

THE ASSOCIATIVITY PROBLEM FOR OWA OPERATORS

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Abstract—

Connectives are usually assumed associative. Associativity allows recursive application of the same binary connective, in such a way that memorizing only one binary connective is enough, no matter the dimension of the real problem. OWA operators are not associative, and its application requires a previously fixed dimension, not to be modified. In this paper we develop the proposal given in [2], where each OWA operator was represented in terms of a family of binary OWA operators. A connective rule will be conceived here as a consistent family of connectives capable of solving arbitrary dimension problems.

Key words: Connective rules, fuzzy sets, linguistic quantifiers.

I. INTRODUCTION.

Decision making usually requires a previous analysis of data, which very often implies some kind of aggregation of information process. In this way, complex issues are simplified in order to fit some available decision making procedure. In particular, information may be passed to an connective operator as an ordered sequence of real numbers, which without loss of generality can be supposed to belong to the unit interval.

Standard connectives are t-norms and t-conorms,

which generalize the notion of conjunction and disjunction of classical logic. They are assumed commutative and associative, so calculus is made simpler (information to be aggregated need not to be presented in any particular way, no matter its dimension). It is well known (see, e.g., [3]) that the *min* operator is the maximal t-norm and the *max* operator is the minimal t-conorm.

In order to fill the gap between *min* and *max*, Yager [5] (see also [6]) proposed *Ordered Weighted Averaging (OWA)* operators, which have been widely used in many different fields. To simplify the formalization of OWA operators let us introduce the notion of *sorting permutation* of a list.

If $L = [a_1, a_2, \dots, a_n]$ is a list of numbers, a sorting permutation σ for L is any permutation of the elements of L that produces a list

$$\sigma(L) = [a_{(1)}, \dots, a_{(n)}]$$

verifying $a_{(i)} \geq a_{(j)}$ for all $i \leq j$.

DEFINITION 1 An OWA operator of dimension n is a connective operator ϕ that has an associated list of weights $W = [w_1, \dots, w_n]$ such that

1. $w_i \in [0, 1]$ for all $1 \leq i \leq n$
2. $\sum_{i=1}^n w_i = 1$
3. for any $L = [a_1, a_2, \dots, a_n]$ and its corresponding $\sigma(L) = [a_{(1)}, \dots, a_{(n)}]$

$$\phi(L) = \sum_{i=1}^n w_i a_{(i)}$$

It is clear that OWA operators are commutative, monotone and idempotent, but in general not associative. Hence, given an OWA operator of dimension n , it can be only applied to aggregation problems of such a dimension n . If the dimension problem is modified, such OWA operator can not be applied. But it is clear that there are considerably many real life decision processes where at different times one has to aggregate (possibly very large) lists of inputs of different dimensions. Connective rules have quite often to be defined before knowing such a list. In many times, we do not know even the number of elements in such a list. A connective rule is in general a rule allowing aggregation of any list, no matter its dimension.

In this paper we focus our attention on the fact that connective rules must allow us aggregation of arbitrary lists. A connective rule must always allow an aggregated value for any possible dimension of the list to be aggregated. In this way, a connective rule should in principle be understood as a family of connective operators, allowing a recursive evaluation.

Sometimes our recursive definition does not depend on the size of inputs itself. This is the case for associative binary rules as t -norms and t -conorms: given $F(x, y)$ associative, and given any n values (a_1, \dots, a_n) , we can apply the same operative definition of $F(x, y)$ to obtain

$$\begin{aligned} F(a_1, \dots, a_n) &= \\ F(F(a_1, \dots, a_{n-1}), a_n) &= \\ F(a_1, F(a_2, \dots, a_n)) & \end{aligned}$$

(in the first case the final value has been obtained by means of a *left recursion* call and in the second case by means of a *right recursion* call).

Although such an associative procedure can not be considered when dealing with OWA operators, both recursive analysis of OWA operators can be applied to the ordered list associated to (a_1, \dots, a_n) . Then, we just need a sequence of OWA operators that takes into account the information about the number of values to be aggregated. Thus, practical aggregation problems, where the number of values to be aggregated is not necessarily previously known, should

be solved by choosing one of these families of OWA operators. Each one of these families solves every aggregation problem for any arbitrary size of the input. Moreover, our recursive definition has also the advantage that aggregation weights can be computed quickly by using a dynamic programming approach (see [1]).

The above arguments also apply to other connective operators different to t -norms, t -conorms and OWA. It is sometimes surprising finding references to connective rules which are not associative, without noticing that such a connective rule should be understood as a family of connective operators. For example, when we talk about the *mean rule* we do not refer to the *mean of n numbers*

$$M_n(a_1, \dots, a_n) = \frac{\sum_{i=1}^n a_i}{n},$$

but to the rule that evaluates such a mean for any set of n numbers. In fact, we can obtain such a *mean rule* by means of any partition of the set of numbers to be aggregated, just keeping in mind the number of items each aggregated value represents (some consequences of this property in group decision making were shown in Montero [4]; see also the general concept of *bag mapping* introduced by Yager [7,8]). For example, if we consider $i + j + k = n$, we can evaluate

$$M_n(a_1, \dots, a_n) = M_{i,j,k}(M_i(\dots, a_i), M_j(\dots, a_{i+j}), M_k(\dots, a_n))$$

where

$$M_{i,j,k}(b_1, b_2, b_3) = \frac{i \cdot b_1 + j \cdot b_2 + k \cdot b_3}{n}.$$

II. RECURSIVE DEFINITION OF CONNECTIVE RULES.

We shall focus our attention on those families of connective operators that can be defined by means of left or right recursive application of their binary connective operators, allowing just a previous rearrangement of the items to be aggregated.

