

Modelling River Channel Formation

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Abstract

A coupled model describing the evolution of the topographic elevation and the depth of the overland water film is here studied when considering the overland flow of water over an erodible sediment. We complete the previous modelling of the problems by SMITH and BRETHERTON (1972) and FOWLER et al. (2007), obtaining a model which involves a degenerate nonlinear parabolic equation (satisfied on the interior of the support of the solution) with a super-linear source term and a prescribed constant mass. The degeneracy of the equation causes the channel width to be self-selecting. We propose here a global formulation of the problem, formulated in the whole space, beyond the support of the solution. An important feature of the model proposed here is that despite of the presence of the superlinear forcing term at the equation, a solution to it can not blow up thanks to the mass constraint.

Key words: River Models, Landscape Evolution, Nonlinear parabolic equations, Free boundaries, singular free boundary flux.

1 Introduction

The understanding of the mechanisms whereby river channels form and which governs their size and transport capacity is challenging. In this work we shall consider the deduction and mathematical analysis of a model describing the river channel formation and the evolution of its depth. A proper model should consider the description of two main process, which are the water flow and the sediment transport. The model here proposed does not need extra conditions at the channel margin in order to determine the channel width, it is the evolving channel which determines its own width. The positive feedback,

consisting on increased flow causing increased erosion, which in turn increases the flow, gives place to the existence of an instability which triggers the mechanism of river channel formation.

The starting point of the model here proposed is the model presented by MEYER-PETER and MÜLLER (1948) and the studies carried out by SMITH and BRETHERTON (1972). Their model consists of a set of partial differential equations describing $s(x, y, t)$, the hillslope elevation, and $h(x, y, t)$, the water depth. Smith and Bretherton found the instability to arise in an ill posed way. The water surface elevation η is related to the hillslope elevation s and the water film thickness h by $\eta = s + \delta h$, where the parameter δ is very small. LOEWENHERZ-LAWRENCE (1994) observed that uniform overland flow is unstable to y -dependent perturbations of small wavelength, and we can examine the nonlinear evolution of these by directly seeking asymptotic expansions in terms of δ . This observation allowed a progress in the modelling. It is supposed that the channels which form are aligned in the X direction, and (sensibly) that the perturbation to the water surface is small, comparable to the overland flow depth: $\eta = \eta_0 + \delta Z$. After linearizing, we find that the nonlinear channel evolution then arises from a rescaling of the hillslope evolution equation and the leading order sediment transport equation takes the form:

$$\frac{\partial H}{\partial T} = S' S^{1/2} H^{3/2} + S^{1/2} \frac{\partial}{\partial Y} \left[\beta H^{1/2} \frac{\partial H}{\partial Y} \right], \quad (1)$$

where $S(X) = |\eta'_0|$ is the unperturbed downhill slope, $S' = dS/dX$ and Y , T and H are the rescaled across stream coordinate, time variable and water film thickness. β is a parameter, typically $\beta = O(1)$. It is important to note that this equation arises through conservation of sediment. Only Y derivatives are present, because the lateral length scale is much smaller than the downslope one. The perturbation Z to the water surface is in fact then determined by quadrature of the water conservation equation, but integration of this equation in the across stream direction yields the integral constraint

$$\int_{-\infty}^{\infty} H^{3/2} dY = \frac{2LrX}{S^{1/2}}, \quad (2)$$

where L is the spacing between channels and r comes from the source term in water the mass conservation and represent the rainfall; the limits in (2) are, however, infinities because the integral is with respect to the much smaller channel width length scale. Suitable initial and boundary conditions for the channel depth are that

$$H \rightarrow 0 \quad \text{as} \quad Y \rightarrow \pm\infty, \quad H = H_0(Y) \quad \text{at} \quad T = 0. \quad (3)$$

