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Twisting, type-N vacuum gravitational fields with symmetries

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The Einstein field equations for twisting, type-N fields in empty space possessing two noncommuting Killing vectors are reduced to a single second-order ordinary differential equation for a complex function. Alternative forms of this basic equation are also presented; in particular, an appropriate Legendre transform provides a partial linearization, leading to a single real, nonlinear, third-order ordinary differential equation.

The problem of finding exact solutions of Einstein's field equations in empty space, representing gravitational radiation from a bounded source (i.e., with appropriate asymptotic behavior), has great physical relevance. The interest in such solutions is twofold: Not only could they have direct physical significance, but they could also provide a means of checking different features of computer codes used in numerical studies of gravitational radiation. Gravitational fields of algebraically degenerate type N in vacuum,[1] which are relevant for the treatment of gravitational radiation far from the sources, have been described completely in the nontwisting case[2-4] (the "twist" is the imaginary part of one of the Newman-Penrose spin coefficients,[5] and represents a searchlightlike variation of the propagation vector[6]). Unfortunately, type-N fields with no twist are not suitable for representing realistic spherical radiation.[6] On the other hand, the field equations in the nonvanishing-twist case are quite involved, being third-order partial differential equations with additional differential constraints, and as a consequence we have a single example of a solution of this type given by Hauser.[7,8] This solution, however, does not exhibit the appropriate asymptotic features.[9]

Hauser's solution possesses a Killing vector; it seems then natural to look for a solution with the maximum possible symmetry as a means of simplifying the field equations. Collinson has shown that the maximum possible number of independent Killing vectors in the twisting, type-N vacuum case is two, in which case the corresponding Lie algebra of isometries has to be non-Abelian.[10] The Killing fields ξ_1 and ξ_2 can be taken in this case so that they satisfy the relation

$$[\xi_1,\xi_2]=\xi_1 , \tag{1}$$

where the bracket is the Lie brackets for vector fields.

In this Rapid Communication, I show how the equations for twisting, type-N vacuum fields possessing two Killing vectors can be considerably reduced, resulting in a *single, second-order* ordinary differential equation for a complex function. For convenience, the vacuum Einstein equations for type-N fields will be formulated here as the following system of matrix-valued differential forms defined on spacetime:[11,12]

$$d\eta-\gamma\wedge\eta+\eta\wedge\gamma^{\dagger}=0 , \tag{2}$$

$$R\wedge\eta=0 , \tag{3}$$

$$pR=0 , \tag{4}$$

$$p\gamma\wedge R=0 , \tag{5}$$

where η is a Hermitian 2×2 matrix of one-forms, representing a null tetrad, and γ is a complex, traceless 2×2 matrix of one-forms (spin connection); the wedge denotes an exterior product, and d exterior differentiation. The curvature R is defined as $R=d\gamma-\gamma\wedge\gamma$, the dagger denotes Hermitian conjugation, and $p=\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$. Equations (2)-(5) are closed under exterior differentiation (i.e., all integrability conditions are taken into account); Eq. (2) expresses the fact that the torsion vanishes, while Eq. (3) comprises the Ricci-flat condition and the integrability condition for Eq. (2). Equation (4) adds the condition [in a particular SL(2,C) gauge] that the field be of type N, while (5) is the integrability condition for Eq. (4). Fields that satisfy (2)-(5) and possess two Killing fields ξ_1 and ξ_2 [which satisfy Eq. (1)] must also satisfy the equations[13]

$$\pounds_{\xi_1}\eta=\chi_1\eta+\eta\chi_1^{\dagger} , \tag{6}$$

$$\pounds_{\xi_2}\eta=\chi_2\eta+\eta\chi_2^{\dagger} , \tag{7}$$

$$\pounds_{\xi_1}\chi_2-\pounds_{\xi_2}\chi_1=[\chi_1,\chi_2]+\chi_1 , \tag{8}$$

where $\pounds$ denotes the Lie derivative, and χ_1 and χ_2 are sl(2,C)-valued functions [Eqs. (6)-(8) are the tetrad equivalent of the Killing equations for the metric[13]]. In order to relate the present form of the equations with the notation for the tetrad one-forms used previously by Hauser,[7,8] η can be written explicitly as

$$\eta=\begin{pmatrix}-k & t\\ \bar{t} & m\end{pmatrix} ,$$

where k and m are real one-forms, while t is a complex one-form, and bars will denote complex conjugation (the notation for the tetrad is related to the Newman-Penrose notation through $k\rightarrow -l$, $m\rightarrow n$, and $t\rightarrow m$). The principal, repeated null eigenform corresponding to the type-

