

ON THE POLYNOMIAL HARDY–LITTLEWOOD INEQUALITY

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ABSTRACT. We investigate the growth of the constants of the polynomial Hardy-Littlewood inequality.

1. INTRODUCTION

Given $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, let $|\alpha| := \alpha_1 + \dots + \alpha_n$ and let \mathbf{x}^α stand for the monomial $x_1^{\alpha_1} \dots x_n^{\alpha_n}$ for $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{K}^n$. The polynomial Bohnenblust–Hille inequality asserts that if P is a homogeneous polynomial of degree m on ℓ_∞^n given by

$$P(x_1, \dots, x_n) = \sum_{|\alpha|=m} a_\alpha \mathbf{x}^\alpha,$$

then

$$\left(\sum_{|\alpha|=m} |a_\alpha|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq D_m^{\mathbb{K}} \|P\|$$

for some positive constant $D_m^{\mathbb{K}}$ which does not depend on n (the exponent $\frac{2m}{m+1}$ is optimal). Precise estimates of the growth of the constants $D_m^{\mathbb{K}}$ are crucial for different applications. The following diagram shows the evolution of the estimates of $D_m^{\mathbb{C}}$:

Authors	Year	Estimate
Bohnenblust and Hille	1931, [6]	$D_m^{\mathbb{C}} \leq m^{\frac{m+1}{2m}} (\sqrt{2})^{m-1}$
Defant, Frerick, Ortega-Cerdá, Ounaies, and Seip	2011, [9]	$D_m^{\mathbb{C}} \leq \left(1 + \frac{1}{m-1}\right)^{m-1} \sqrt{m} (\sqrt{2})^{m-1}$
Bayart, Pellegrino, and Seoane-Sepúlveda	2013, [5]	$D_m^{\mathbb{C}} \leq C(\varepsilon) (1 + \varepsilon)^m,$

where, in the table above, $C(\varepsilon) (1 + \varepsilon)^m$ means that given $\varepsilon > 0$, there is a constant $C(\varepsilon) > 0$ such that $D_m^{\mathbb{C}} \leq C(\varepsilon) (1 + \varepsilon)^m$ for all m .

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For real scalars it is shown in [7] that

$$(1.1)^m \leq D_m^{\mathbb{R}} \leq C(\varepsilon) (2 + \varepsilon)^m,$$

and this means that for real scalars the inequality is hypercontractive and this result is optimal.

When replacing ℓ_∞^n by ℓ_p^n the extension of the Bohnenblust–Hille inequality is called Hardy–Littlewood inequality and the optimal exponents are $\frac{2mp}{mp+p-2m}$ for $p \geq 2m$. This is a consequence of the multilinear Hardy–Littlewood inequality (see [2, 10]). More precisely, if P is a homogeneous polynomial of degree m on ℓ_p^n , with $p \geq 2m$, given by

$$P(x_1, \dots, x_n) = \sum_{|\alpha|=m} a_\alpha \mathbf{x}^\alpha,$$

then

$$\left(\sum_{|\alpha|=m} |a_\alpha|^{\frac{2mp}{mp+p-2m}} \right)^{\frac{mp+p-2m}{2mp}} \leq D_{m,p}^{\mathbb{K}} \|P\|,$$

and $D_{m,p}^{\mathbb{K}}$ does not depend on n .

In this paper we look for upper and lower estimates for $D_{m,p}^{\mathbb{K}}$.

2. FIRST (AND PROBABLY BAD) UPPER ESTIMATES FOR $D_{m,p}^{\mathbb{K}}$

Given $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, let us define

$$\binom{m}{\alpha} := \frac{m!}{\alpha_1! \cdots \alpha_n!},$$

for $|\alpha| = m \in \mathbb{N}^*$. A straightforward consequence of the multinomial formula yields the following relationship between the coefficients of a homogeneous polynomial and the polar of the polynomial (this lemma appears in [8] and is essentially *folklore*).

Lemma 2.1. *If P is a homogeneous polynomial of degree m on \mathbb{K}^n given by*

$$P(x_1, \dots, x_n) = \sum_{|\alpha|=m} a_\alpha \mathbf{x}^\alpha,$$

and L is the polar of P , then

$$L(e_1^{\alpha_1}, \dots, e_n^{\alpha_n}) = \frac{a_\alpha}{\binom{m}{\alpha}},$$

where $\{e_1, \dots, e_n\}$ is the canonical basis of \mathbb{K}^n and $e_k^{\alpha_k}$ stands for e_k repeated α_k times.