The equation (1), together with the integral constraint (2) and initial/boundary conditions (3), forms the basis of our study. We will assume that $S' > 0$, so that the nonlinear

term in (1) is a source. Since the downslope coordinate only appears in the coefficients, it can be scaled out of the problem. According to FOWLER et al. (2007), if we define

$$H = \left(\frac{6}{\beta}\right)^{1/3} (LrX)^{2/3}u, \quad T = \left(\frac{\beta}{6}\right)^{1/6} \frac{t}{S^{1/2}S'(LrX)^{1/3}}, \quad Y = \left(\frac{2\beta}{3S'}\right)^{1/2} x, \quad (4)$$

the problem to be studied consists of a non-linear partial differential equation of diffusive type to describe the depth of an evolving channel :

$$\begin{aligned} u_t &= u^{3/2} + (u^{3/2})_{xx}, \\ \int_{-\infty}^{\infty} u^{3/2} dx &= 1, \quad u \rightarrow 0 \quad \text{as} \quad x \rightarrow \pm\infty, \\ u &= u_0(x) \quad \text{at} \quad t = 0. \end{aligned} \quad (5)$$

2 Mathematical analysis

In this section we shall deal with the analysis of the problem (5) assuming that the initial thickness perturbation $u_0(x)$ is a bounded and nonnegative function with a compact and connected support $[0, \zeta_0]$ such that $\int_0^{+\infty} u_0^m(x)dx = M/2$, for $m > 1$, generalizing the particular case of (5), $m = 3/2$.

To precise, in this section and the following ones, we shall deal with the analysis of the *global solvability* (in time) of the following problem: find a continuous curve $\zeta : [0, +\infty) \rightarrow \mathbb{R}^+$ and a function $u : \mathcal{P} \rightarrow [0, +\infty)$ (regular enough) such that

$$(SL) \left\{ \begin{array}{ll} u_t = (u^m)_{xx} + u^m, & \text{in } \mathcal{D}'(\mathcal{P}), \\ u(x, 0) = u_0(x) & \text{a.e. } x \in \Omega_0, \\ u(x, t) > 0, & \text{a.e. } (x, t) \in \mathcal{P}, \\ u(x, t) \equiv 0, & \text{a.e. } (x, t) \notin \mathcal{P}, \\ u(\zeta(t), t) = 0, \quad (u^m)_x(0, t) = 0 & \text{a.e. } t \in (0, +\infty), \\ \zeta(0) = \zeta_0 \text{ and } \zeta(t) > 0 & \text{for any } t \geq 0, \\ \int_0^{\zeta(t)} u^m(x, t)dx = \frac{M}{2} & \text{a.e. } t \in (0, +\infty). \end{array} \right.$$

where $\Omega_0 = (0, \zeta_0)$, $\Omega_t = (0, \zeta(t)) \times \{t\}$, $\mathcal{P} = \cup_{t>0} \Omega_t$. Notice that $\mathcal{D}'(\mathcal{P})$ denotes the space of *distributions* on \mathcal{P} and \mathcal{P} is the *positivity subset* of the solution. Later on we shall make more precise the (minimal) regularity of the solution. The function $\zeta(t)$ is called *the interface* separating the (connected) region where $u(x, t) > 0$ from the region where $u(x, t) = 0$. It is unknown and it is usually called the *free* or *moving boundary* of the problem. Due to the free boundary we shall refer to the strong formulation (SL) as the

strong-local formulation. We emphasize that the mass conservation constraint in (SL),

$$\int_0^{\zeta(t)} u^m(x, t) dx = \frac{M}{2} \quad \text{a.e. } t \in (0, +\infty)$$

prevents possible blow-up phenomena which could arise (without this condition) due to the presence of the source term u^m in the equation.

In order to get a global formulation (i.e. extended to the whole domain $(x, t) \in (0, +\infty) \times (0, +\infty)$, and not only on $(x, t) \in \mathcal{P}$), we shall see that it is necessary to provide a suitable description of the flux $-(u^m)_x(\zeta(t), t)$ at the free boundary. We shall first analyze this consideration in the associated stationary problem to obtain a new constrained global formulation for this case. The parabolic case will be treated later on.

