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SOCIAL WELFARE FUNCTIONS IN A FUZZY ENVIRONMENT

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In this paper, fuzzy preference relations are considered and the concept of "fuzzy acyclity" is defined in order to measure their rationality. Such a concept is applied to study the properties of social welfare function in a fuzzy environment. In particular, the existence of non-irrational aggregation rules is assured, in such a way that they can be used as a group decision aid.

INTRODUCTION

The axiomatic approach to Social Welfare Functions proposed by Arrow[1] shows that some desirable properties for social choice are incompatible. Such a result is known as the impossibility Theorem[2] and, in spite of some possibility results, Arrow's negative philosophy remains valid.[3]

In his classical work, Arrow considered the aggregation problem of preferences in a group. It is assumed that each individual defines a preference ordering over a given set of actions, and the problem is how to define Social Welfare Functions assigning a group preference ordering to each conceivable individual preference orderings. For example, formalization of the following conditions leads us to contradiction in non-trivial cases[1], as follows:

(1) Unrestricted Domain: a social welfare function is defined for every possible set of individual orderings.

(2) Non-negative Response: social ordering responds positively—or at least not negatively—to alterations in individual values.

(3) Independence of Irrelevant alternatives: social ordering from any given set of alternatives depends only on the orderings of individuals among those alternatives.

(4) Citizen Sovereign: any possible ordering can be reached by varying individual orderings.

(5) Non-Dictatorship: there is no individual such that social ordering is always his individual ordering, independently of other individuals.

These ethical conditions seem valid at first sight, and many efforts have been done to discuss where does the difficult lie. We will prove that such a conflict lies in the previous concept of rationality: each individual—or group—is considered "rational" if and only if its preference relation R is an ordering relation (a complete, reflexive and transitive relation). It is well known that one way to overcome Arrow's Impossibility Theorem is to drop the requirement of transitiveness. For example, if collective preference is supposed to be quasi-transitive (strict preferences are transitive) an oligarchy emerges; if it is only acyclic, oligarchy can be avoided, but decision rules are poorly decisive.

The aim of this paper is to study the concept of Social Welfare Functions when preferences are stablished as Fuzzy Preference Relations. We must remember that Fuzzy Set Theory was introduced by Zadeh[4] in order to generalize classical Set Theory. A Fuzzy Set is defined through a "membership" function over a given family of objects, $p: x \to [0, 1]$, in such a way that $p(x)$ means the degree in which element x belongs to such a fuzzy set. When $p(x) \in \{0, 1\}$ for all x in X, such a set is called "crisp" set or unfuzzy set, and $p: X \to \{0, 1\}$ is the characteristic function of some classical set $A \subset X (x \in A$ if and only if $p(x) = 1)$.

Fuzzy Binary Relations in a given family of objects X are defined through a membership function $p: X \times X \to [0, 1]$. We will call "Fuzzy Preference Relation" any fuzzy binary relation verifying $p(x, y) + p(y, x) \geqslant 1$ for all $x, y \in X$. Such a fuzzy binary relation is called "complete" by Basu[5] and "connected" by Dutta *et al.*[6] Each value $p(x, y)$ will represent the degree in which alternative x is not worst than alternative y. From now on, we will assume that each individual defines a fuzzy preference relation over the given set of alternatives, in such a way that the problem is how to define a group fuzzy preference relation for any conceivable individual fuzzy preference relations.

FUZZY ACYCLITY

The concept of rationality has traditionally been explained in terms of an acyclic binary preference.[3] In a few words, an acyclic relation is a complete binary relation such that there is no sequence $x_1, x_2, \ldots, x_k$ (a "cycle", being $x_{k+1} = x_1$) with $x_j R x_{j+1} \forall j = 1, \ldots, k$. In this section we will translate the idea of acyclity to fuzzy preference relations in order to measure their rationality. In this way, it will be proposed an alternative definition of fuzzy rationality to that of Basu.[5]

Let $p: X \times X \to [0, 1]$ be fuzzy preference relation over a family of alternatives X, in such a way that each value $p_I(x, y) = p_I(y, x) = p(x, y) + p(y, x) - 1$ can be viewed as the degree of indifference (xIy) between both alternatives, and $p_S(x, y) = p(x, y) - p_I(x, y) = 1 - p(y, x)$ means the degree of verification of strict preference (xSy). We will denote $p_R(x, y) = p(x, y)$ the degree of weak preference (xRy), and it is clear that $p_S(x, y) + p_I(x, y) + pS(y, x) = 1$.

