \left[R(t)\right]^2,$$

d being a parameter, and no maximum or minimum capacity constraints for any landfill. To solve the problem, we need to obtain the solution for each possible value of K, and select that providing the least value of the objective function. We analyse the $K = 2$ case, whose attached optimization problem is the following:

$$\begin{aligned}\min_{\{Y_0, Y_1, R(t), T_1\}} &\left[a + bY_0\right] + \int_0^{T_1} e^{-\delta t}\left\{c_0\left[Q - R(t)\right] + d\left[R(t)\right]^2\right\} dt \\ &+ e^{-\delta T_1}\left[a + bY_1\right] + \int_{T_1}^{\tau} e^{-\delta t}\left\{c_1\left[Q - R(t)\right] + d\left[R(t)\right]^2\right\} dt\end{aligned}$$

subject to

$$\begin{array}{ll} \dot{Y}_0 = -Q + R(t) & 0 \le t \le T_1, \\ \dot{Y}_1 = -Q + R(t) & T_1 \le t \le \tau, \\ Y_0(0) = Y_0, \quad Y_1(T_1) = Y_1, & \\ Y_0(T_1) = Y_1(\tau) = 0. & \\ R(t) \le Q & \end{array}$$

The first step is to solve the sub-problem attached to landfill 1, that is,

$$\min_{Y_1, R(t)} e^{-\delta T_1}\left[a + bY_1\right] + \int_{T_1}^{\tau} e^{-\delta t}\left\{c_1\left[Q - R(t)\right] + d\left[R(t)\right]^2\right\} dt \tag{18}$$

subject to

$$\dot{Y}_1 = -Q + R(t) \qquad T_1 \le t \le \tau,$$

$$Y_1(T_1) = Y_1, \qquad Y_1(\tau) = 0,$$

taking T_1 and τ as given. The attached current value Hamiltonian is

$$\mathcal{H}_1 = c_1\left[Q - R(t)\right] + d\left[R(t)\right]^2 + \Psi_1(t)\left[Q - R(t)\right].$$

As shown in the appendix, the solution to this problem is

$$\left.\begin{array}{l} Y_1^* = \left[Q - \frac{c_1}{2d}\right](\tau - T_1) + \frac{b}{2d\delta}\left[1 - e^{\delta(\tau - T_1)}\right] \\ Y_1^*(t) = \left[Q - \frac{c_1}{2d}\right](\tau - t) + \frac{b}{2d\delta}\left[e^{\delta(t - T_1)} - e^{\delta(\tau - T_1)}\right] \\ \Psi_1^*(t) = be^{\delta(t - T_1)} \\ R^*(t) = \frac{c_1}{2d} + \frac{b}{2d}e^{\delta(t - T_1)} \end{array}\right\} T_1 \le t \le \tau, \tag{19}$$

from which, we obtain

$$J_1^*(T_1) = e^{-\delta T_1}\left[a + bY_1^*\right] + \int_{T_1}^{\tau} e^{-\delta t}\left\{c_1\left[Q - R^*(t)\right] + d\left[R^*(t)\right]^2\right\} dt,$$

which only depends on parameters of the problem and the variable T_1. Afterwards, it is necessary to solve the problem corresponding to landfill $k = 0$, which is the following:

$$\min_{Y_0, R(t), T_1}\left[a + bY_0\right] + \int_0^{T_1} e^{-\delta t}\left\{c_0\left[Q - R(t)\right] + d\left[R(t)\right]^2\right\} dt + J_1^*(T_1)$$

subject to

$$\begin{array}{ll} \dot{Y}_0 = -Q + R(t) & 0 \le t \le T_1, \\ Y_0(0) = Y_0, \quad Y_0(T_1) = 0, & \\ R(t) \le Q & \end{array}$$

[7] We say "the saving that may be obtained" and not "the saving that is obtained" because, in this problem, amount of waste dumped is a decision variable and it is not sure, *a priori*, that a higher value of Y_k implies a delay of future landfils.

