

Isospin-Breaking quark condensates in Chiral Perturbation Theory

A. Gómez Nicola* and R. Torres Andrés†

Departamento de Física Teórica II. Univ. Complutense. 28040 Madrid. Spain.

We analyze the isospin-breaking corrections to quark condensates within one-loop $SU(2)$ and $SU(3)$ Chiral Perturbation Theory including $m_u \neq m_d$ as well as electromagnetic (EM) contributions. The explicit expressions are given and several phenomenological aspects are studied. We analyze the sensitivity of recent condensate determinations to the EM low-energy constants (LEC). If the explicit chiral symmetry breaking induced by EM terms generates a ferromagnetic-like response of the vacuum, as in the case of quark masses, the increasing of the order parameter implies constraints for the EM LEC, which we check with different estimates in the literature. In addition, we extend the sum rule relating quark condensate ratios in $SU(3)$ to include EM corrections, which are of the same order as the $m_u \neq m_d$ ones, and we use that sum rule to estimate the vacuum asymmetry within ChPT. We also discuss the matching conditions between the $SU(2)$ and $SU(3)$ LEC involved in the condensates, when both isospin-breaking sources are taken into account.

PACS numbers: 12.39.Fe, 11.30.Rd,

I. INTRODUCTION

The low-energy sector of QCD has been successfully described over recent years within the chiral lagrangian framework. Chiral Perturbation Theory (ChPT) is based on the spontaneous breaking of chiral symmetry $SU_L(N_f) \times SU_R(N_f) \rightarrow SU_V(N_f)$ with $N_f = 2, 3$ light flavours and provides a consistent and systematic model-independent scheme to calculate low-energy observables [1–3]. The effective ChPT lagrangian is constructed as the more general expansion $\mathcal{L} = \mathcal{L}_{p^2} + \mathcal{L}_{p^4} + \dots$ compatible with the QCD underlying symmetries, where p denotes derivatives or meson mass and external momentum below the chiral scale $\Lambda_\chi \sim 1$ GeV.

The $SU_V(N_f)$ group of vector transformations corresponds to the isospin symmetry for $N_f = 2$. In the $N_f = 3$ case, the vector group symmetry is broken by the strange-light quark mass difference $m_s - m_{u,d}$, although m_s can still be considered as a perturbation compared to Λ_χ , leading to $SU(3)$ ChPT [3]. In the $N_f = 2$ case, the isospin symmetric limit is a very good approximation in Nature. However, there are several known examples where isospin breaking is phenomenologically relevant at low energies, such as sum rules for quark condensates [3], meson masses and corrections to Dashen's theorem [4], pion-pion [5, 6] and pion-kaon [7, 8] scattering in connection with mesonic atoms [9, 10], CP violation [11], $a_0 - f_0$ mixing [12], kaon decays [13, 14] and other hadronic observables (see [15] for a recent review).

The two possible sources of isospin breaking are the $m_d - m_u$ light quark mass difference and electromagnetic (EM) interactions. Both can be accommodated within the ChPT framework. The former is accounted for by modifying the quark mass matrix and generates a $\pi^0\eta$ mixing term in the $SU(3)$ lagrangian [3]. The expected corrections from this source are of order $(m_d - m_u)/m_s$. On the other hand, EM interactions, which in particular induce mass differences between charged and neutral light mesons, can be included in ChPT via the external source method and give rise to new terms in the effective lagrangian [4–6, 16–18] of order \mathcal{L}_{e^2} , $\mathcal{L}_{e^2p^2}$ and so on, with e the electric charge. These terms fit into the ChPT power counting scheme by considering formally $e^2 = \mathcal{O}(p^2/F^2)$, with F the pion decay constant in the chiral limit.