Let us consider a cycle $(x_1, x_2, \ldots, x_k)$ with k distinct elements in X, being $x_{k+1} = x_1$. Given any pair of consecutive alternatives, they can be connected by any basic relation: $x_j S x_{j+1}$, $x_j I x_{j+1}$ or $x_{j+1} S x_j$. And given a path $t = x_1 R_1 x_2 R_2 x_3 \ldots x_k R_k x_1$, it seems natural to define its weight

$$W(t) = \prod_{j=1}^{k} p_{R_j}(x_j, x_{j+1}).$$

It is clear that such a path is non-acyclic if and only if $x_j R x_{j+1} \forall_j$, or $x_{j+1} R x_j \forall_j$, with some strict preference. Therefore, we can define the value which adds the weights of all acyclic paths in a given cycle:

$$A_p(x_1, x_2, \ldots, x_k) = \sum_{(R_j) \in A} \prod_j p_{R_j}(x_j, x_{j+1})$$

where addition ranges over all acyclic paths $(R_1, \ldots, R_k)$ in the given cycle.

THEOREM 1.—Let $(x_1, \ldots, x_k)$ be cycle with k distinct elements. Then

$$A_p(x_1, \ldots, x_k) = 1 - \left[\prod_{j=1}^{k} p_R(x_j, x_{j+1}) + \prod_{j=1}^{k} p_R(x_{j+1}, x_j) - 2 \prod_{j=1}^{k} p_I(x_j, x_{j+1}) \right]$$

Proof: Let A' be the set of all non-acyclic paths in the given cycle. Since

$$1 = \prod_{j=1}^{k} (p_S(x_j, x_{j+1}) + p_I(x_j, x_{j+1}) + p_S(x_{j+1}, x_j))$$

$$= \sum_{(R_j) \in A} \prod_j p_{R_j}(x_j, x_{j+1}) + \sum_{(R_j) \in A'} \prod_j p_{R_j}(x_j, x_{j+1})$$

we obtain

$$\begin{aligned} A_p(x_1, \ldots, x_k) &= 1 - \sum_{(R_j) \in A'} \prod_j p_{R_j}(x_j, x_{j+1}) \\ &= 1 - \Big(\prod_j [p_S(x_j, x_{j+1}) + p_I(x_j, x_{j+1})] \\ &\quad + \prod_j [p_I(x_j, x_{j+1}) + p_S(x_{j+1}, x_j)] \\ &\quad - 2 \cdot \prod_j p_I(x_j, x_{j+1}) \Big) \end{aligned}$$

from the previous considerations about non acyclic paths. Moreover, it is clear that $A_p(x_1, \ldots, x_k) \in [0, 1]$.

For example, given a fuzzy preference p, we obtain

$$\begin{aligned} A_p(x) &= 1 - (W(xSx) + W(xTx)) \\ &= 1 - (p_S(x, y) + p_S(y, x)) \\ &= p_I(x, x) \\ &= W(xIx) \end{aligned}$$

$$\begin{aligned} A_p(x, y) = {} & 1 - (W(xSySx) + W(xSyIx) \\ & + W(xIySx) + W(xIyTx) \\ & + W(xTyIx) + W(xTyTx)) \\ = {} & 1 - 2(p_S(x, y) \cdot p_S(y, x) \\ & + p_S(x, y) \cdot p_I(y, x) + p_I(x, y) \cdot p_S(x, y)) \\ = {} & p_S^2(x, y) + p_I^2(x, y) + p_S^2(y, x) \\ = {} & W(xSyTx) + W(xIyIx) + W(xTySx) \end{aligned}$$

where xTy means ySx, and both elements are assumed as different. Analogously,

$$\begin{aligned} A_p(x, y, z) = {} & 1 - (p_R(x, y) \cdot p_R(y, z) \cdot p_R(z, x) \\ & + p_R(z, y) \cdot p_R(y, x) \cdot p_R(x, z) \\ & - 2 \cdot p_I(x, y) \cdot p_I(y, z) \cdot p_I(z, x)) \end{aligned}$$

for all distinct x, y, z.

