MEASURING THE RATIONALITY OF A FUZZY PREFERENCE RELATION

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Abstract

In this paper we deal with fuzzy preference relations and its rationality, which is conceived as a fuzzy property. A measure of this "rationality" is proposed, and some results are given.

Key words: Acyclity, Fuzzy Preference relation, Rationality.

Introduction

Let us suppose an individual who must define his preferences over a finite set X of alternatives. Such preferences may be of a fuzzy nature, and we can suppose that such an individual is able to define a "fuzzy opinion":

DEFINITION 1 .- A "Fuzzy Opinion" is a fuzzy preference relation

$$\mu : X \times X \longrightarrow [0,1]$$
$$(x,y) \longrightarrow \mu(x,y)$$

such that

$$\mu(x,y) + \mu(y,x) \geq 1 \qquad \forall\, x, y \in X$$

in such a way that $\mu(x,y)$ represents the degree in which alternative x is not worse (x P y) than alternative y.

On the one hand, a Fuzzy Opinion μ can be viewed as an "outranking" relation in the sense of Roy (1977), in such a way that

$$\mu^I(x,y) = \mu(x,y) + \mu(y,x) - 1$$

represents the degree of "indifference" (x I y) between both alternatives ($\mu^I(x,y) = \mu^I(y,x)$) and

$$\mu^S(x,y) = 1 - \mu(y,x)$$

represents the degree of "strict preference" (x S y) of alternative x over alternative y, in such a way that

$$\mu^S(x,y) + \mu^I(x,y) + \mu^S(y,x) = 1$$

On the other hand, since last property can be viewed as an orthogonality condition, a Fuzzy Opinion defines a "Fuzzy Partition" (Ruspini, 1969) of the cartesian product X x X: the family of three fuzzy sets with μ^S, μ^I and μ^{-S} ($\mu^{-S}(x,y) = \mu^S(y,x) \quad \forall\, x, y \in X$) as their respective membership functions.

The Concept of Acyclity

One can ask when a given fuzzy opinion can be considered as "rational". Classical works on fuzzy preference relation propose conditions like "reflexivity" ($\mu(x,x) = 1 \quad \forall\, x \in X$, due to Zadeh, 1971) or any type of fuzzy transitivity (see, for example, the book of Dubois-Prade, 1980). Max-min transitivity ($\mu(x,y) \geq \min\{\mu(x,z), \mu(z,y)\} \quad \forall\, x,y,z \in X$, proposed by Zadeh, 1971) is the usual condition of transitivity. The idea lying behind it is that the shorter the chain, the stronger the relation, in such a way that the strength of the link between two elements must be greater than or equal to the strength of any indirect chain. Though reflexivity and max-min transitivity can be justified in order to decision-making (Montero-Tejada, unpublished paper), they are not real conditions for being rational, since the set of fuzzy relations verifying each property has well-defined boundaries: intuitively we see that there are fuzzy relations with more or less rationality, so that it seems natural to consider "rationality" as a fuzzy property.

One way for measuring the rationality of a fuzzy opinion, based on classical concept of "acyclity" is the following: consider the set of alternatives $X = \{x_1, x_2, x_3\}$ and let

$$A(\{x_i\}) = \{x_i \; I \; x_i\} \qquad i = 1,2,3$$

$$A(\{x_i, x_j\}) = \{x_i \; S \; x_j \; , \; x_i \; I \; x_j \; , \; x_j \; S \; x_i\} \qquad \forall \; i \neq j$$

$$A(X) = \{x_1 \; I \; x_2 \; I \; x_3, \; x_1 \; S \; x_2 \; S \; x_3, \; x_1 \; S \; x_2 \; I \; x_3, \; x_1 \; I \; x_2 \; S \; x_3$$
$$x_1 \; S \; x_3 \; S \; x_2, \; x_1 \; I \; x_3 \; S \; x_2, \; x_2 \; S \; x_3 \; S \; x_1,$$
$$x_2 \; S \; x_3 \; I \; x_1, \; x_2 \; I \; x_3 \; S \; x_1, \; x_2 \; S \; x_1 \; S \; x_3,$$
$$x_3 \; S \; x_1 \; S \; x_2, \; x_3 \; S \; x_1 \; I \; x_2, \; x_3 \; S \; x_2 \; S \; x_1\}$$