From now on, for any map $f : \mathbb{R} \rightarrow \mathbb{R}$ we define

$$f(\infty) := \lim_{p \rightarrow \infty} f(p).$$

The following result is also essentially known. We present here the details of its proof for the sake of completeness of the paper.

Proposition 2.2. *If P is a homogeneous polynomial of degree m on ℓ_p^n , with $p \geq 2m$, given by*

$$P(x_1, \dots, x_n) = \sum_{|\alpha|=m} a_\alpha \mathbf{x}^\alpha,$$

then

$$\left(\sum_{|\alpha|=m} |a_\alpha|^{\frac{2mp}{mp+p-2m}} \right)^{\frac{mp+p-2m}{2mp}} \leq D_{m,p}^{\mathbb{K}} \|P\|$$

with

$$D_{m,p}^{\mathbb{K}} = C_{m,p}^{\mathbb{K}} \frac{m^m}{(m!)^{\frac{mp+p-2m}{2mp}}},$$

where $C_{m,p}^{\mathbb{K}}$ are the constants of the multilinear Hardy-Littlewood inequality.

Proof. From Lemma 2.1 we have

$$\begin{aligned} \sum_{|\alpha|=m} |a_\alpha|^{\frac{2mp}{mp+p-2m}} &= \sum_{|\alpha|=m} \left(\binom{m}{\alpha} |L(e_1^{\alpha_1}, \dots, e_n^{\alpha_n})| \right)^{\frac{2mp}{mp+p-2m}} \\ &= \sum_{|\alpha|=m} \binom{m}{\alpha}^{\frac{2mp}{mp+p-2m}} |L(e_1^{\alpha_1}, \dots, e_n^{\alpha_n})|^{\frac{2mp}{mp+p-2m}}. \end{aligned}$$

However, for every choice of α , the term

$$|L(e_1^{\alpha_1}, \dots, e_n^{\alpha_n})|^{\frac{2mp}{mp+p-2m}}$$

is repeated $\binom{m}{\alpha}$ times in the sum

$$\sum_{i_1, \dots, i_m=1}^n |L(e_{i_1}, \dots, e_{i_m})|^{\frac{2mp}{mp+p-2m}}.$$

Thus

$$\sum_{|\alpha|=m} \binom{m}{\alpha}^{\frac{2mp}{mp+p-2m}} |L(e_1^{\alpha_1}, \dots, e_n^{\alpha_n})|^{\frac{2mp}{mp+p-2m}} = \sum_{i_1, \dots, i_m=1}^n \binom{m}{\alpha}^{\frac{2mp}{mp+p-2m}} \frac{1}{\binom{m}{\alpha}} |L(e_{i_1}, \dots, e_{i_m})|^{\frac{2mp}{mp+p-2m}}$$

and, since

$$\binom{m}{\alpha} \leq m!$$

we have

$$\sum_{|\alpha|=m} \binom{m}{\alpha}^{\frac{2mp}{mp+p-2m}} |L(e_1^{\alpha_1}, \dots, e_n^{\alpha_n})|^{\frac{2mp}{mp+p-2m}} \leq (m!)^{\frac{mp-p+2m}{mp+p-2m}} \sum_{i_1, \dots, i_m=1}^n |L(e_{i_1}, \dots, e_{i_m})|^{\frac{2mp}{mp+p-2m}}.$$

We finally obtain

$$\begin{aligned} \left(\sum_{|\alpha|=m} |a_\alpha|^{\frac{2mp}{mp+p-2m}} \right)^{\frac{mp+p-2m}{2mp}} &\leq \left((m!)^{\frac{mp-p+2m}{mp+p-2m}} \sum_{i_1, \dots, i_m=1}^n |L(e_{i_1}, \dots, e_{i_m})|^{\frac{2mp}{mp+p-2m}} \right)^{\frac{mp+p-2m}{2mp}} \\ &= (m!)^{\frac{mp-p+2m}{2mp}} \left(\sum_{i_1, \dots, i_m=1}^n |L(e_{i_1}, \dots, e_{i_m})|^{\frac{2mp}{mp+p-2m}} \right)^{\frac{mp+p-2m}{2mp}} \\ &\leq (m!)^{\frac{mp-p+2m}{2mp}} C_{m,p}^{\mathbb{K}} \|L\|. \end{aligned}$$