The purpose of this paper is to study the isospin-breaking corrections to quark condensates, whose main importance is their relation to the symmetry properties of the QCD vacuum. The singlet contributions $\langle \bar{u}u + \bar{d}d \rangle$ for $SU(2)$ and $\langle \bar{u}u + \bar{d}d + \bar{s}s \rangle$ for $SU(3)$ are order parameters for chiral symmetry, while the isovector one $\langle \bar{u}u - \bar{d}d \rangle$ behaves as an order parameter for isospin breaking, which is not spontaneously broken [19]. We will calculate the condensates within one-loop ChPT, which ensures the model independency of our results, and will address several phenomenological consequences. The two sources of isospin-breaking will be treated consistently on the same footing, which will allow us to test the sensibility of previous phenomenological analysis to the EM low-energy constants (LEC). Moreover, the EM corrections induce an explicit breaking of chiral symmetry which will lead to lower bounds for certain combinations of the LEC involved, provided the vacuum response is ferromagnetic, as in the case of quark masses. In addition, in

*Electronic address: gomez@fis.ucm.es

†Electronic address: rtandres@fis.ucm.es

$SU(3)$ one can derive a sum rule relating the different condensate ratios for $m_u \neq m_d$ [3] which, as we will show here, receives an EM correction not considered before and of the same order as that proportional to $m_u - m_d$. The latter is useful to estimate the vacuum asymmetry $\langle \bar{d}d \rangle / \langle \bar{u}u \rangle$ reliably within ChPT. An additional aspect that we will discuss is the matching of the $SU(2)$ and $SU(3)$ LEC combinations appearing in the condensates when both isospin-breaking sources are present, comparing with previous results in the literature. The analysis carried out in the present work will serve also to establish a firm phenomenological basis for its extension to finite temperature, in order to study different aspects related to chiral symmetry restoration [20].

With the above motivations in mind, the paper is organized as follows: in section II we briefly review the effective lagrangian formalism needed for our present work, paying special attention to several theoretical issues and to the numerical values of the parameters and LEC needed here. Quark condensates for $SU(2)$ are calculated and analyzed in section III, where we discuss the general aspects of the bounds for the EM LEC based on chiral symmetry breaking. In that section we also comment on the analogy with lattice analysis. The $SU(3)$ case is separately studied in section IV. In that section, we perform first a numerical analysis of the isospin-breaking corrections, paying special attention to the effect of the EM LEC in connection with previous results in the literature. In addition, we obtain the EM corrections to the sum rule for condensate ratios, which we use to estimate the vacuum asymmetry within ChPT. We also provide the LEC bounds for this case, checking them with previous LEC estimates and, finally, we discuss the matching conditions for the LEC involved. In Appendix A we collect the lagrangians of fourth order and the renormalization of the LEC used in the main text.

II. FORMALISM: EFFECTIVE LAGRANGIANS FOR ISOSPIN BREAKING

The effective chiral lagrangian up to fourth order is given schematically by

$$\mathcal{L}_{eff} = \mathcal{L}_{p^2+e^2} + \mathcal{L}_{p^4+e^2p^2+e^4}. \quad (1)$$

The second order lagrangian is the familiar non-linear sigma model, including now the gauge coupling of mesons to the EM field through the covariant derivative, plus an extra term:

$$\begin{aligned} \mathcal{L}_{p^2+e^2} = & \frac{F^2}{4} \text{tr} [D_\mu U^\dagger D^\mu U + 2B_0 \mathcal{M} (U + U^\dagger)] \\ & + C \text{tr} [QUQU^\dagger]. \end{aligned} \quad (2)$$

Here, F is the pion decay constant in the chiral limit and $U(x) \in SU(N_f)$ is the Goldstone Boson (GB) field in the exponential representation $U = \exp[i\Phi/F]$ with :

$$\begin{aligned} SU(2) : \Phi &= \begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{pmatrix}, \\ SU(3) : \Phi &= \begin{pmatrix} \pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ \sqrt{2}\pi^- & -\pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}K^0 \\ \sqrt{2}K^- & \sqrt{2}K^0 & \frac{2}{\sqrt{3}}\eta \end{pmatrix}, \end{aligned} \quad (3)$$