2.1 The stationary case

Let M be a positive, fixed, real number and define $v = u^m$, then the *strong* formulation of the stationary problem associated to (SL) can be written in the following way:

$$-v_{xx} - v = 0, \quad v'(0) = 0, \quad \lim_{x \rightarrow +\infty} v(x) = 0, \quad \int_0^{+\infty} |v(x)| dx = \frac{M}{2}.$$

It is not difficult to observe that the formulation does not correspond to a standard constrained problem of the Calculus of Variations and that, therefore, the integral constraint must be carefully considered. Moreover, since the solutions of the ODE, $v_{xx} + v = 0$, are explicitly given by $v(x) = A \cos x + B \sin x$, we see that none of them can satisfy $v_x(\zeta_\infty) = 0$ if $\{v > 0\} = (0, \zeta_\infty)$. So, necessarily, the limit $\lim_{x \nearrow \zeta_\infty} v_x(x)$ is strictly negative (since the function is passing from positive values to zero). Hence, if we extend v by zero to the rest of $(0, +\infty)$, we get that $v_x(x)$ has a discontinuity at $x = \zeta_\infty$. In particular, v_{xx} is not an integrable function on $(0, +\infty)$ but a measure with a non-zero singular part. This introduces our formulation of the constraint by means of the ‘‘measure’’:

$$\mu = -v_{xx} \in \mathcal{M}(0, +\infty),$$

where $\mathcal{M}(0, +\infty)$ is the space of Radon measures (see, for instance, EVANS and GARIEPY, 1992). In fact, from the identity $v_{xx} = -v$ on $\{v > 0\}$ we see that the (signed) Jordan decomposition of μ (in the form $\mu = \mu_+ - \mu_-$, with $\mu_+ \perp \mu_-$) is given by

$$\mu_+ = v \text{ (which is in } L^1(0, \infty)) \quad \text{and} \quad \mu_- = -c\delta_{\zeta_\infty} \text{ for some } c > 0,$$

where δ_{ζ_∞} is the Dirac delta distribution located at interface $\zeta_\infty \in \mathbb{R}^+$, i.e. where $v(\zeta_\infty) = 0$ (sometimes we shall use the alternative notation $\delta_{\zeta_\infty} = \delta_{\partial\{v=0\}}$). So, μ_- is a singular measure with respect to the Lebesgue measure. Notice that,

$$-v_{xx} - v = c\delta_{\partial\{v=0\}}, \tag{6}$$

where, c can be obtained from the following:

$$0 = \int_0^{+\infty} d\mu = \int_0^{+\infty} v dx - \int_0^{+\infty} d\mu_- = \frac{M}{2} - c < \delta_{\zeta_\infty}, 1 > = \frac{M}{2} - c,$$

i.e., $c = M/2$ and thus, necessarily, $\mu_- = -\frac{M}{2}\delta_{\zeta_\infty}$. Moreover, integrating in (6) we have:

$$0 = \int_0^{+\infty} d\mu = - \int_0^{\zeta_\infty} v_{xx} dx + \int_{\zeta_\infty}^{+\infty} d\mu_- = -v_x(\zeta_\infty) - \frac{M}{2}$$

and we deduce that, in the stationary case, the flux is determined by the integral constraint ($-v_x(\zeta_\infty) = M/2$) and reciprocally. Notice also that:

$$\|\mu\|_{\mathcal{M}(0,+\infty)} = \|\mu_+\|_{\mathcal{M}(0,+\infty)} + \|\mu_-\|_{\mathcal{M}(0,+\infty)} = \int_0^{+\infty} v dx + \frac{M}{2} = M.$$

As a result of the above considerations, we derive the following (symmetric) global formulation: *Find a stationary state $v(x)$ and a point $\zeta_\infty \in \mathbb{R}^+$ satisfying*

$$(SP) \begin{cases} v_{xx} + v = (M/2)\delta_{\zeta_\infty}, & \text{in } \mathcal{D}'(0, +\infty), \\ v(x) > 0, & x \in [0, \zeta_\infty), \\ v(x) \equiv 0, & x \geq \zeta_\infty, \\ v_x(0) = 0. \end{cases}$$

Problems of this type arise in fluid mechanics (problems of the Bernoulli type), in combustion and in plasma physics (see, e.g., DÍAZ et al., 2007, and its references).