On the other hand, it is well-known that

$$\|L\| \leq \frac{m^m}{m!} \|P\|$$

and, hence,

$$\begin{aligned} \left(\sum_{|\alpha|=m} |a_\alpha|^{\frac{2mp}{mp+p-2m}} \right)^{\frac{mp+p-2m}{2mp}} &\leq C_{m,p}^{\mathbb{K}} (m!)^{\frac{mp-p+2m}{2mp}} \frac{m^m}{m!} \|P\| \\ &= C_{m,p}^{\mathbb{K}} \frac{m^m}{(m!)^{\frac{mp+p-2m}{2mp}}} \|P\|. \end{aligned}$$

□

Remark 2.3. Let us define the polarization constants for polynomials on ℓ_p spaces as

$$\mathbb{K}(m, p) := \inf\{M > 0 : \|L\| \leq M\|P\|\},$$

where the infimum is taken over all $P \in \mathcal{P}(^m \ell_p^n)$ and L is the polar of P . Notice that $\mathbb{K}(m, p)$ may improve the inequality

$$\|L\| \leq \frac{m^m}{m!} \|P\|$$

used in the proof of Proposition 2.2. Using $\mathbb{K}(m, p)$ instead of $\frac{m^m}{m!}$ in the proof of Proposition 2.2 we end up with

$$D_{m,p}^{\mathbb{K}} = (m!)^{\frac{mp-p+2m}{2mp}} C_{m,p}^{\mathbb{K}} \mathbb{K}(m, p).$$

The calculation of polarization constants has been a fruitful problem since the 1970's and some facts are known about them. Specifically, Harris proved in [12] that

$$\mathbb{C}(m, p) \leq \left(\frac{m^m}{m!} \right)^{\frac{|p-2|}{p}},$$

for all $p \geq 1$, whenever m is a power of 2. Also, Harris [12] proved that

$$\mathbb{C}(m, \infty) \leq \frac{m^{\frac{m}{2}} (m+1)^{\frac{m+1}{2}}}{2^m m!}.$$

One more example was provided by Sarantopoulos [15] who proved that

$$\mathbb{K}(m, p) = \frac{m^{\frac{m}{p}}}{m!},$$

whenever $1 \leq p \leq \frac{m}{m-1}$.

3. THE REAL POLYNOMIAL HARDY–LITTLEWOOD INEQUALITY: LOWER BOUNDS FOR THE CONSTANTS

For $m \geq 2$ and $p \geq 2m$, let us denote by $H_{\mathbb{K}, m, p}^{\text{pol}}$ the optimal constants satisfying the polynomial Hardy-Littlewood inequality with scalars in \mathbb{K} . In this section we show that for real scalars the polynomial Hardy-Littlewood inequality has at least an hypercontractive growth.

Theorem 3.1. For all positive integers $m \geq 2$ and $2m \leq p < \infty$ we have

$$\left(\sqrt[16]{2} \right)^m \leq 2^{\frac{mp+p-6m+4}{4p} \cdot \frac{m-1}{m}} \leq H_{\mathbb{R}, m, p}^{\text{pol}}.$$

Proof. Let m be an even integer. Consider the m -homogeneous polynomial $P_m : \ell_p^m \rightarrow \mathbb{R}$ given by

$$P_m(x_1, \dots, x_m) = (x_1^2 - x_2^2)(x_3^2 - x_4^2) \cdots (x_{m-1}^2 - x_m^2).$$

Notice that

$$\|P_m\| = P_m \left(\frac{1}{\sqrt[p]{m/2}}, 0, \frac{1}{\sqrt[p]{m/2}}, \dots, \frac{1}{\sqrt[p]{m/2}}, 0 \right) = \left(\frac{1}{\sqrt[p]{m/2}} \right)^m.$$

From the Hardy-Littlewood inequality for P_m we have

$$\left(\sum_{|\alpha|=m} |a_\alpha|^{\frac{2mp}{mp+p-2m}} \right)^{\frac{mp+p-2m}{2mp}} \leq H_{\mathbb{R},m,p}^{\text{pol}} \|P_m\|,$$

i.e.,

$$H_{\mathbb{R},m,p}^{\text{pol}} \geq \frac{\left(2^{\frac{m}{2}}\right)^{\frac{mp+p-2m}{2mp}}}{\left(\frac{1}{\sqrt[p]{m/2}}\right)^m} = 2^{\frac{mp+p-2m}{4p}} \left(\frac{m}{2}\right)^{\frac{m}{p}} = 2^{\frac{mp+p-6m}{4p}} m^{\frac{m}{p}} \geq 2^{\frac{mp+p-6m}{4p}}.$$