the current value Hamiltonian being

$$\mathcal{H}_0 = c_0 [Q - R(t)] + d [R(t)]^2 + \Psi_0(t) [Q - R(t)],$$

whose solution, as shown in the appendix, is given by

$$\left.\begin{array}{l} Y_0^* = \left[Q - \frac{c_0}{2d}\right] T_1 + \frac{b}{2d\delta}\left[1 - e^{\delta T_1}\right] \\ Y_0^*(t) = \left[Q - \frac{c_0}{2d}\right](T_1 - t) + \frac{b}{2d\delta}\left[e^{\delta t} - e^{\delta T_1}\right] \\ \Psi_0^*(t) = be^{\delta t} \\ R^*(t) = \frac{c_0}{2d} + \frac{b}{2d}e^{\delta t} \end{array}\right\} 0 \leq t \leq T_1 \tag{20}$$

Finally, we need to find the optimal value of T_1, which is obtained from the transversality condition

$$\mathcal{H}_0(T_1^-) = \delta C(Y_1^*) - C'(Y_1^*)\frac{\partial Y_1^*}{\partial T_1} + \mathcal{H}_1(T_1^+)$$

that, using (19) and (20), becomes

$$\underbrace{c_0\left[Q - \frac{c_0}{2d} - \frac{b}{2d}e^{\delta T_1}\right] + d\left[\frac{c_0}{2d} + \frac{b}{2d}e^{\delta T_1}\right]^2 + be^{\delta T_1}\left[Q - \frac{c_0}{2d} - \frac{b}{2d}e^{\delta T_1}\right]}_{\mathcal{H}_0(T_1^-)}$$

$$= \underbrace{\delta\left\{a + b\left(\left[Q - \frac{c_1}{2d}\right](\tau - T_1) + \frac{b}{2d\delta}\left[1 - e^{\delta(\tau - T_1)}\right]\right)\right\}}_{C(Y_1^*)} + b\underbrace{\left\{\left[Q - \frac{c_1}{2d}\right] - \frac{b}{2d}e^{\delta(\tau - T_1)}\right\}}_{-\frac{\partial Y_1^*}{\partial T_1}}$$

$$\underbrace{+c_1\left[Q - \frac{(c_1 + b)}{2d}\right] + d\left[\frac{(c_1 + b)}{2d}\right]^2 + b\left[Q - \frac{(c_1 + b)}{2d}\right]}_{\mathcal{H}_1(T_1^+)}$$

or, simplifying,

$$\begin{array}{rl} & 4dc_0Q - c_0^2 - 2c_0be^{\delta T_1} - b^2e^{2\delta T_1} + 4dQbe^{\delta T_1} \\ = & 4da\delta + 4db\delta Q(\tau - T_1) - 2c_1b\delta(\tau - T_1) - b^2 + 2b \\ & -2b^2e^{\delta(\tau - T_1)} - 2be^{\delta(\tau - T_1)} + 8dbQ - 4c_1b + 4dc_1Q - c_1^2. \end{array} \tag{21}$$

To illustrate the results, let us show the solution for the following parameters value[8]:

$$\begin{array}{lll} a = 10, & c_1 = 3, & Q = 20, \\ b = 0.8, & d = 4, & \tau = 30, \\ c_0 = 2, & \delta = 0.04, & \end{array} \tag{22}$$

which is given by

$$\begin{array}{rcll} T_1^* & = & 15, & \\ Y_0^* & = & 294.2, & \\ Y_1^* & = & 292.3, & \\ R^*(t) & = & \frac{1}{4} + \frac{1}{10}e^{0.04t} & 0 \leq t < 15, \\ R^*(t) & = & \frac{3}{8} + \frac{1}{10}e^{\delta(t-15)} & 15 \leq t \leq 30. \end{array} \tag{23}$$

[8]By the bisection method, the numerical value T_1^* that solves (21) is obtained, and from T_1^* and the parameter values, the optimal value of Y_0 and Y_1 is obtained using (19) and (20).