be the sets of acyclic paths with groups of one, two and three alternatives. Given a fuzzy opinion defined over X, it seems natural to define the weight W of each acyclic path as follows:

$$W(x_i \; I \; x_i) = \mu^I(x_i, x_i) \qquad i = 1,2,3$$

$$W(x_i \; R \; x_j \; (-R) \; x_i) = \mu^R(x_i, x_j)^2 \qquad \forall \; i \neq j$$

where R represents any basic relation S, I or $-S$ $(-(-S) = S, -I = I)$, and

$$W(x_1 \; R_1 \; x_2 \; R_2 \; x_3 \; R_3 \; x_1) = \mu^{R_1}(x_i, x_j) \cdot \mu^{R_2}(x_j, x_k) \cdot \mu^{R_3}(x_k, x_i)$$

where each R_i represents the appropiate relation S, I or $-S$, in such a way that the considered path is acyclic (for example, if $R_1 = I$ and $R_2 = I$ it must be $R_3 = I$; if $R_1 = S$ and $R_2 = S$, it must be $R_3 = -S$). Therefore, we can define

$$A_\mu(G) = \sum_{c \in A(G)} W(c)$$

as a measure of acylity in the path G of alternatives, and

$$A(\mu) = \min_{G \subset X} A_\mu(G)$$

as a measure of acyclity of the fuzzy opinion μ.

Now we can propose a general definition:

DEFINITION 2 .- Let μ be a fuzzy opinion defined over a finite set of alternatives X, and let A(G) be the set of acyclic paths with lenght card (G) concerning all alternatives in $G \subset X$.

We will call "acyclity" of μ to the value

$$A(\mu) = \min_{G \subset X} A_\mu(G)$$

where $A_\mu(G)$ is trivially defined as above, from

$$W(c) = \prod_i \mu^{R(c^i)}(x_i, x_{i+1}) \qquad \forall \; c \in A(G)$$

with $R(c^i)$ the appropiate relation between x_i and x_{i+1} for the given path $x_1 \; R(c^1) \; x_2 \; R(c^2) \; \ldots \; x_k \; R(c^k) \; x_1$ through G.

THEOREM 1 .- Let $G = \{x_1, \ldots, x_k\}$ be a path of alternatives. Then

$$A_\mu(G) = 1 - \left\{ \prod_{i=1}^{k} \mu(x_i, x_{i+1}) + \prod_{i=1}^{k} \mu(x_{i+1}, x_i) - 2 \cdot \prod_{i=1}^{k} \mu^I(x_i, x_{i+1}) \right\}$$

with $x_{k+1} = x_1$.

Proof: on the one hand, since

$$1 = \prod_i (\mu^S(x_i, x_{i+1}) + \mu^I(x_i, x_{i+1}) + \mu^{-S}(x_i, x_{i+1})) \; .$$

$$= \sum_{c \in A(G)} \prod_i \mu^{R(c^i)}(x_i, x_{i+1}) + \sum_{c \notin A(G)} \prod_i \mu^{R(c^i)}(x_i, x_{i+1})$$

we get

$$A_\mu(G) = 1 - \sum_{c \in A(G)} \prod_i \mu^{R(c^i)}(x_i, x_{i+1})$$

where

$$A'_\mu(G) = \sum_{c \notin A(G)} \prod_i \mu^{R(c^i)}(x_i, x_{i+1})$$

represents a measure of non-acyclity in G.