If m is odd, define

$$Q_m(x_1, \dots, x_m) = (x_1^2 - x_2^2)(x_3^2 - x_4^2) \cdots (x_{m-2}^2 - x_{m-1}^2) x_m.$$

Then

$$\|Q_m\| \leq \|P_{m-1}\| = \left(\frac{1}{\sqrt[p]{(m-1)/2}} \right)^{m-1}.$$

From the Hardy-Littlewood inequality for Q_m we have

$$\left(\sum_{|\alpha|=m} |a_\alpha|^{\frac{2mp}{mp+p-2m}} \right)^{\frac{mp+p-2m}{2mp}} \leq H_{\mathbb{R},m,p}^{\text{pol}} \|Q_m\|,$$

i.e.,

$$\begin{aligned} H_{\mathbb{R},m,p}^{\text{pol}} &\geq \frac{\left(2^{\frac{m-1}{2}}\right)^{\frac{mp+p-2m}{2mp}}}{\left(\frac{1}{\sqrt[p]{(m-1)/2}}\right)^{m-1}} \\ &= 2^{\frac{(mp+p-2m)(m-1)}{4mp}} \left(\frac{m-1}{2}\right)^{\frac{m-1}{p}} \\ &= 2^{\frac{mp+p-6m+4}{4p}, \frac{m-1}{m}} (m-1)^{\frac{m-1}{p}} \\ &\geq 2^{\frac{mp+p-6m+4}{4p}, \frac{m-1}{m}}. \end{aligned}$$

□

Remark 3.2. *From the estimates of the last proof note that if m is even, and $m \geq 4$, then*

$$\begin{aligned}
 H_{\mathbb{R},m,p}^{\text{pol}} &\geq 2^{\frac{mp+p-6m}{4p}} m^{\frac{m}{p}} \\
 (3.1) \qquad \qquad &\geq 2^{\frac{mp+p-6m}{4p}} \left(2^{\frac{3}{2}}\right)^{\frac{m}{p}} \\
 &= \left(\sqrt[4]{2}\right)^{m+1}.
 \end{aligned}$$

If m is odd, and $m \geq 5$, then

$$\begin{aligned}
 H_{\mathbb{R},m,p}^{\text{pol}} &\geq 2^{\frac{mp+p-6m+4}{4p}, \frac{m-1}{m}} (m-1)^{\frac{m-1}{p}} \\
 (3.2) \qquad \qquad &\geq 2^{\frac{mp+p-6m+4}{4p}, \frac{m-1}{m}} \left(2^{\frac{3}{2}}\right)^{\frac{m-1}{p}} \\
 &= 2^{\frac{pm^2+4m-p-4}{4mp}} \\
 &\geq \left(\sqrt[4]{2}\right)^{m-\frac{1}{m}}.
 \end{aligned}$$

Thus, by (3.1) and (3.2), if $m \geq 4$

$$H_{\mathbb{R},m,p}^{\text{pol}} \geq \left(\sqrt[4]{2}\right)^{m-\frac{1}{m}}.$$

4. REAL VERSUS COMPLEX ESTIMATES

As it happens with the constants of the Bohnenblust-Hille inequality, we observe that

$$(4.1) \qquad H_{\mathbb{R},m,p}^{\text{pol}} \leq 2^{m-1} H_{\mathbb{C},m,p}^{\text{pol}}.$$

In fact, from [13] we know that if $P : \ell_p \rightarrow \mathbb{R}$ is an m -homogeneous polynomial and $P_{\mathbb{C}} : \ell_p \rightarrow \mathbb{C}$ is the same polynomial, then

$$\|P_{\mathbb{C}}\| \leq 2^{m-1} \|P\|.$$

We thus obtain (4.1). So if one succeeds in proving that the constants of the complex Hardy-Littlewood polynomial inequality have a subexponential growth (as it happens with the constants of the complex polynomial Bohnenblust-Hille inequality) then we immediately conclude that the constants of the real polynomial Hardy littlewood inequality have an exponential growth and this result is optimal (due to Theorem 3.1).

5. THE COMPLEX POLYNOMIAL HARDY-LITTLEWOOD INEQUALITY: TRYING TO FIND UPPER ESTIMATES

In this section we try to improve the estimates for the upper bounds of the complex polynomial Hardy-Littlewood inequality from Section 2. However, for $p < \infty$ the validity of our estimates depend on a conjecture (see (5.1)). For the case $p = \infty$ our results are exactly those from ([5, Remark 2.2]).