with η the octet member with $I_3 = S = 0$. The covariant derivative is $D_\mu = \partial_\mu + iA_\mu[Q, \cdot]$ with A the EM field. \mathcal{M} and Q are the quark mass and charge matrices, i.e., in $SU(3)$ $\mathcal{M} = \text{diag}(m_u, m_d, m_s)$ and $Q = \text{diag}(e_u, e_d, e_s)$ with $e_u = 2e/3, e_d = e_s = -e/3$ for physical quarks. The additional term in (2), the one proportional to C , can be understood as follows: the QCD lagrangian for $m_u = m_d$ coupled to the EM field is not invariant under an isospin transformation $q \rightarrow gq$ with $g \in SU(N_f)$ and q the quark field. However, it would be isospin invariant if the quark matrix Q is treated as an external field transforming as $Q \rightarrow g^\dagger Q g$. Therefore, the low-energy effective lagrangian has to include all possible terms compatible with this new symmetry, in addition to the standard QCD symmetries. The lowest order $\mathcal{O}(e^2)$ is the C -term in (2), since U transforms as $U \rightarrow g^\dagger U g$. Actually, one allows for independent ‘‘spurion’’ fields $Q_L(x)$ and $Q_R(x)$ transforming under $SU_L(N_f) \times SU_R(N_f)$ so that one can build up the new possible terms to any order in the chiral lagrangian expansion, taking in the end $Q_L = Q_R = Q$ [4].

In the previous expressions, $F, B_0 m_{u,d,s}, C$ are the low-energy parameters to this order. Working out the kinetic terms, they can be directly related to the leading-order tree level values for the decay constants and masses of the

GB. In $SU(2)$, the tree level masses to leading order are:

$$\begin{aligned} M_{\pi^+}^2 &= M_{\pi^-}^2 = 2\hat{m}B_0 + 2C\frac{e^2}{F^2}, \\ M_{\pi^0}^2 &= 2\hat{m}B_0, \end{aligned} \quad (4)$$

with $\hat{m} = (m_u + m_d)/2$ the average light quark mass. Note that both terms contributing to the charged pion mass are of the same order in the chiral power counting, although numerically $(M_{\pi^\pm}^2 - M_{\pi^0}^2)/M_{\pi^0}^2 \simeq 0.1$, which we will use in practice as a further perturbative parameter to simplify some of the results.

In the $SU(3)$ case, the mass term in (2) induces a mixing contribution between the π^0 and the η meson fields given by $\mathcal{L}_{mix} = (B_0/\sqrt{3})(m_d - m_u)\pi^0\eta$. Therefore, the kinetic term has to be brought to the canonical form before identifying the GB masses, which is performed by the field rotation [3]:

$$\begin{aligned} \pi^0 &= \bar{\pi}^0 \cos \varepsilon - \bar{\eta} \sin \varepsilon, \\ \eta &= \bar{\pi}^0 \sin \varepsilon + \bar{\eta} \cos \varepsilon, \end{aligned} \quad (5)$$

where the mixing angle is given by:

$$\tan 2\varepsilon = \frac{\sqrt{3} m_d - m_u}{2 m_s - \hat{m}}. \quad (6)$$

Once the above $\pi^0\eta$ rotation is carried out, the $SU(3)$ tree level meson masses to leading order read:

$$\begin{aligned} M_{\pi^+}^2 &= M_{\pi^-}^2 = 2\hat{m}B_0 + 2C\frac{e^2}{F^2}, \\ M_{\pi^0}^2 &= 2B_0 \left[\hat{m} - \frac{2}{3}(m_s - \hat{m})\frac{\sin^2 \varepsilon}{\cos 2\varepsilon} \right], \\ M_{K^+}^2 &= M_{K^-}^2 = (m_s + m_u)B_0 + 2C\frac{e^2}{F^2}, \\ M_{K^0}^2 &= (m_s + m_d)B_0, \\ M_\eta^2 &= 2B_0 \left[\frac{1}{3}(\hat{m} + 2m_s) + \frac{2}{3}(m_s - \hat{m})\frac{\sin^2 \varepsilon}{\cos 2\varepsilon} \right]. \end{aligned} \quad (7)$$

The above five equations are the extension of the Gell-Mann-Oakes-Renner (GOR) relations [21] to the isospin asymmetric case and allow to relate the four constants $B_0 m_{u,d,s}$ and C (ε is given in terms of quark masses in (6)) with the five meson masses or combinations of them. The additional equation provides the following relation between the tree level LO masses:

$$(M_{K^\pm}^2 - M_{\pi^\pm}^2)^2 - 3(M_\eta^2 - M_{K^0}^2)(M_{K^0}^2 - M_{\pi^0}^2) = 0, \quad (8)$$