In figure number 4.1 the optimal shape of $R(t)$ is shown. Now sensibility analysis experiments are performed by changing one parameter each time and holding the rest at the benchmark values given at (22). In figures 4.1 the effect of different parameters on optimal values of Y_0 and Y_1 are shown.

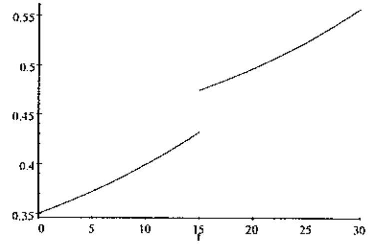

Figure 4.1 Solution for $R(t)$

For "low" values of a it is optimal to build two landfills with capacities Y_0^* and Y_1^*, while from a certain threshold value a, the fixed cost attached to the building of a landfill is so high that it is not optimal to build two, but only one with enough capacity to dispose of all the waste generated throughout the period $[0, \tau]$. From that threshold value of a, we have $Y_1^* = 0$.

The higher the value of parameter b, which represents the marginal cost of each landfill built capacity, the lower the optimal value for Y_0 and Y_1, and hence, the total landfilled amount of waste. For the solution to be still feasible, total recycled waste throughout $[0, \tau]$ must increase as b increases.

The higher the value of parameter c_0, measuring the unit disposal cost of the first landfill, the lower the optimal value of Y_0 and the higher the value of Y_1.

As for parameter c_1, when it is below a certain threshold value, small increments do not affect the optimal value Y_0 and produce a slight decrease in Y_1 (the scale of the plot do not allow the latter effect to be perceived visually). So that, the solution does not change in the interval $[0, T_1)$, while recycling is more intensively used, and landfilling less, in the interval $[T_1, \tau]$. When c_1 exceeds a certain value, the second landfill ceases to be profitable, and it becomes optimal to build a single landfill for all the waste generated throughout $[0, \tau]$.

Increasing parameter d makes recycling more expensive as compared with landfilling, and it leads to an increase in both landfills capacity in order to allow more waste landfilling and less recycling.

Because of the linearity of building costs, and given $c_1 > c_0$, for very low values of δ there is no reason to use two landfills, bearing twice the fixed cost a, but it is better to build a single landfill. So, for low values of δ, we find $Y_1^* = 0$. For "medium" values of δ, Y_0^* and Y_1^* approximately have the values given in (23), with Y_0 slightly decreasing and Y_1 slightly increasing (in a range that can not be visually perceived with the plot scale). Finally, for a certain threshold value of δ, a negative (positive) leap happens for Y_0 (Y_1).

The instantaneous waste generation, represented by Q affects the optimal values of Y_0 and Y_1 in a linear and positive way.

The time horizon variable τ has a two-piece effect: for low values of τ, it is optimal to build a single landfill, and hence, $Y_1 = 0$. For "small" τ increments, the optimal value of Y_0 increases and that of Y_1 stays at zero. For a high enough τ increment, a leap happens in the solution: building two landfills becomes optimal, so that Y_0 sharply decreases and Y_1 switches from zero to a strictly positive value. From that point, further τ increments leads to increase in both landfills' capacity.