The following multi-index notation will come in handy for us: for positive integers m, n , we set

$$\begin{aligned}\mathcal{M}(m, n) &:= \{\mathbf{i} = (i_1, \dots, i_m); i_1, \dots, i_m \in \{1, \dots, n\}\}, \\ \mathcal{J}(m, n) &:= \{\mathbf{i} \in \mathcal{M}(m, n); i_1 \leq i_2 \leq \dots \leq i_m\},\end{aligned}$$

and for $k = 1, \dots, m$, $\mathcal{P}_k(m)$ denotes the set of the subsets of $\{1, \dots, m\}$ with cardinality k . For $S = \{s_1, \dots, s_k\} \in \mathcal{P}_k(m)$, its complement will be $\widehat{S} := \{1, \dots, m\} \setminus S$, and \mathbf{i}_S shall mean $(i_{s_1}, \dots, i_{s_k}) \in \mathcal{M}(k, n)$. For a multi-index $\mathbf{i} \in \mathcal{M}(m, n)$, we denote by $|\mathbf{i}|$ the cardinality of the set of multi-indices $\mathbf{j} \in \mathcal{M}(m, n)$ such that there is a permutation σ of $\{1, \dots, m\}$ with $i_{\sigma(k)} = j_k$, for every $k = 1, \dots, m$. The equivalence class of \mathbf{i} is denoted by $[\mathbf{i}]$. When we write $c_{[\mathbf{i}]}$ for $\mathbf{i} \in \mathcal{M}(m, n)$ we mean $c_{\mathbf{j}}$ for $\mathbf{j} \in \mathcal{J}(m, n)$ and \mathbf{j} equivalent to \mathbf{i} .

The following very recent generalization of the famous Blei inequality will be crucial for our estimates (see [5, Remark 2.2]).

Lemma 5.1 (Bayart, Pellegrino, Seoane, [5]). *Let m, n positive integers, $1 \leq k \leq m$ and $1 \leq s \leq q$, satisfying $\frac{m}{p} = \frac{k}{s} + \frac{m-k}{q}$. Then for all scalar matrix $(a_{\mathbf{i}})_{\mathbf{i} \in \mathcal{M}(m, n)}$,*

$$\left(\sum_{\mathbf{i} \in \mathcal{M}(m, n)} |a_{\mathbf{i}}|^p \right)^{\frac{1}{p}} \leq \prod_{S \in \mathcal{P}_k(m)} \left(\sum_{\mathbf{i}_S} \left(\sum_{\mathbf{i}_{\widehat{S}}} |a_{\mathbf{i}}|^q \right)^{\frac{s}{q}} \right)^{\frac{1}{s} \cdot \frac{1}{\binom{m}{k}}}.$$

Let us use the following notation: $S_{\ell_p^n}$ denotes the unit sphere on ℓ_p^n if $p < \infty$, and $S_{\ell_\infty^n}$ denotes the n -dimensional torus. More precisely: for $p \in (0, \infty)$

$$S_{\ell_p^n} := \left\{ \mathbf{z} = (z_1, \dots, z_n) \in \mathbb{C}^n : \|\mathbf{z}\|_{\ell_p^n} = 1 \right\},$$

and

$$S_{\ell_\infty^n} := \mathbb{T}^n = \{ \mathbf{z} = (z_1, \dots, z_n) \in \mathbb{C}^n : |z_i| = 1 \}.$$

For $p \in [1, \infty]$, let μ_p^n be the normalized Lebesgue measure on $S_{\ell_p^n}$. For the sake of simplicity, μ_p^n will be simply denoted by μ^n . The following inequality is a variant of a result due to Bayart [4, Theorem 9] (see, for instance, [5, Lemma 5.1]).

Lemma 5.2 (Bayart). *Let $1 \leq s \leq 2$. For every m -homogeneous polynomial $P(\mathbf{z}) = \sum_{|\alpha|=m} a_\alpha \mathbf{z}^\alpha$ on \mathbb{C}^n with values in \mathbb{C} , we have*

$$\left(\sum_{|\alpha|=m} |a_\alpha|^2 \right)^{\frac{1}{2}} \leq \left(\frac{2}{s} \right)^{\frac{m}{2}} \left(\int_{\mathbb{T}^n} |P(\mathbf{z})|^s d\mu^n(\mathbf{z}) \right)^{\frac{1}{s}}.$$