N character of the space will be k in the gauge determined by Eq. (4). The nonvanishing-twist condition can be expressed as

$$dk\wedge k\neq 0 \ .$$

Following Hauser, part of the gauge freedom that remains after Eq. (4) is imposed can be used in order to simplify the form of γ. In spacetime coordinates $\{u,\sigma,\zeta,\bar\zeta\}$ (where u and σ are real, while ζ is complex), γ can be brought to the form

$$\gamma=\begin{pmatrix}0 & d\zeta\\ \Phi d\zeta & 0\end{pmatrix}, \tag{9}$$

where Φ is a complex function. By manipulating Eqs. (2)-(8), it can be shown[13] that the Killing fields can be expressed [within the gauge where Eq. (9) holds] in the form

$$\xi_1=i\frac{\partial}{\partial\zeta}-i\frac{\partial}{\partial\bar\zeta},\quad \xi_2=\zeta\frac{\partial}{\partial\zeta}+\bar\zeta\frac{\partial}{\partial\bar\zeta} \tag{10}$$

(the auxiliary matrices χ_i are not needed in the following; their explicit form is given in Ref. 13). The tetrad one-forms k, m, t, and $\bar t$ can be determined according to (2)-(8) with (10). They read

$$k=-(\zeta+\bar\zeta)du-Dd\zeta-\bar D d\bar\zeta \ ,$$
$$m=(\zeta+\bar\zeta)^{-1}d\sigma+(-\sigma+F\bar D-M)(\zeta+\bar\zeta)^{-2}d\zeta$$
$$+(-\sigma+\bar F D-\bar M)(\zeta+\bar\zeta)^{-2}d\bar\zeta \ ,$$
$$t=(1-\bar D_u)du-\sigma(\zeta+\bar\zeta)^{-1}d\zeta+\bar M(\zeta+\bar\zeta)^{-1}d\bar\zeta \ ,$$

where F, D, and M are complex functions of the variable u only, and a subscript denotes differentiation with respect to the corresponding variable. The field and symmetry equations (2)-(8) reduce now to the following final equations for F, D, and M:

$$M_u=-F \ , \tag{11}$$
$$F_u=-2F\bar D^{-1} \ , \tag{12}$$
$$D_u=1-D^{-1}M \ , \tag{13}$$
$$2M-2\bar M-2F\bar D+2\bar F D+F\bar M-\bar F M=0 \ . \tag{14}$$

Equations (11)-(14) can be used to derive a third-order ordinary differential equation for one of the complex functions (e.g., F). Equation (14) is then seen to be a (compatible) second-order differential constraint on that particular function. Similar sets, consisting of a complex, third-order equation coupled to a real, second-order equation for the same complex function have been considered previously;[13-16] in the following, however, an alternative approach which leads to a single second-order equation will be used.

I define a coordinate change $u\to v=v(u)$ for the independent variable u by means of the following indefinite integral:

$$v(u)=\int F(u)\bar F(u)du \ . \tag{15}$$

It is readily seen that

$$(2F\bar D-2M-F\bar M)'=1 \tag{16}$$

(primes will denote derivatives with respect to v). From (16) we get $2F\bar D-2M-F\bar M=v+v_0$, where the integration constant v_0 is real by (14), and can be made to vanish by appropriately choosing the constant of integration in (15); in the following, this choice will be assumed. Equations (11)-(14) reduce now to

$$M'=-\bar F^{-1} \ , \tag{17}$$
$$F'=-2\bar F^{-1}\bar D^{-1} \ , \tag{18}$$
$$D'=(D-M)D^{-1}F^{-1}\bar F^{-1} \ , \tag{19}$$
$$2F\bar D-2M-F\bar M=v \ . \tag{20}$$

Finally, it is easy to check that Eqs. (17)-(20) can be used to explicitly solve for D and F as functions of M, $\bar M$, M', and $\bar M'$; the remaining function M is forced to satisfy a single, second-order ordinary differential equation. Introducing $\omega=2M$ for convenience, the resulting equation is

$$\omega''=\frac{2\omega'^2\bar\omega'}{\omega-\bar\omega\omega'-v\omega'} \ . \tag{21}$$