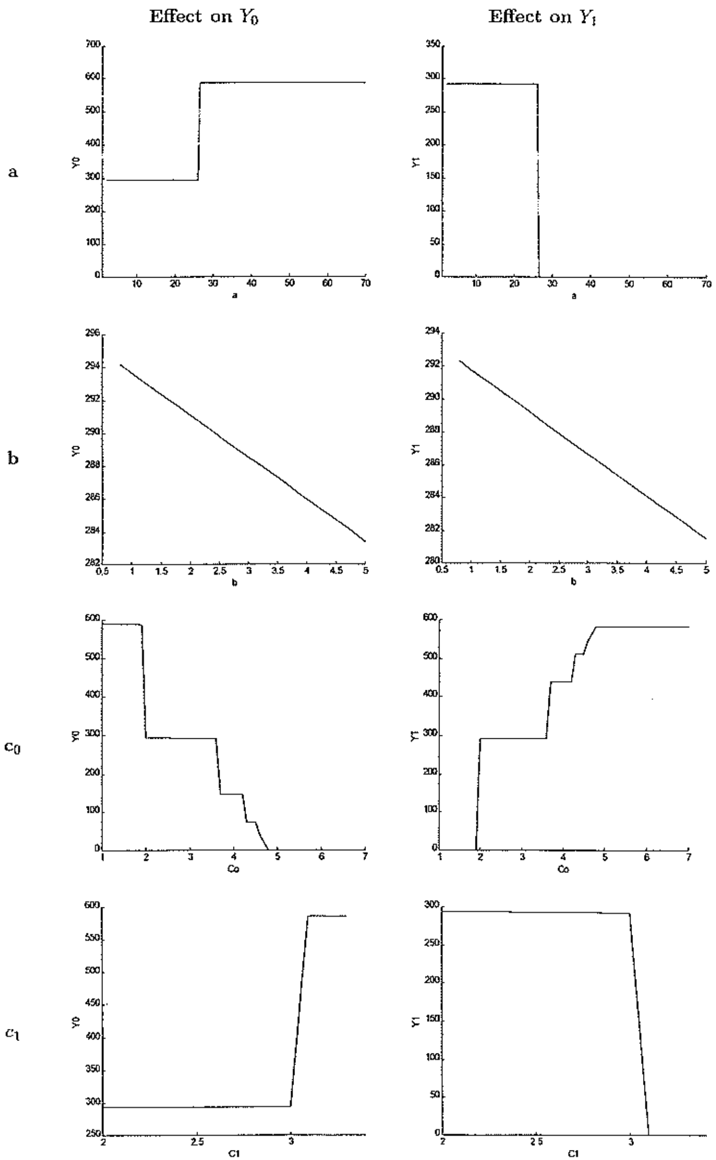

Figure 4.2a. Effect of parameters on Y_0 and Y_1

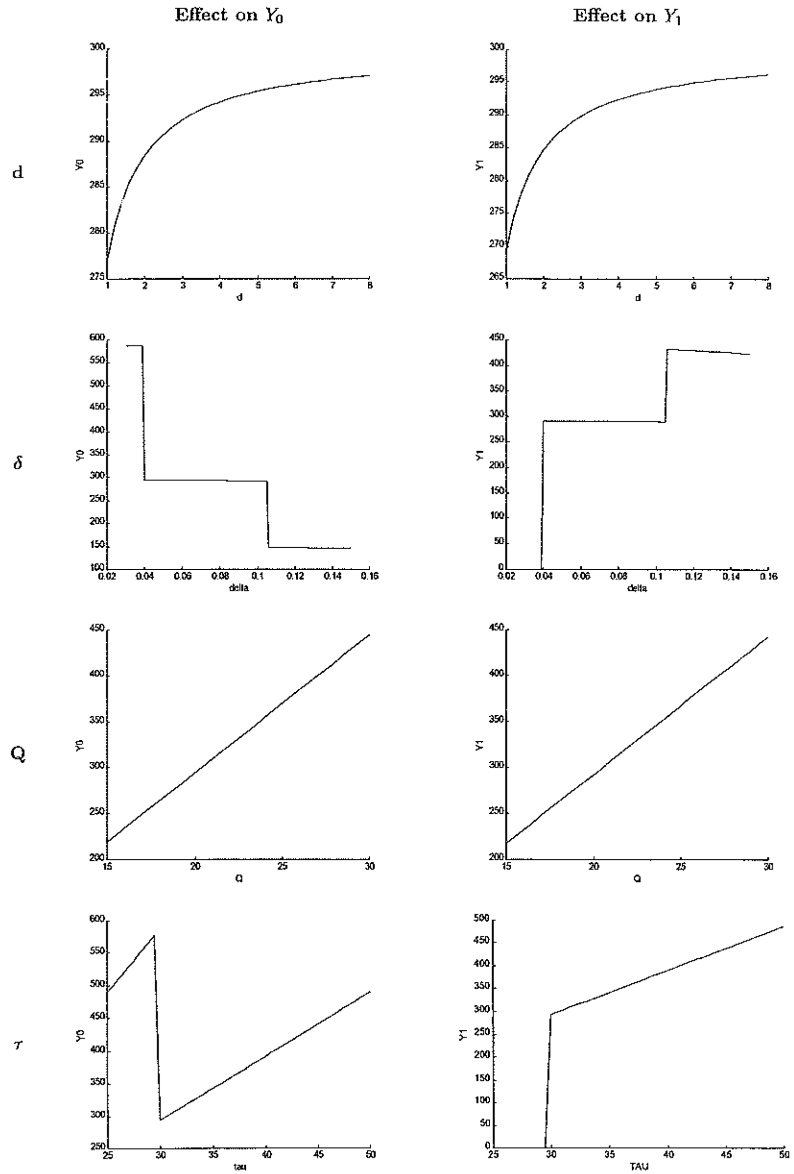

Figure 4.2b. Effect of parameters on Y_0 and Y_1

5 Conclusions and Future Research

The optimal capacity of landfills, which is a rather relevant economic decision and is usually taken as given in most economic articles, has been studied in the present paper within a dynamic framework. The basic dynamic nature of the problem has been pointed out and several specific cases have been explored. To deal with this matter, a class of optimal control problems, sharing some continuous time and some discrete time features, have been stated and solved. The mathematical structure of those problems could also be useful to model some other economic situations, such as investment decisions or durable consumption goods decisions.

Given that landfills are depletable and replaceable resources, their capacity and useful life should not be individually (landfill by landfill) decided, but the whole sequence of necessary landfills would rather be jointly designed. If the only difference among the various places available for the building of landfills is the attached unit waste management cost, then it is optimal to make use of such places beginning from the lowest cost one and following in the order of increasing unit cost.

In an interior solution, the optimal capacity of a certain landfill k is determined according to the so-called *Optimal Capacity Condition*, which states the equality between such capacity marginal cost and marginal gain. The marginal cost is given by the building cost plus the management cost attached to landfill k, while the marginal gain comes from all the discounted cost saving attached to future landfills that can be achieved by increasing the capacity of landfill k. Optimal capacity depends positively on the expected future waste management cost increment from the present landfill to the following one, in such a way that the more management costs increase, the more decreasing the sequence of capacities decreases.