For $m \in [2, \infty]$ let us define $p_0(m)$ as the infimum of the values of $p \in [2m, \infty]$ such that for all $1 \leq s \leq \frac{2p}{p-2}$ there is a $K_{s,p} > 0$ such that

$$(5.1) \quad \left(\sum_{|\alpha|=m} |a_\alpha|^{\frac{2p}{p-2}} \right)^{\frac{p-2}{2p}} \leq K_{s,p}^m \left(\int_{S_{\ell_p^n}} |P(\mathbf{z})|^s d\mu^n(\mathbf{z}) \right)^{\frac{1}{s}}$$

for all positive integers n and all m -homogeneous polynomials $P : \mathbb{C}^n \rightarrow \mathbb{C}$. For the sake of simplicity $p_0(m)$ will be simply denoted by p_0 . From Bayart's lemma we know that this definition makes sense, since from Bayart's lemma we know that (5.1) is valid for $p = \infty$. We conjecture that $p_0 \leq m^2$.

Now, let us state and prove the main result of this section. The argument of the proof follows the lines of that in [9, 5]. We will use the following result due L. Harris (see [12, Theorem 1]):

Lemma 5.3 (Harris). *Let X be a complex normed linear space. If P is a homogeneous polynomial of degree m on X and L is the polar of P , then, for any nonnegative integers m_1, \dots, m_k with $m_1 + \dots + m_k = m$ and for any $x^{(1)}, \dots, x^{(k)}$ unit vectors in X ,*

$$|L(\underbrace{x^{(1)}, \dots, x^{(1)}}_{m_1 \text{ times}}, \dots, \underbrace{x^{(k)}, \dots, x^{(k)}}_{m_k \text{ times}})| \leq \frac{m_1! \cdots m_k! \cdot m^m}{m_1^{m_1} \cdots m_k^{m_k} \cdot m!} \|P\|.$$

Theorem 5.4. *Let $m \in [2, \infty]$ and $1 \leq k \leq m-1$. If $p_0(m-k) < p \leq \infty$ (and $p = \infty$ if $p_0(m-k) = \infty$) then, for every m -homogeneous polynomial $P : \ell_p^n \rightarrow \mathbb{C}$, defined by $P(\mathbf{z}) = \sum_{|\alpha|=m} a_\alpha \mathbf{z}^\alpha$, we have*

$$\begin{aligned} & \left(\sum_{|\alpha|=m} |a_\alpha|^{\frac{2mp}{mp+p-2m}} \right)^{\frac{mp+p-2m}{2mp}} \\ & \leq K_{\frac{2kp}{kp+p-2k}, p}^{m-k} \cdot \frac{m^m}{(m-k)^{m-k}} \cdot \left(\frac{(m-k)!}{m!} \right)^{\frac{p-2}{2p}} \left(\frac{2}{\sqrt{\pi}} \right)^{\frac{2k(k-1)}{p}} \cdot (B_{\mathbb{C},k}^{\text{mult}})^{\frac{p-2k}{p}} \|P\|, \end{aligned}$$

where $B_{\mathbb{C},k}^{\text{mult}}$ is any constant satisfying the k -linear Bohnenblust-Hille inequality.

Proof. We can also write

$$P(\mathbf{z}) = \sum_{\mathbf{i} \in \mathcal{J}(m,n)} c_{\mathbf{i}} z_{i_1} \cdots z_{i_m}.$$

Consider

$$\begin{aligned} \rho &= \frac{2mp}{mp+p-2m}, \\ s_k &= \frac{2kp}{kp+p-2k}, \\ q &= \frac{2p}{p-2}. \end{aligned}$$

Note that

$$\begin{aligned} s_k &\leq 2 < q, \\ \frac{m}{\rho} &= \frac{mp+p-2m}{2p}, \end{aligned}$$

and

$$\begin{aligned} \frac{k}{s_k} + \frac{m-k}{q} &= \frac{kp+p-2k}{2p} + \frac{(m-k)(p-2)}{2p} \\ &= \frac{kp+p-2k}{2p} + \frac{mp-kp-2m+2k}{2p} \\ &= \frac{mp+p-2m}{2p}. \end{aligned}$$

and thus

$$\frac{m}{\rho} = \frac{k}{s_k} + \frac{m-k}{q}$$

and we can use Lemma 5.1.