The problem of finding twisting, type-N fields with two Killing vectors reduces to one of finding solutions of Eq. (21) with nonvanishing twist. For completeness, the tetrad and other relevant quantities are given below in terms of ω, $\bar\omega$, ω', and $\bar\omega'$ $(\Delta\equiv\omega-\bar\omega\omega'-v\omega')$:

$$k=-\tfrac14\omega'\bar\omega'(\zeta+\bar\zeta)dv-\tfrac14\Delta d\zeta-\tfrac14\bar\Delta d\bar\zeta \ ,$$
$$m=(\zeta+\bar\zeta)^{-1}d\sigma+\tfrac12[v-\bar\omega(\bar\omega')^{-1}-2\sigma](\zeta+\bar\zeta)^{-2}d\zeta$$
$$+\tfrac12[v-\omega(\omega')^{-1}-2\sigma](\zeta+\bar\zeta)^{-2}d\bar\zeta \ ,$$
$$t=\tfrac12\bar\omega\omega'\bar\omega'\bar\Delta^{-1}dv-\sigma(\zeta+\bar\zeta)^{-1}d\zeta+\tfrac12\bar\omega(\zeta+\bar\zeta)^{-1}d\bar\zeta \ ,$$
$$dk\wedge k=\tfrac18\omega'\bar\omega'\Delta^{-1}\bar\Delta^{-1}(\bar\omega\Delta^2-\omega\bar\Delta^2)dv\wedge d\zeta\wedge d\bar\zeta \ ,$$
$$\Phi=-2(\zeta+\bar\zeta)^{-2}(\bar\omega')^{-1} \ .$$

The curvature two-form R has the expression

$$R=\begin{pmatrix}0 & 0\\ \Psi k\wedge t & 0\end{pmatrix},$$

with

$$\Psi=32(\zeta+\bar\zeta)^{-2}(\bar\omega')^{-1}(\bar\omega\Delta+2\sigma\bar\Delta)^{-1} \ .$$

No explicit exact, closed-form solutions of Eq. (21) of a nontrivial nature (i.e., with nonvanishing twist) are known at present. It is possible, however, to add the following remarks, which may help in the search for solutions: In the first place, the main equation (21) can be transformed into one not explicitly involving the independent variable, by defining $z=v^{-1}\omega$ and taking $\ln v$ as the new independent variable. By denoting by a dot the derivative with respect to the latter, the resulting equation reads

$$\ddot z+\dot z+\frac{2(z+\dot z)^2(\bar z+\dot{\bar z})}{z\bar z+\bar z\dot z+\dot z}=0 \ . \tag{22}$$

Secondly, one notices that when $\omega = 2f$ (f being a real function; this case gives zero twist), Eq. (21) can be *linearized* by means of the Legendre transform

$$h \equiv f - vf', \quad x \equiv f', \quad v = -h_x \ , \qquad (23)$$

which brings (21) over into the equation

$$h_{xx} + \tfrac{1}{2} x^{-1} h_x + \tfrac{1}{4} x^{-3}(1-2x)h = 0 \ .$$

Guided by this linearization, one can proceed in the following way: By expressing ω and ω' in terms of their real and imaginary parts, according to $\omega = 2f + 2ig$, $\omega' = 2x + 2iy$ (f, g, x, and y real), Legendre-transformed variables h and x are defined as in (23); Eq. (21) is then equivalent to the following set of equations, *linear* in g and h:

$$g_x + y h_{xx} = 0 \ , \qquad (24)$$

$$Ag + Bh + Lh_x + Qh_{xx} = 0 \ , \qquad (25)$$

$$ag + bh + lh_x + qh_{xx} = 0 \ , \qquad (26)$$

where the coefficients A, B, L, Q, a, b, l, and q are real functions of x, y, and y_x. By applying the compatibility conditions for the system (24)–(26), it is found that the problem reduces to a single, real, ordinary differential equation of the third order for the real function $y(x)$:

$$\mathcal{F}(x, y, y_x, y_{xx}, y_{xxx}) = 0 \ . \qquad (27)$$

Once Eq. (27) is solved, h is given by a quadrature, and g is given in finite terms. Unfortunately, (27) is a complicated equation with polynomial nonlinearities; its length prevents its reproduction here in full. Details of the derivation of Eq. (27) and of its properties will be published elsewhere. Further work is being done in searching for closed-form solutions of Eqs. (21) and (22).

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[1]For a description of the algebraic classification of gravitational fields, and references to the original papers, see D. Kramer, H. Stephani, E. Herlt, and M. MacCallum, *Exact Solutions of Einstein's Field Equations* (Cambridge Univ. Press, Cambridge, 1980).

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