If instant waste generation is not constant, but follows a certain time evolution, a solution method is suggested, based on discretizing the continuous time problem by summing up the generated amount of waste between two consecutive (endogenously determined) landfill switching times. This strategy allows us to avoid the temporal nature of the switching time variable, that becomes a state variable of the problem. The time-variable role is played by the landfill index ($k = 0, 1, \dots, K-1$).

Selecting management technologies and building landfills are related decisions. When both decisions are jointly considered, a multiple-stage optimal control results, whose solution requires the use of dynamic continuous time techniques (Pontryagin Maximum Principle) for every landfill sub-problem and discrete time procedure (Dynamic Programming) to manage the whole problem.

Under the assumptions made for the landfilling and recycling problem, the recycled amount of waste is time increasing within every landfill's useful life.

The following are some plausible future research lines:

- Enriching the optimal capacity problem, relaxing assumptions and adding new elements such as
  - Considering the possibility of stochastic future generation of waste
  - Exploring other waste treatment technologies, such as incineration or composting.
- Joint study of optimal capacity and optimal location of landfills.

6 Appendix: Mathematical Conditions

6.1 Proof of Proposition 1

For each of the possible values of K, the corresponding control problem can be solved either by dynamic programming, as illustrated in the case $K = 2$, or by the Lagrange method[9]. As the state equation is very simple, it is possible to work in the following way, for a generic value of K. Applying recurrently the formula $T_{i+1} = T_i + \frac{Y_i}{Q}$, and assuming that all the landfills' capacity get exhausted under the optimal solution, we obtain

$$T_i = \frac{1}{Q}\sum_{j=0}^{i-1} Y_j, \qquad i = 1, 2, \dots, K-1 \tag{24}$$

[9] See Chow (1997) for a comparison of both methods.