Let $L : \ell_p^n \times \cdots \times \ell_p^n \rightarrow \mathbb{C}$ be the unique symmetric m -linear map associated to P . Note that

$$L(\mathbf{z}^{(1)}, \dots, \mathbf{z}^{(m)}) = \sum_{\mathbf{i} \in \mathcal{M}(m, n)} \frac{c[\mathbf{i}]}{|\mathbf{i}|} z_{i_1}^{(1)} \cdots z_{i_m}^{(m)}.$$

Thus

$$\begin{aligned} \sum_{|\alpha|=m} |a_\alpha|^{\frac{2mp}{mp+p-2m}} &= \sum_{\mathbf{i} \in \mathcal{J}(m, n)} |c_{\mathbf{i}}|^{\frac{2mp}{mp+p-2m}} \\ &= \sum_{\mathbf{i} \in \mathcal{M}(m, n)} |\mathbf{i}|^{\frac{-p}{mp+p-2m}} \left(\frac{|c[\mathbf{i}]|}{|\mathbf{i}|^{\frac{1}{q}}} \right)^{\frac{2mp}{mp+p-2m}} \\ &\leq \sum_{\mathbf{i} \in \mathcal{M}(m, n)} \left(\frac{|c[\mathbf{i}]|}{|\mathbf{i}|^{\frac{1}{q}}} \right)^{\frac{2mp}{mp+p-2m}} \end{aligned}$$

Using Lemma 5.1 with $s_k = \frac{2kp}{kp+p-2k}$ and $q = \frac{2p}{p-2}$, we get

$$\begin{aligned} &\left(\sum_{|\alpha|=m} |a_\alpha|^{\frac{2mp}{mp+p-2m}} \right)^{\frac{mp+p-2m}{2mp}} \\ &\leq \left[\prod_{S \in \mathcal{P}_k} \left(\sum_{\mathbf{i}_S \in \mathcal{M}(k, n)} \left(\sum_{\mathbf{i}_{\hat{S}} \in \mathcal{M}(m-k, n)} \left(\frac{|c[\mathbf{i}]|}{|\mathbf{i}|^{\frac{1}{q}}} \right)^q \right)^{\frac{s_k}{q}} \right)^{\frac{1}{s_k}} \right]^{\frac{1}{\binom{m}{k}}} \end{aligned}$$

Note that $|\mathbf{i}| \leq |\mathbf{i}_{\hat{S}}| \frac{m!}{(m-k)!}$, and thus

$$\begin{aligned} &\left(\sum_{|\alpha|=m} |a_\alpha|^{\frac{2mp}{mp+p-2m}} \right)^{\frac{mp+p-2m}{2mp}} \\ &\leq \left(\frac{m!}{(m-k)!} \right)^{\frac{q-1}{q}} \left[\prod_{S \in \mathcal{P}_k} \left(\sum_{\mathbf{i}_S \in \mathcal{M}(k, n)} \left(\sum_{\mathbf{i}_{\hat{S}} \in \mathcal{M}(m-k, n)} \frac{|c[\mathbf{i}]|^q}{|\mathbf{i}|} \left(\frac{|\mathbf{i}_{\hat{S}}|}{|\mathbf{i}|} \right)^{q-1} \right)^{\frac{s_k}{q}} \right)^{\frac{1}{s_k}} \right]^{\frac{1}{\binom{m}{k}}} \\ &= \left(\frac{m!}{(m-k)!} \right)^{\frac{q-1}{q}} \left[\prod_{S \in \mathcal{P}_k} \left(\sum_{\mathbf{i}_S \in \mathcal{M}(k, n)} \left(\sum_{\mathbf{i}_{\hat{S}} \in \mathcal{M}(m-k, n)} \frac{|c[\mathbf{i}]|^q}{|\mathbf{i}|^q} |\mathbf{i}_{\hat{S}}|^{q-1} \right)^{\frac{s_k}{q}} \right)^{\frac{1}{s_k}} \right]^{\frac{1}{\binom{m}{k}}} \end{aligned}$$

Let us fix $S \in \mathcal{P}_k(m)$. There is no loss of generality in supposing $S = \{1, \dots, k\}$. We then fix some $\mathbf{i}_S \in \mathcal{M}(k, n)$ and we introduce the following $(m-k)$ -homogeneous polynomial on ℓ_p^n :

$$P_{\mathbf{i}_S}(\mathbf{z}) = L(e_{i_1}, \dots, e_{i_k}, \mathbf{z}, \dots, \mathbf{z}).$$