DEFINITION 2 A right connective rule is a family of connective operators

$$\phi_2, \dots, \phi_n \dots$$

such that each ϕ_i is a connective operator of dimension i .

$$\phi_i : [0, 1]^i \rightarrow [0, 1]$$

in such a way that there exists a family of binary (i.e., 2-dimension) connective operators

$$F_2, \dots, F_n, \dots$$

verifying that

$$\phi_n(a_1, \dots, a_n) = F_n(\dots, F_3(F_2(b_1, b_2), b_3), \dots, b_n)$$

for each n , being

$$\{b_1, \dots, b_n\}$$

some permutation of $\{a_1, \dots, a_n\}$.

Left connective rules can be defined analogously, by means of a family of binary connective operators G_2, \dots, G_n, \dots such that for each n we have

$$\phi_n(a_1, \dots, a_n) = G_n(b_1, \dots, (b_{n-3}, G_3(b_{n-2}, G_3(b_{n-1}, b_n))) \dots)$$

where

$$\{b_1, \dots, b_n\}$$

is some permutation of $\{a_1, \dots, a_n\}$.

Most connective rules $\phi_2, \dots, \phi_n, \dots$ that can be found in the literature are defined by means of a unique commutative and associative binary operator

$$\phi : [0, 1]^2 \rightarrow [0, 1]$$

such that

$$\begin{aligned} \phi_n(a_1, \dots, a_n) &= \\ \phi(\dots \phi(\phi(b_1, b_2), b_3), \dots, b_n) &= \\ \phi(b_1, \dots, (b_{n-3}, \phi(b_{n-2}, \phi(b_{n-1}, b_n))) \dots) & \end{aligned}$$

for any permutation

$$\{b_1, \dots, b_n\}$$

of $\{a_1, \dots, a_n\}$. Hence, when we refer to some t -norm or a t -conorm as a connective rule we really mean the

family of connective operators in this a way univocally defined. The whole family of connective operators is fully characterized by its first connective operator of dimension 2.

Obviously, the above definition represents a consistency assumption to be imposed on connective rules. Not every family of connective operators will be consistent in this sense, as shown in the next section when dealing with OWA rules.

III. OWA RULES.

OWA operators are not in general associative, so when the concept of OWA rule is to be introduced, we refer to a consistent family of OWA operators. We can check that once an OWA operator of dimension n has been fixed, all OWA operators of lower dimension belonging to its right and left OWA rules can be univocally defined. In fact, every OWA operator can be recursively defined, both left and right, once the values to be aggregated have been properly ordered, and these two recursive representations are unique.

A consistent OWA rule will be then given by a sequence of OWA operators that can be explained in terms of a sequence of binary OWA operators allowing its right or left recursive representation. It is then natural to characterize each consistent OWA rule by means of the sequence of weights associated to its right or left recursive representation (see [2]).

DEFINITION 3 A basis function is any mapping f that to any integer n associates a number in the unit interval (that is, $f(n) \in [0, 1]$ for all n) with $f(1) = 1$.

Each basis function f will then allow the recursive definition of two families of OWA operators. If for any $n \geq 2$, we shall denote by F_n and F'_n the two-dimension OWA operators such that

$$F_n(b_1, b_2) = (1 - f(n))b_{[1]} + f(n)b_{[2]}$$

and

$$F'_n(b_1, b_2) = f(n)b_{[1]} + (1 - f(n))b_{[2]}$$

It is then easy to check that not every family $\{\phi_2, \dots, \phi_n, \dots\}$ of OWA operators can be recursively defined as above by means of OWA operators of

dimension 2. Anyway, its consistency can be easily characterized by means of the weights of the OWA operators.

Two particular cases to be analyzed are the *mean rule* and those rules characterized by a constant base function (i.e., when there exists a value $a \in [0, 1]$ such that $f(a) = a$ for all $a \geq 2$).

IV. FINAL COMMENTS

Following a basic result obtained by the authors in [2] for OWA operators, we have proposed a general approach to arbitrary connective rules. These connective rules are conceived here as consistent families of connective operators, allowing its representation in terms of right or left recursion of binary connective operators. Each one of these elements of the connective rule can be characterized, as here shown for consistent OWA rules. Non-associative operators can not be considered in most practical applications if a consistent rule has not been defined (unless the number of items to be aggregated is known and remains constant through out our study).

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REFERENCES

- [1] T.H. Cormen, C.E. Leiserson and R.R. Rivest. *Introduction to Algorithms*. MIT Press, Cambridge, MA, 1990.
- [2] V. Cutello and J. Montero. Recursive families of OWA operators. *Proceedings of the Third IEEE Conference on Fuzzy Systems* (Orlando, June 26-29, 1994), pp 1136-1141.
- [3] G.J. Klir and T.A. Folger. *Fuzzy sets, Uncertainty and Information*. Prentice Hall, Englewood Cliffs, NJ, 1988.
- [4] J. Montero. Aggregation of fuzzy opinions in a non-homogeneous group. *Fuzzy sets and systems*, 25:15-20, 1988.
- [5] R.R. Yager. On ordered weighted averaging aggregation operators in multi-criteria decision making. *IEEE transactions on Systems Man and Cybernetics*, 18:183-190, 1988.
- [6] R.R. Yager. Families of OWA operators. *Fuzzy sets and systems*, 59:125-148, 1993.
- [7] R.R. Yager. MAM and MOM operators for aggregation. *Information Sciences*, 69:259-273, 1993.
- [8] R.R. Yager. Aggregation operators and fuzzy systems modeling. *Fuzzy sets and systems*, 67:129-145, 1994.