On the other hand, solving the integral of the objective function of (P) and using the equation (24), we have

$$\int_{T_i}^{T_{i+1}} e^{-\delta(t-T_i)}\phi_i Q dt = \frac{Q\phi_i}{\delta}\left[1 - e^{-\delta(T_{i+1}-T_i)}\right] = \frac{Q\phi_i}{\delta}\left[1 - e^{-\delta\frac{Y_i}{Q}}\right],$$

and therefore the problem (P) consists of finding the sequence of capacities $\{Y_0, Y_1, \dots, Y_{K-1}\}$ which minimise

$$C(Y_0) + \frac{Q\phi_0}{\delta}\left[1 - e^{-\delta\frac{Y_0}{Q}}\right] + \sum_{i=1}^{K-1} e^{-\frac{\delta}{Q}\sum_{j=0}^{i-1} Y_j}\left[C(Y_i) + \frac{Q\phi_i}{\delta}\left(1 - e^{-\delta\frac{Y_i}{Q}}\right)\right]$$

subject to the overall capacity constraint

$$Y_0 + Y_1 + \cdots + Y_{K-1} = \tau Q. \tag{25}$$

The Lagrangean of this problem is

$$C(Y_0) + \frac{Q\phi_0}{\delta}\left[1 - e^{-\delta\frac{Y_0}{Q}}\right] + \sum_{i=1}^{K-1} e^{-\frac{\delta}{Q}\sum_{j=0}^{i-1} Y_j}\left[C(Y_i) + \frac{Q\phi_i}{\delta}\left(1 - e^{-\delta\frac{Y_i}{Q}}\right)\right] - \lambda\left[\tau Q - \sum_{i=1}^{K-1} Y_{K-1}\right]$$

being λ the Lagrange multiplier attached to the constraint (25).

The first order conditions for $Y_0, Y_1, \dots, Y_{K-1}$ are

$$C'(Y_0) + \phi_0 e^{-\delta\frac{Y_0}{Q}} - \frac{\delta}{Q}\sum_{i=1}^{K-1} e^{-\frac{\delta}{Q}\sum_{j=0}^{i-1} Y_j}\left[C(Y_i) + \frac{Q\phi_i}{\delta}\left(1 - e^{-\delta\frac{Y_i}{Q}}\right)\right] = \lambda,$$

$$\begin{aligned} &e^{-\frac{\delta}{Q}\sum_{j=0}^{k-1} Y_j}\left[C'(Y_k) + \phi_k e^{-\delta\frac{Y_k}{Q}}\right] - \\ &-\tfrac{\delta}{Q}\textstyle\sum_{i=k+1}^{K-1} e^{-\frac{\delta}{Q}\sum_{j=0}^{i-1} Y_j}\left[C(Y_i) + \tfrac{Q\phi_i}{\delta}\left(1 - e^{-\delta\frac{Y_i}{Q}}\right)\right] = \lambda, \qquad k = 1, 2 \dots, K-2 \end{aligned}$$

and

$$e^{-\frac{\delta}{Q}\sum_{j=0}^{K-1} Y_j}\left[C'(Y_K) + \phi_{K-1} e^{-\delta\frac{Y_{K-1}}{Q}}\right] = \lambda$$

jointly with (25).

