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CONSISTENT VALUED PREFERENCE MODELS

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***Abstract*— As shown by Fodor, Ovchinnikov and Roubens [3,4,7,14], a binary preference relation should be always understood as a structure which explicits how strict preference, indifference, weak preference and even incomparability are defined. Some particular solutions have been axiomatically characterized by these authors. In this paper we shall discuss some of their basic assumptions and comment on the real degree of freedom we have in order to define consistent families of these four basic valued preference relations.**

Keywords: valued preferences, fuzzy preferences.

I. Introduction.

In classical preference modelling it is given a finite set A of feasible alternatives with a weak binary preference relation defined on it. Associated to such weak binary relation, three binary relations (strict preference P, indifference I and incomparability J) are defined on $A \times A$. Translating this scheme into valued binary preference relations appears as an interesting task (see [3,4,6,14]). If each value $R(a,b) \in [0,1]$ represents the degree to which the assertion "alternative a is not worse than alternative b" is true, then three associated valued preference relations have to be defined: valued strict preference (P), valued indifference (I) and valued incomparability (J). The axiomatic approach initially proposed in Ovchinnikov-Roubens [14] lead to a unique family of solutions. Fodor [3] improves the model, but in fact basically the same family of solutions is characterized. Some more particular solutions can be found in [4,5,6,7].

In this paper we shall discuss basic assumptions in Fodor-Roubens-Ovchinnikov (FOR) model, introducing an alternative approach that allows to overcome some technical difficulties. The key questions about the way new consistent structures can be found is fully solved in this context.

II. Fodor-Ovchinnikov-Roubens (FOR) model.

Fodor, Ovchinnikov and Roubens (see [3,4,6,14] but mainly [7]) propose to consider standard fuzzy conjunctions and disjuntions in order to get appropriate translations from *crisp* mathematical expressions into *valued* equations. Following most fuzzy literature existence of some t-conorm S and some t-norm T are directly imposed in order to represent the *valued* union (disjunction) and the *valued* intersection (conjunction). These two operators should be related by means of a strict negation funtion n, leading to a De Morgan triple

$$(T, S, n)$$

where

$$S(x,y) = n^{-1}(T(n(x), n(y)))$$

for each $x, y \in [0,1]$. Continuity of t-norm T -and therefore continuity of the associated t-conorm S- is justified from practical arguments (in fact, we would rather accept a stronger analytical stability than just continuity).

A second set of key assumptions refer to the functional form of the basic *valued* strict preference, indifference and incomparability relations. Fodor and Roubens propose three general axioms, named Independence of Irrelevant Alternatives, Positive Association and Symmetry, in such a way that it is assumed the existence of three functions p, i, j from $[0,1]^2$ into $[0,1]$ such that

- $P(a,b) = p(R(a,b), R(b,a))$,
- $I(a,b) = i(R(a,b), R(b,a))$ and
- $J(a,b) = j(R(a,b), R(b,a))$,

where $p(x,y)$ is nondecreasing with respect to its first argument but nonincreasing with respect to its second argument, $i(x,y)$ is symmetric and nondecreasing with respect to both arguments, and $j(x,y)$ is also symmetric and nonincreasing with respect to both arguments. We shall accept these conditions, although they will turn out to be partially redundant. Moreover, practical considerations may again suggest that the mappings p, i, j should be continuous.

The third set of key assumptions in Fodor-Roubens model is given by two key mathematical expressions, both translated to this valued framework from the same crisp formula:

(A) $S(p(x,y), i(x,y)) = x$

(B) $S(p(y,x), j(y,x)) = n(x)$

Last assumptions (A) and (B) will then imply that T is an Archimedean t-norm (i.e., $T(x,x) < x$ for all $x \in (0,1)$) with zero divisors (i.e., $T(x,y) = 0$ for some $x, y > 0$). Hence, being T continuous, the representation obtained in [13] holds. That is,

$$T(x,y) = \phi^{-1}(max\{\phi(x) + \phi(y) - 1, 0\})$$

for some automorphism in the unit interval ϕ. Moreover, from basic assumptions in Fodor-Roubens model it is implied that the strict negation n is in fact a strong negation (i.e., $n(n(x)) = x$ for all x), since it must be

$$n(x) = \phi^{-1}(1 - \phi(x))$$

for all x, with ϕ the same automorphism defining T.

III. An alternative approach.

We highlight two main problems with the above FOR formalization. On one hand, we notice that conceptually different intensities may require different aggregation operators: the aggregation of strict preferences and indifference should not be by definition done by the same aggregation rule that gives the degree of comparability. On the other hand, it is not clear why those two aggregation rules must be t-conorms.