Therefore, given any fuzzy preference relation, we can define its "Acyclity" as the value of the cycle with lowest degree of acyclity in the previous sense:

DEFINITION 1.—Let $\mathscr{R}(X)$ be the family of all fuzzy preference relations defined over X. The "acyclity" is a fuzzy property with membership function A: $\mathscr{R}(X) \to [0, 1]$ which assigns to each preference relation p the value

$$A(p) = \min_{(x_1, \ldots, x_k)} Ap(x_1, \ldots, x_k)$$

which minimum along all cycles in X with distinct elements.

THEOREM 2.—Let R be a complete binary relation on X and p its crisp membership function ($p(x, y) \in \{0, 1\} \forall x, y \in X$). Then R is an ordering relation if and only if $A(p) = 1$.

Proof: trivial, since reflexivity ($xIx \forall x \in X$) holds if and only if $A_p(x) = 1 \forall x \in X$, in such a way that reflexivity means that every cycle with one element is acyclic. Analogously, antisymmetry ($xRy, yRx \Rightarrow xIy$) refers to acyclity of cycles with two elements, and transitivity ($xRy, yRz \Rightarrow xRz$) means that every cycle with three elements is acyclic. Therefore, $A(p) \in \{0, 1\}$ for any given complete binary relation, and $A(p) = 1$ if and only if reflexivity, antisymmetry and transivity hold.

Analogous treatment can be applied to define quasi-acyclity as another fuzzy property on $\mathscr{R}(X)$. Moreover, other measures of acyclity, based on any aggregation operation distinct of minimization, can be proposed (for instance, it seems advisable to weight $A_p(x_1, \ldots, x_k)$ through the length k of such a cycle).

EXAMPLE

Let us consider $X = \{x, y, z\}$ and the fuzzy opinion p defined as follows:

$$p(x, y) = 0.7 \quad p(y, x) = 0.8$$
$$p(y, z) = 0.5 \quad p(z, y) = 0.6$$
$$p(z, x) = 0.4 \quad p(x, z) = 0.9$$

and

$$p(x, x) = p(y, y) = p(z, z) = 1.$$

Hence

$$p_I(x, y) = 0.7 + 0.8 - 1 = 0.5$$
$$p_I(y, z) = 0.5 + 0.6 - 1 = 0.1$$
$$p_I(z, x) = 0.4 + 0.9 - 1 = 0.3$$

and

$$p_I(x, x) = p_I(y, y) = p_I(z, z) = 1$$

Applying the above considerations, we obtain, for example,

$$\begin{aligned} W(xSx) &= p_S(x, x) = 0 \\ W(xSySx) &= p_S(x, y) \cdot p_S(y, x) \\ &= (0.7 - 0.5) \cdot (0.8 - 0.5) 0.06 \\ W(xIySzTx) &= p_I(x, y) \cdot p_S(y, z) \cdot p_S(x, z) \\ &= 0.5(0.5 - 0.1) \cdot (0.9 - 0.3) = 0.12 \end{aligned}$$