Equating the first order equations for two consecutive arbitrary landfills, k and $k+1$, ($k = 1, 2, \dots, K-3$[10]) we obtain

$$\begin{aligned} &e^{-\frac{\delta}{Q}\sum_{j=0}^{k-1} Y_j}\left[C'(Y_k) + \phi_k e^{-\delta\frac{Y_k}{Q}}\right] - \frac{\delta}{Q}\sum_{i=k+1}^{K-1} e^{-\frac{\delta}{Q}\sum_{j=0}^{i-1} Y_j}\left[C(Y_i) + \frac{Q\phi_i}{\delta}\left(1 - e^{-\delta\frac{Y_i}{Q}}\right)\right] \\ = \; &e^{-\frac{\delta}{Q}\sum_{j=0}^{k} Y_j}\left[C'(Y_{k+1}) + \phi_{k+1} e^{-\delta\frac{Y_{k+1}}{Q}}\right] - \frac{\delta}{Q}\sum_{i=k+2}^{K-1} e^{-\frac{\delta}{Q}\sum_{j=0}^{i-1} Y_j}\left[C(Y_i) + \frac{Q\phi_i}{\delta}\left(1 - e^{-\delta\frac{Y_i}{Q}}\right)\right]. \end{aligned}$$

Multiplying both sides by $e^{\frac{\delta}{Q}\sum_{j=0}^{k-1} Y_j}$, adding $\frac{\delta}{Q}\sum_{i=k+2}^{K-1} e^{-\frac{\delta}{Q}\sum_{j=0}^{i-1} Y_j}\left[C(Y_i) + \frac{Q\phi_i}{\delta}\left(1 - e^{-\delta\frac{Y_i}{Q}}\right)\right]$ to both sides and rearranging, we obtain (2).

[10] The intermediate expressions for $k = 0$ and $k = K-2$ are slightly different, but it is easy to show that equation (2) also holds for these two cases.

6.2 Solution of Example 4.1

As T_1 is given, $e^{-\delta T_1}$ is constant and minimizing the objective function (18) is the same as minimizing

$$[a + bY_1] + \int_{T_1}^{\tau} e^{-\delta(t-T_1)} \left\{ c_1 [Q - R(t)] + d [R(t)]^2 \right\} dt$$

which, making the variable change $\omega = t - T_1$, may be expressed as

$$[a + bY_1] + \int_{0}^{\tau - T_1} e^{-\delta\omega} \left\{ c_1 [Q - R(\omega)] + d [R(\omega)]^2 \right\} d\omega$$

and the problem constraints become

$$\dot{Y}_1(\omega) = -Q + R(\omega),$$

$$Y_1(0) = Y_1, \qquad Y_1(\tau - T_1) = 0,$$

$$R(t) \leq Q$$

with T_1 and τ given. The current value Hamiltonian is defined as

$$\mathcal{H}_1 = c_1 [Q - R(\omega)] + d [R(\omega)]^2 + \Psi_1(\omega) [Q - R(\omega)].$$

The necessary first order conditions (Pontryagin Maximum Principle) are the following:

$$\frac{\partial \mathcal{H}_1}{\partial R(\omega)} = -c_1 + 2dR(\omega) - \Psi_1(\omega) = 0, \tag{26}$$

$$\dot{\Psi}_1(\omega) = \delta\Psi_1(\omega), \tag{27}$$

$$\dot{Y}_1(\omega) = -Q + R(\omega). \tag{28}$$

Solving equation (27), we have $\Psi_1(\omega) = \Psi_1(0) e^{\delta\omega}$ and, substituting in (26) and rearranging, we have

$$R(\omega) = \frac{c_1}{2d} + \frac{\Psi_1(0)}{2d} e^{\delta\omega}. \tag{29}$$

Substituting (29) in (28) and solving the resulting differential equation, whose general solution for Y_1 is

$$Y_1(\omega) = \left[\frac{c_1}{2d} - Q\right]\omega + \frac{\Psi_1(0)}{2d\delta} e^{\delta\omega} + A,$$