We have the following existence and uniqueness result:

Given $M > 0$ there exists a unique solution $(v(x), \zeta_\infty)$ of (SP) given by

$$\zeta_\infty = \frac{\pi}{2} \quad \text{and} \quad v(x) = \frac{M}{2} \cos x \left[1 - H\left(x - \frac{\pi}{2}\right) \right], \quad (7)$$

where $H(x - \pi/2)$ denotes the Heaviside function located at $\pi/2$ i.e.,

$$v(x) = \begin{cases} (M/2) \cos x & \text{if } x \in [0, \pi/2], \\ 0 & \text{if } x \in (\pi/2, +\infty). \end{cases}$$

2.2 Parabolic case

Next, we shall show that in order to generalize the global formulation, obtained in the stationary case, to the parabolic case, it is not enough to consider the presence of the Dirac delta. In the parabolic case, the Dirac delta does not prevent the blow-up phenomenon. In fact, we are able to find an infinite number of solutions which are not globally defined in time (well-known for the case of zero, continuous free boundary flux, see SAMARSKI et al., 1995, Chapter IV, Section 1.1).

Consider the naturally associated problem: given $T > 0$ (arbitrary) and a continuous, symmetric, non negative initial data $u_0(x)$, with compact support $[0, \zeta_0]$ and such that $\int_0^{+\infty} u_0^m(x)dx = M/2$, we look for a continuous curve $\zeta : [0, T] \rightarrow \mathbb{R}^+$ and a function $u : \mathbb{R}^+ \times [0, T] \rightarrow [0, +\infty)$ such that u satisfies the strong-local formulation (SL). As in the stationary case, the global formulation on the whole domain $\mathbb{R}^+ \times [0, T]$ of the partial differential equation includes a Dirac delta distribution located, for each $t \in (0, T]$, at the free moving boundary $x = \zeta(t)$ since the free boundary flux is discontinuous there (due to the mass constraint). We introduce the notation $\delta_{\partial\{u(t, \cdot)=0\}}$ to design the Dirac delta distribution located at the interface $x = \zeta(t)$ for each $t \in (0, T)$ (i.e. $\delta_{\partial\{u(t, \cdot)=0\}} = \delta_{(\zeta(t), t)}$). Then, we have the following problem:

$$(P_0) \begin{cases} u_t = (u^m)_{xx} + u^m - \frac{M}{2}\delta_{\partial\{u(t, \cdot)=0\}}, & \mathcal{D}'(\mathbb{R}^+ \times (0, T)), \\ u(x, 0) = u_0(x) & \text{a.e. } x \in (0, \zeta_0) \\ u(x, t) > 0, & \text{a.e. } (x, t) \in \mathcal{P}_T, \\ u(x, t) \equiv 0, & \text{a.e. } (x, t) \notin \mathcal{P}_T, \\ u(\zeta(t), t) = 0, \quad u_x(0, t) = 0 & \text{a.e. } t \in (0, T), \\ \zeta(0) = \zeta_0 \text{ and } \zeta(t) > 0 & \text{for any } t \in [0, T], \end{cases} \quad (8)$$

where \mathcal{P}_T (the *positivity subset of u*) is defined by $\mathcal{P}_T = \{(x, t) \in \mathbb{R}^+ \times [0, T] : 0 \leq x < \zeta(t)\}$. The above problem has blow up solutions. Indeed, it is possible to construct an infinite number of initial data such that the corresponding solutions $\{u_{T_e}\}$, with T_e be a positive parameter, of problem (P_0) (i.e. with a discontinuous free boundary flux condition) are not globally defined in time (the solution u_{T_e} being defined on a finite time interval $[0, T_e)$). Moreover, u_{T_e} verifies that

$$\int_0^{+\infty} u^m(x, t)dx = \frac{MT_e^{1/(m-1)}}{2(T_e - t)^{1/(m-1)}} \text{ for } t \in [0, T_e).$$

We shall look for separable solutions of the form

$$u(x, t) = (T_e - t)^{-1/(m-1)}\theta(x). \quad (9)$$

Then, the following result holds:

Theorem. *For any $c > 0$, the problem:*

$$(P_w) \begin{cases} w'' + w - \frac{1}{m-1}w^{1/m} = c\delta_{\zeta_0}, & \mathcal{D}'(0, +\infty), \\ w(x) = 0 & x > \zeta_0, \\ w'(0) = 0, \end{cases}$$

admits a unique nonnegative solution w such that

$$\int_0^{+\infty} w(x)dx = c. \quad (10)$$

If we take $\theta^m(x) = w(x)$ and $c = \frac{M}{2T_e^{m/(m-1)}}$, then the pair $u_{T_e}(x, t) = (T_e - t)^{-1/(m-1)}\theta(x)$ and $\zeta(t) \equiv \zeta_0$ satisfies (P_0) for $u_0(x) := T_e^{-1/(m-1)}w^{1/m}(x)$.

Proof. Consider the ODE $w'' + w - \frac{1}{m-1}w^{\frac{1}{m}} = 0$, written in terms of the variable $\theta = w^{\frac{1}{m}}$:

$$(\theta^m)'' + \theta^m - \frac{1}{m-1}\theta = 0,$$

and let us define the new variable

$$y = \frac{1}{m-1}(\theta^{m-1})'.$$

Then, in the variables (θ, y) , we obtain the following dynamical system:

$$(S_1) := \begin{cases} \theta' = y\theta^{2-m}, \\ my'\theta = -\theta^m + \frac{1}{m-1}\theta - my^2\theta^{2-m} \end{cases}$$

Consider $f = \theta^{m-1}$ and assume $1 < m < 2$. The dynamical system written in the new variables results to be:

$$(S_2) := \begin{cases} f' = (m-1)y, \\ y'f = -\frac{1}{m}f^2 + \frac{1}{m(m-1)}f - y^2. \end{cases}$$

Re-scaling the spatial variable x , such that $x = \tau f$, and considering the derivatives with respect the new spatial coordinate τ , we get the equivalent system of ODEs:

$$(S_3) := \begin{cases} f_\tau = (m-1)yf, \\ y_\tau = -\frac{1}{m}f^2 + \frac{1}{m(m-1)}f - y^2, \end{cases}$$

where $f_\tau = \partial_\tau f$ and the same for y_τ .

If one is looking for an orbit such that the derivative $\theta' \rightarrow 0$ for $\theta \rightarrow 0$, or equivalently for $f \rightarrow 0$, it is necessary to consider $y \rightarrow \infty$, due to the identity $\theta' = y\theta^{2-m}$. Because of the above considerations, we proceed to analyze the orbits $y \rightarrow \infty$ when $f \rightarrow 0$. In this case, we can approximate the system (S_3) by the asymptotically equivalent system:

$$(S_4) := \begin{cases} f_\tau = (m-1)yf, \\ y_\tau = -y^2, \end{cases}$$

from which we deduce the ODE of separable variables:

$$\frac{dy}{df} = \frac{-y}{(m-1)f},$$

whose solutions are of the form $y = k_1 f^{\frac{-1}{m-1}}$ with k_1 a non zero real constant. Hence,

$$f' = (m-1)k_1 f^{\frac{-1}{m-1}} \text{ and } f^{\frac{m}{m-1}} = mk_1 x + k_2,$$

k_2 a real constant. Writing the solution in terms of the original variables, we find that the dynamical system admits solutions that in the neighborhood of the free boundary behaves in the following way:

$$\theta(x) \sim (mk_1x + k_2)^{\frac{1}{m}}.$$

Therefore, in terms of the pressure variable $\theta^m(x)$, we find $\theta^m(x) \sim mk_1x + k_2$.

Notice that $\theta(0) > 0$ and we wish the solution θ satisfies $\theta(x) \geq 0$ and $\theta'(x) \leq 0$. These requirements suggests that the constants k_1 and k_2 should satisfy the following:

$$k_1 < 0, \quad k_2 > 0, \quad k_2 = u_0(0)T_e^{\frac{1}{m-1}} \text{ and } k_1m\zeta_0 = -k_2.$$

notice also that, $(\theta^m)' = mk_1 < 0$, i.e., non zero derivative at the free boundary.