On the order hand, it is clear that any given path

$$x_1 \; R(c^1) \; x_2 \ldots R(c^{k-1}) \; x_k \; R(c^k) \; x_1$$

is in fact non - acyclic if and only if it is not the path of indifferences ($R(c^i) = I \;\; \forall i$) and

$$\exists \; R(c^i) = S \quad , \quad \exists \; R(c^j) = -S$$

do not hold simultaneously. Therefore,

$$A'_\mu(G) = \prod_i \mu(x_i, x_{i+1}) + \prod_i \mu(x_{i+1}, x_i) - 2 \cdot \prod_i \mu^I(x_i, x_{i+1})$$

and the theorem follows immediately.

Moreover, it follows that

$$0 \leq A_\mu(G) \leq 1$$

in all cases, and therefore

$$0 \leq A(\mu) \leq 1$$

in such a way that from now on we can talk about "acyclity" as a fuzzy property

$$A : F(X) \longrightarrow [0,1]$$

according to definition 2.

THEOREM 2. - Let us suppose $\mu \in F(X)$ a non-fuzzy opinion (in other words, $\mu(x_i, x_j) \in \{0,1\} \;\; \forall i,j$). Then μ is acyclic if and only if $A(\mu) = 1$.

Proof: trivial, since $A_\mu(G) = 1$ for each G if and only if all paths in G are acyclic.

Moreover, we can observe that

$$W(c) \in \{0,1\} \qquad \forall c \in A(G)$$

for any given acyclic non-fuzzy relation, with only one path inside A(G) such that $W(c) = 1$, and $A(\mu) = 0$ when μ is non-acyclic.

<u>Example</u>

Let us consider $X = \{x_1, x_2, x_3\}$ and the fuzzy opinion μ defined as follows:

$\mu(x_1, x_2) = 0.6$ $\qquad$ $\mu(x_2, x_1) = 0.9$

$\mu(x_2, x_3) = 0.5$ $\qquad$ $\mu(x_3, x_2) = 0.8$

$\mu(x_3, x_1) = 0.4$ $\qquad$ $\mu(x_1, x_3) = 0.7$

and $\mu(x_i, x_i) = 1 \quad \forall i = 1,2,3$. Hence,

$$\mu^{I}(x_1, x_2) = 0.6 + 0.9 - 1 = .5$$

$$\mu^{I}(x_2, x_3) = 0.5 + 0.8 - 1 = 0.3$$

$$\mu^{I}(x_3, x_1) = 0.4 + 0.7 - 1 = 0.1$$

and $\mu^{I}(x_i, x_i) = 1 \quad \forall i$, and we get

$$A_\mu(\{x_i\}) = 1 \qquad \forall\, i = 1,2,3$$

$$A_\mu(\{x_1, x_2\}) = \mu^{S}(x_1,x_2)^2 + \mu^{I}(x_1,x_2)^2 + \mu^{S}(x_2, x_1)^2$$

$$= (0.6 - 0.5)^2 + 0.5^2 + (0.9 - 0.5)^2 =$$

$$= 0.42$$

$$A_\mu(\{x_2, x_3\}) = \mu^{S}(x_2, x_3)^2 + \mu^{I}(x_2,x_3)^2 + \mu^{S}(x_3, x_2)^2 =$$

$$= (0.5 - 0.3)^2 + 0.3^2 + (0.8 - 0.3)^2 =$$

$$= 0.38$$

$$A_\mu(\{x_3, x_1\}) = \mu^{S}(x_3,x_1)^2 + \mu^{I}(x_3,x_1) + \mu^{S}(x_1, x_3)^2$$

$$= (0.4 - 0.1)^2 + 0.1^2 + (0.7 - 0.1)^2 =$$

$$= 0.46$$

$$A_\mu(\{x_1, x_2, x_3\}) = 1 - \{0.6 \cdot 0.5 \cdot 0.4 + 0.9 \cdot 0.8 \cdot 0.7 -$$

$$- 2 \cdot 0.5 \cdot 0.3 \cdot 0.1\} = 0.406$$

and therefore,

$$A(\mu) = 0.38$$

which means that the pair $\{x_2, x_3\}$ is the group of alternatives with the lowest acyclity.

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