Observe that

$$\begin{aligned} P_{\mathbf{i}_S}(\mathbf{z}) &= \sum_{\mathbf{i}_S \in \mathcal{M}(m-k, n)} \frac{c[\mathbf{i}]}{|\mathbf{i}|} z_{\mathbf{i}_S} \\ &= \sum_{\mathbf{i}_S \in \mathcal{J}(m-k, n)} \frac{c[\mathbf{i}]}{|\mathbf{i}|} |\mathbf{i}_S| z_{\mathbf{i}_S} \end{aligned}$$

so that

$$\|P_{\mathbf{i}_S}(\mathbf{z})\|_q = \left(\sum_{\mathbf{i}_S \in \mathcal{J}(m-k, n)} \frac{|c[\mathbf{i}]|^q}{|\mathbf{i}|^q} |\mathbf{i}_S|^q \right)^{\frac{1}{q}} = \left(\sum_{\mathbf{i}_S \in \mathcal{M}(m-k, n)} \frac{|c[\mathbf{i}]|^q}{|\mathbf{i}|^q} |\mathbf{i}_S|^{q-1} \right)^{\frac{1}{q}}.$$

By the definition of p_0 we have

$$\|P_{\mathbf{i}_S}(\mathbf{z})\|_q^{s_k} \leq K_{s_k, p}^{(m-k)s_k} \int_{S_{\ell_p^n}} |L(e_{i_1}, \dots, e_{i_k}, \mathbf{z}, \dots, \mathbf{z})|^{s_k} d\mu^n(\mathbf{z}).$$

Thus,

$$\begin{aligned} & \sum_{\mathbf{i}_S} \left(\sum_{\mathbf{i}_S} \frac{|c[\mathbf{i}]|^q}{|\mathbf{i}|^q} |\mathbf{i}_S|^{q-1} \right)^{\frac{1}{q} \times s_k} \\ & \leq K_{s_k, p}^{(m-k)s_k} \int_{S_{\ell_p^n}} \sum_{\mathbf{i}_S} |L(e_{i_1}, \dots, e_{i_k}, \mathbf{z}, \dots, \mathbf{z})|^{s_k} d\mu^n(\mathbf{z}). \end{aligned}$$

Now fixing $\mathbf{z} \in S_{\ell_p^n}$ we apply the multilinear Hardy-Littlewood inequality to the k -linear form

$$\left(\mathbf{z}^{(1)}, \dots, \mathbf{z}^{(k)} \right) \mapsto L \left(\mathbf{z}^{(1)}, \dots, \mathbf{z}^{(k)}, \mathbf{z}, \dots, \mathbf{z} \right)$$

and we obtain, from [3, Theorem 1.1] and Lemma 5.3,

$$\begin{aligned} & \sum_{\mathbf{i}_S} |L(e_{i_1}, \dots, e_{i_k}, \mathbf{z}, \dots, \mathbf{z})|^{s_k} \\ & \leq \left(\left(\frac{2}{\sqrt{\pi}} \right)^{\frac{2k(k-1)}{p}} \cdot (B_{\mathbb{C}, k}^{\text{mult}})^{\frac{p-2k}{p}} \cdot \sup_{\mathbf{z}^{(1)}, \dots, \mathbf{z}^{(k)} \in S_{\ell_p^n}} \left| L \left(\mathbf{z}^{(1)}, \dots, \mathbf{z}^{(k)}, \mathbf{z}, \dots, \mathbf{z} \right) \right| \right)^{s_k} \\ & \leq \left(\left(\frac{2}{\sqrt{\pi}} \right)^{\frac{2k(k-1)}{p}} \cdot (B_{\mathbb{C}, k}^{\text{mult}})^{\frac{p-2k}{p}} \cdot \frac{(m-k)! \cdot m^m}{(m-k)^{m-k} \cdot m!} \|P\| \right)^{s_k}. \end{aligned}$$

Thus

$$\begin{aligned} & \left(\sum_{|\alpha|=m} |a_\alpha|^{\frac{2mp}{mp+p-2m}} \right)^{\frac{mp+p-2m}{2mp}} \\ & \leq \left(\frac{m!}{(m-k)!} \right)^{\frac{q-1}{q}} \cdot K_{s_k, p}^{m-k} \cdot \frac{(m-k)! \cdot m^m}{(m-k)^{m-k} \cdot m!} \cdot \left(\frac{2}{\sqrt{\pi}} \right)^{\frac{2k(k-1)}{p}} \cdot (B_{\mathbb{C}, k}^{\text{mult}})^{\frac{p-2k}{p}} \|P\|. \end{aligned}$$

□

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