Note that the uniqueness of the solution to the problem (P_w) -(10) is deduced from the fact that necessarily all solution w , $w \neq$ leads to a equation in terms of θ . Then, as a result of the uniqueness of solution to the equation for θ and of the fact that there is a only way to satisfy the integral constraint on w , i.e. $\int_0^\infty w(x)dx = c$, we get that the solution w to the problem (P_w) -(10) is unique as well.

As a conclusion, we get the existence of separable solutions in time and space given by (9), that present blow up and admit discontinuous derivative at the free boundary. Note that the jump of the derivative at the free boundary would be given by a multiple of the Heaviside function, whose derivative is precisely a multiple of the Dirac delta distribution. Therefore, we shall need some additional condition in order to give a global formulation to the whole domain $\mathbb{R}^+ \times (0, T)$. Notice that if we define (for a.e. $t \in (0, T)$ fixed) the spatial distribution

$$\mu(t, \cdot) := u_t(t, \cdot) - (u^m)_{xx}(t, \cdot)$$

then we must expect to know that, in fact, such a distribution is a bounded measure $\mathcal{M}(0, +\infty)$ (with compact support) since

$$\mu(t, \cdot) = u^m(t, \cdot) - \frac{M}{2}\delta_{\partial\{u(t, \cdot)=0\}}.$$

Moreover its signed (Jordan) decomposition, $\mu(t, \cdot) = \mu_+(t, \cdot) - \mu_-(t, \cdot)$, must be given by $\mu_+(t, \cdot) = u^m(t, \cdot)$ and $\mu_-(t, \cdot) = (M/2)\delta_{\partial\{u(t, \cdot)=0\}}$. Now, as in the stationary case we have that the mass constraint $\int_0^{+\infty} u^m(x, t)dx = M/2$ is equivalent to the ‘‘zero total measure’’ condition

$$\int_0^{+\infty} d\mu(t, \cdot) = 0, \quad \text{for a.e. } t \in (0, T). \quad (11)$$

So, we arrive to the global formulation: find a nonnegative function $u : \mathbb{R}^+ \times [0, T) \rightarrow$

$[0, +\infty)$ such that

$$(P) \begin{cases} u_t = (u^m)_{xx} + u^m - \frac{M}{2}\delta_{\partial\{u(t,\cdot)=0\}}, & \mathcal{D}'(\mathbb{R}^+ \times (0, T)), \\ u(x, 0) = u_0(x) & \text{a.e. } x \in (0, +\infty), \\ u_x(0, t) = 0, u(x, t) \rightarrow 0 \text{ as } x \rightarrow +\infty & \text{a.e. } t \in (0, T), \\ \mu(t, \cdot) := u_t(t, \cdot) - (u^m)_{xx}(t, \cdot) \text{ satisfies (11)} & \text{a.e. } t \in (0, T). \end{cases} \quad (12)$$

Notice that now the compact support condition is not explicitly required. In fact, following the numerical experiences of FOWLER et al., 2007, we conjecture that problem (P) can be solved for suitable, strictly positive initial data $u_0(x)$ such that $u_0(x) \rightarrow 0$ as $x \rightarrow +\infty$. Notice also that if a solution u of (P) gives rise a free boundary $\zeta(t) := \partial\{u(t, \cdot) = 0\}$ then, the *zero total measure condition* (11) implies that the free boundary flux must be given by

$$-(u^m)_x(\zeta(t), t) = \frac{M}{2} - \int_0^{\zeta(t)} u_t(x, t) dx \quad \text{a.e. } t \in (0, +\infty).$$

Here (and, in fact, also in (11)) there is a slight abuse of notation since, a priori, $u_t(x, t)$ (respectively $\mu(t, \cdot)$) does not need to be a $L^1(\mathbb{R}^+)$ function but merely a bounded measure. Nevertheless we keep the classical notation for simplicity reasons. In any case, we see that in the transient regime the boundary flux at the free boundary is unknown (being also discontinuous), as opposed to the stationary case where the flux (also discontinuous) can be explicitly known. Moreover, the above considerations allow us to conclude that any solution of the *strong-local* formulation (SL) solves problem (P) and that any (regular enough) solution of (P) with compact support satisfies the *strong-local* formulation (SL).

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