(xSx and $xSySx$ are non-acyclic paths, but $xIySzTx$ is an acyclic path) and moreover,

$$\begin{aligned} A_p(x) &= A_p(y) = A_p(z) = 1 \\ A_p(x, y) &= (0.7 - 0.5)^2 + 0.5^2 + (0.8 - 0.5)^2 \\ &= 0.38 \\ A_p(y, z) &= (0.5 - 0.1)^2 + 0.1^2 + (0.6 - 0.1)^2 \\ &= 0.42 \\ A_p(z, x) &= (0.4 - 0.3)^2 + 0.3^2 + (0.9 - 0.3)^2 \\ &= 0.46 \\ A_p(x, y, z) &= A_p(x, z, y) = 1 - (0.7.0.5.0.4 \\ &\quad + 0.8.0.9.0.6 - 2.0.5.0.1.0.3) \\ &= 0.458 \end{aligned}$$

Therefore, $A(p) = 0.38$ and the lowest acyclity is reached in the (x, y)-cycle.

EXISTENCE OF SOCIAL WELFARE FUNCTIONS

Following Arrow's formulation, we can define a fuzzy social welfare function as a process or rule which assigns a social fuzzy preference relation to each set of fuzzy preference relations (one fuzzy preference relation for each individual). Here we propose a family of fuzzy ethical conditions, analogous to those of Arrow, which seem valid to be assumed.

We will suppose a finite set of individuals $D = \{1, 2, \ldots, n\}$ and a finite set of alternatives X. Let $\mathscr{R}_0(X)$ be the family of all fuzzy preference relations on X with no-absolute irrationality, and $\mathscr{R}_0^n(X) = \mathscr{R}_0(X) x \ldots x \mathscr{R}_0(X)$ the Cartesian product (in other words, each individual i defines a fuzzy preference $p^i \in \mathscr{R}(X)$ such that $A(p^i) \neq 0$).

DEFINITION 2.—Any mapping

$$S: \mathscr{R}_0^n(X) \to \mathscr{R}_0(X)$$

is called a Social Welfare Function.

It must be pointed out that a condition of unrestricted domain is assumed in this definition, in the sense that if individual orderings p^i are not absolutely irrational, then the same must be true for social ordering $S(p^1, \ldots, p^n)$. In other words,

$$A(S(p^1, \ldots, p^n)) \neq 0 \quad \forall (p^1, \ldots, p^n) \in \mathscr{R}_0^n(X)$$

For example, we can define the following social welfare function $\bar{p}$:

$$\bar{p}(x, y) = \sum_{i=1}^{n} p^i(x, y)/n \quad \forall x, y \in X$$

It is clear that such a mapping defines a social fuzzy preference for any given family of individual fuzzy preferences, since

$$\bar{p}(x, y) \in [0, 1] \quad \bar{p}(x, y) + \bar{p}(y, x) \geq 1 \quad \forall x, y \in X$$

and $A(\bar{p}) \neq 0$ always is verified. In fact: Let us suppose a cycle $(x_1, \ldots, x_k)$. Then there exists an acyclic path $(R_j^i) \in A$ such that

$$\prod_j p_{R_j}(x_j, x_{j+1}) > 0 \text{ for any individual } i.$$

Therefore,

$$\sum_i p^i_{R_j}(x_j, x_{j+1}) > 0 \quad \forall j,$$

and

$$\prod_j \bar{p}_{R_j}(x_j, x_{j+1}) > 0$$

in such a way that

$$A_{\bar{p}}(x_1, \ldots, x_k) = \sum_{(R_j) \in A} \prod_j \bar{p}_{R_j}(x_j, x_{j+1}) > 0$$

Hence, $A(\bar{p}) \neq 0$ (since X is finite, the family of cycles in X is finite too).

Moreover, it is easy to prove that the social welfare function $\bar{p}$ verifies some ethical conditions, in such a way that it is assured the existence of non-irrational and ethical solutions for the problem of aggregation of preferences:

THEOREM 3.—Let us consider $S(p^1, \ldots, p^n) = \bar{p}$. Then the following properties are verified:

(1) Non-negative response: Let us suppose $(p^1, \ldots, p^n), (q^1, \ldots, q^n) \in \mathcal{R}_0^n(X)$ such that

$$p^i(x, y) \geq q^i(x, y) \quad \forall i \in D, \quad \forall x, y \in X$$

with some strict inequality. Then

$$\bar{p}(x, y) \geq \bar{q}(x, y) \quad \forall x, y \in X$$

(2) Independence of irrelevant alternatives: Given in a non-empty subset of alternatives Y,

$$p^i(x, y) = q^i(x, y) \quad \forall i \in D, \forall x, y \in Y$$
$$\Rightarrow \bar{p}(x, y) = \bar{q}(x, y) \quad \forall x, y \in Y$$

(3) Citizen sovereign: Given any $p \in \mathcal{R}_0(X)$, there exists $(p^1, \ldots, p^n) \in \mathcal{R}_0^n(X)$ such that $\bar{p}(x, y) = p(x, y)$ $\forall x, y \in X$.

(4) Non-dictatorship: There is no individual i such that $\bar{p}(x, y) = p^i(x, y)$ $\forall x, y \in X$, without taking into account the preferences of other individuals in D.

In this way the aggregation rule $\bar{p}$ defines social preferences relations with no-absolute irrationality, verifying a basic ethical condition of non-negative responsiveness and independence of irrelevant alternatives. A condition of unrestricted domain is verified by the definition of the Social Welfare Function, and citizen sovereign can be deduced from unanimity: if $p^i(x, y) = p(x, y)$ $\forall x, y \in X$ for every individual i, then $\bar{p}(x, y) = p(x, y)$ $\forall x\, y \in X$ holds. Moreover, it is clear there is no dictatorship—though dictatorship can be defined as a fuzzy property.[7]

In any case, classical Social Welfare Functions can be analyzed through definition 2, since classical preference relations on X are included in $\mathcal{R}_0(X)$. Moreover, classical ethical conditions can be deduced when the individual preferences are supposed crisp relations (for example, classical unanimity means that if $p^i(x, y) = p(x, y) \in \{0, 1\}$ $\forall x, y \in X$ for all individuals, then $S(p^1, \ldots, p^n) = p$). Therefore, Arrow's Theorem can be explained in this context as follows: There is no ethical social welfare function assigning social preference relations with absolute rationality. But we have proved that we can avoid absolute irrationality.

It must be pointed out that it is not easy to find social welfare functions in our sense. For example,

$$\hat{p}(x, y) = \max_i p^i(x, y) \quad \forall x, y \in X$$

defines a mapping onto $\mathcal{R}(X)$ such that

$$\hat{p}(x, y) \in [0, 1] \; \hat{p}(x, y) + \hat{p}(y, x) \geq 1 \quad \forall x, y \in X$$

and it verifies ethical condition of Theorem 3, but $A(\hat{p}) \neq 0$ is not assured, and therefore $\hat{p}$ is not a mapping onto $\mathcal{R}_0(X)$. For instance, given a group with two individuals and p^1, p^2 their respective preferences such that

$$p^1(x, y) = p^2(y, x) = 0.5$$
$$p^1(y, z) = p^2(z, y) = 1$$
$$p^1(z, x) = p^2(z, x) = 1$$
$$p^1(y, x) = p^2(x, y) = 1$$
$$p^1(z, y) = p^2(y, z) = 0.5$$
$$p^1(x, z) = p^2(x, z) = 0$$

then

$$\hat{p}(x, y) = 1 \quad \hat{p}(y, x) = 1$$
$$\hat{p}(y, z) = 1 \quad \hat{p}(z, y) = 1$$
$$\hat{p}(z, x) = 1 \quad \hat{p}(x, z) = 0$$

in such a way that $\hat{p}_I(x, y) = \hat{p}_I(y, z) = 1$, $\hat{p}_S(z, x) = 1$ and $A(\hat{p}) = 0$.

CONCLUDING REMARKS

The treatment here given is developed from a welfare point of view, not from a decision one. Thus, the results obtained in this paper should be viewed only as a first attempt of justifying some usual fuzzy preference aggregation rules; though our preferences are often vague, exact choices must be made in any particular decision problem.[3,6] In any case, non-irrational rules can be useful in group decision-making problems.

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