A being a constant. Using the initial condition $Y_1(0) = Y_1$ and rearranging we obtain the value of $A = Y_1 - \frac{\Psi_1(0)}{2d\delta}$, so that the solution for $Y_1(\omega)$ becomes

$$Y_1(\omega) = Y_1 + \left[\frac{c_1}{2d} - Q\right]\omega + \frac{\Psi_1(0)}{2d\delta}\left[e^{\delta\omega} - 1\right] \tag{30}$$

and, using the final condition $Y_1(\tau - T_1) = 0$ and rearranging, provides the following expression for Y_1:

$$Y_1 = \left[Q - \frac{c_1}{2d}\right](\tau - T_1) + \frac{\Psi_1(0)}{2d\delta}\left[1 - e^{\delta(\tau - T_1)}\right]. \tag{31}$$

From the Y_1 optimality condition, the following value of $\Psi_1(0)$ is obtained:

$$C'(Y_1) = b = \Psi_1(0) \tag{32}$$

and substituting (31) and (32) in (30) the following final expression for $Y_1(\omega)$ is obtained:

$$Y_1(\omega) = \left[Q - \frac{c_1}{2d}\right](\tau - T_1) + \frac{b}{2d\delta}\left[1 - e^{\delta(\tau - T_1)}\right] + \left[\frac{c_1}{2d} - Q\right]\omega + \frac{b}{2d\delta}\left[e^{\delta\omega} - 1\right]$$

and the solution for $\Psi_1(\omega)$ and $R(\omega)$ is given by

$$\Psi_1(\omega) = be^{\delta\omega},$$

$$R(\omega) = \frac{c_1}{2d} + \frac{b}{2d}e^{\delta\omega}.$$

Undoing the variable change $\omega = t - T_1$, the expressions in (19) are obtained.
The current value Hamiltonian of the sub-problem linked to landfill 0 is defined as

$$\mathcal{H}_0 = c_0 [Q - R(t)] + d [R(t)]^2 + \Psi_0(t) [Q - R(t)].$$

The Pontryagin Maximum Principle conditions are

$$\frac{\partial \mathcal{H}_0}{\partial R(t)} = -c_0 + 2dR(t) - \Psi_0(t) = 0, \tag{33}$$

$$\dot{\Psi}_0(t) = \delta\Psi_0(t), \tag{34}$$

$$\dot{Y}_0(t) = -Q + R(t). \tag{35}$$

Solving equation (34), we have $\Psi_0(t) = \Psi_0(0) e^{\delta t}$ and, substituting in (33) and rearranging, we obtain

$$R(t) = \frac{c_0}{2d} + \frac{\Psi_0(0)}{2d} e^{\delta t}. \tag{36}$$

Using (36) in (35) and solving the resulting differential equation the following general solution for Y_0 is obtained:

$$Y_0(t) = \left[\frac{c_0}{2d} - Q\right] t + \frac{\Psi_0(0)}{2d\delta} e^{\delta t} + A,$$

A being a constant. Using the boundary condition $Y_0(0) = Y_0$ and rearranging the value $A = Y_0 - \frac{\Psi_0(0)}{2d\delta}$ is obtained, so that the solution for $Y_0(t)$ becomes

$$Y_0(t) = Y_0 + \left[\frac{c_0}{2d} - Q\right] t + \frac{\Psi_0(0)}{2d\delta}\left[e^{\delta t} - 1\right] \tag{37}$$

and, using the final condition $Y_0(T_1) = 0$ and rearranging, the following expression is obtained for Y_0:

$$Y_0 = \left[Q - \frac{c_0}{2d}\right] T_1 + \frac{\Psi_0(0)}{2d\delta}\left[1 - e^{\delta T_1}\right]. \tag{38}$$

From the Y_0 optimality condition, we obtain the value for $\Psi_0(0)$,

$$C'(Y_0) = b = \Psi_0(0) \tag{39}$$

and using (38) and (39) in (37) and rearranging, we have the following final expression for $Y_0(t)$:

$$Y_0(t) = \left[Q - \frac{c_0}{2d}\right](T_1 - t) + \frac{b}{2d\delta}\left[e^{\delta t} - e^{\delta T_1}\right]$$

and the solution for $\Psi_0(t)$ and $R(t)$ is obtained by substituting (39) in (34) and (36).

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