What we all can accept is that once a weak valued preference relation has been defined by the decision maker, we should be able to evaluate all other three basic valued preference relations. That is, from each pair of values $R(a,b), R(b,a)$ we should be able to derive the degree of strict preference $P(a,b)$, the degree of indifference $I(a,b)$ and the degree of incomparability $J(a,b)$.

Then, the aggregated value obtained from $P(a,b)$ and $I(a,b)$ should give us the intensity of the weak preference $R(a,b)$, being $R(a,b)$ the set of data given by the decision maker. Incomparability is in fact defined just as the negation of comparability, which has to be measured as an aggregated value from the weak intensity value $R(a,b)$ and $P(b,a)$. In this way, the aggregation of comparability and incomparability degrees should cover the whole intensity, and our negation can be assumed to be a strong negation (negation of negation of comparability is again comparability). These were the underlying intuitive arguments in the model of Montero (see [1,2,9,10,11], where such an Arrow-like amalgamation model is considered under no incomparability, i.e., $J(a,b) = 0$ for all a, b).

Axioms formalizing the above ideas can be summarized -once we assume that intensities will take values in the unit interval $[0,1]$, having both extreme points 0, 1 the obvious intended meaning- in the existence of three aggregation functions S_w, S_{pj}, S_c and a (strong) negation n such that weak intensity (R), comparability (C) and incomparability (J) are defined in the following way (see [12] for a detailed exposition):

(R)

$$\begin{aligned} R(a,b) &= S_w(P(a,b), I(a,b)) \\ &= n(S_{pj}(P(b,a), J(a,b))) \end{aligned}$$

(C) $C(a,b) = S_c(R(a,b), P(b,a))$

(J) $J(a,b) = n(C(a,b))$

Obviously, condition (R) is just another way of writing conditions (A) and (B). Moreover, the mapping n has been naturally assumed to be a strong negation in the unit interval.

In fact, condition (J) contains the definition of the comparability relation in terms of such a negation n and those two aggregation functions S_c, S_w. Each one of these aggregation functions are in principle mappings S that can naturally be assumed to be monotonic: $S(a,b) \geq S(c,d)$ whenever $a \geq c$ and $b \geq d$. In order to assure that such a mapping is in fact an aggregation function, some minimal boundary requirements should be added. For example, it is natural to accept that the agregation of "nothing" to "something" does not introduce any change, that is, condition $S(x,0) = S(x,0) = x$. But it seems to be arbitrary to impose symmetry or associativity on these aggregation functions (as already pointed out, we can always assume that every mapping is continuous).

Condition (C) turns out to be the cornerstone under our approach. Once j (and therefore c) has been assumed symmetric, we have $C(a,b) = C(b,a)$, that is,

$$S_c(S_w(P(a,b), I(a,b)), P(b,a)) = S_c(S_w(P(b,a), I(b,a)), P(a,b))$$

Such an expression can be re-written as

$$S_c(S_w(P(a,b), I(a,b)), P(b,a)) = S_c'(P(a,b), S_w'(I(b,a), P(b,a))$$

We are then suggesting that our model should verify a generalized functional equation of associativity. In particular, since i is symmetric, we can impose the functional equation

$$S_c(S_w(x,y), z) = S_c'(x, S_w'(y,z))$$

assuming the above relationship between continuous and nondecreasing aggregation functions. The above functional equation was solved by Koopmans [8] assuming that each aggregation function is continuous and verifies a stronger strict monotonicity condition, that in fact can be easily assumed for our aggregation functions S_w and S_w', but also for S_c, S_c'. We do expect that as soon as one of the basic intensities increases, the aggregated valued should increase. Each aggregation function should be strictly increasing in its real range, i.e., when restricted to those values $p(x,y), i(x,y)$ (or $x, p(y,x)$) to be aggregated by S_w (or S_c). In order to apply Koopman's result, we shall impose that our aggregation functions are strictly increasing in the cartesian product of appropriate (non-trivial, non-negative) compact real intervals, but verifying that $S(x,y) \in [0,1]$ for every pair (x,y) in the real range. Under such a condition, it can be proven that all our aggregation functions can be represented by means of a unique automorphism ϕ in the unit interval, which will additionally define the negation function n.

Notice that our approach is absolutely coherent with the FOR model: if the representation

$$S(x,y) = \phi^{-1}(min(\phi(x) + \phi(y), 1))$$

holds for some automorphism ϕ in the unit interal, such a t-conorm with zero divisors S is strictly increasing whenever

$$\phi(x) + \phi(y) \leq 1.$$

It can be then proven that functions p, i, c, j can not be arbitrarily defined.

THEOREM III.1 *Let a strict preference intensity function p be given (alternatively, function i, c or j). Then the remaining intensity functions are fixed up to some automorphism ϕ.*

General conditions on how these i, p, c, j have to be defined follow from our FORM model. As a consequence, it can be shown that some strict preference relations that can be found in the literature do not allow a consistent preference structure.

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