The above equation is compatible with the one obtained in [22] neglecting $\mathcal{O}(m_u - m_d)^2$ terms. Actually, note that although all terms in (7) are formally of the same chiral order, numerically (see below) we expect $\varepsilon \sim (\sqrt{3}/4)(m_d - m_u)/m_s \ll 1$ and hence the mixing-angle corrections to the squared masses to be $\mathcal{O}(M_\pi^2 \varepsilon)$ and $\mathcal{O}(M_\eta^2 \varepsilon^2)$ for the neutral pion and eta respectively. On the other hand, in the isospin-symmetric limit ($m_u = m_d$ and $e = 0$) (8) is nothing but the Gell-Mann-Okubo formula $4M_{K^\pm}^2 - 3M_\eta^2 - M_{\pi^0}^2 = 0$ [23]. Neglecting only the $m_d - m_u$ mass difference in (7) leads to Dashen's theorem $M_{K^\pm}^2 - M_{K^0}^2 = M_{\pi^\pm}^2 - M_{\pi^0}^2$ [24] and then eq.(8) reduces to $4M_{K^0}^2 - 3M_\eta^2 - M_{\pi^0}^2 = 0$, i.e., the Gell-Mann-Okubo formula for neutral states. However, the violation of Dashen's theorem at tree level due to those quark mass differences is significant numerically for kaons. In our present treatment we consider those differences on the same footing as the EM corrections to the masses. For pions, the main effect in the $\pi^0 - \pi^\pm$ mass difference comes from the EM contribution [25].

All the previous expressions hold for tree level LO masses M_a^2 with $a = \pi^\pm, \pi^0, K^\pm, \eta$, in terms of which we will write all of our results. They coincide with the physical masses to leading order in ChPT, i.e., $M_{a,phys}^2 = M_a^2(1 + \mathcal{O}(M^2))$. Calculating the ChPT corrections to a given order allows then to determine the numerical values of the tree level masses, knowing their physical values and to that order of approximation. The same holds for F , which coincides with the meson decay constants in the chiral limit $F_{a,phys}^2 = F^2(1 + \mathcal{O}(M^2))$. Next to leading order $\mathcal{O}(M^2)$ corrections to

meson masses and decay constants were given in [2, 3] for $e^2 = 0$. EM corrections to the masses can be found in [4] for $SU(3)$ and in [6, 18] for $SU(2)$ including both $e^2 \neq 0$ and $m_u \neq m_d$ isospin-breaking terms.

The fourth-order lagrangian in (1) consists of all possible terms compatible with the QCD symmetries to that order, including the EM ones. The \mathcal{L}_{p^4} lagrangian is given in [2] for the $SU(2)$ case, $h_{1,2,3}$ (contact terms) and $l_{1\dots7}$ denoting the dimensionless LEC multiplying each independent term, and in [3] for $SU(3)$ the LEC named $H_{1,2}$ and $L_{1\dots10}$. The EM $\mathcal{L}_{e^2 p^2}$ and \mathcal{L}_{e^4} for $SU(2)$ are given in [5, 6], $k_{1,\dots,13}$ denoting the corresponding EM LEC, and in [4] for $SU(3)$ with the $K_{1\dots17}$ EM LEC. For completeness, we give in Appendix A the relevant terms needed in this work. The LEC are renormalized in such a way that they absorb all the one-loop ultraviolet divergences coming from \mathcal{L}_{p^2} and \mathcal{L}_{e^2} , according to the ChPT counting, and depend on the \overline{MS} low-energy renormalization scale μ in such a way that the physical quantities are finite and scale-independent. The renormalization conditions for all the LEC can be found in [2, 4, 6, 17] and we collect in Appendix A only those needed in the present work.

As customary, we denote the scale-dependent and renormalized LEC by an “ r ” superscript. The renormalized LEC are independent of the quark masses by definition, although their finite parts are unknown, i.e., they are not provided within the low-energy theory. The numerical values of the LEC at a given scale can be estimated by fitting meson experimental data, theoretically by matching the underlying theory under some approximations, or from the lattice. These procedures allow to obtain estimates for the “real-world” LEC at the expense of introducing residual dependencies of those LEC on the parameters of the approximation procedure, which typically involves a truncation of some kind. Examples of these are the m_s dependence on the $SU(2)$ LEC when matching the $SU(3)$ ones, the correlations between LEC, masses and decay constants through the fitting procedure, the QCD renormalization scale and gauge dependence of some of the EM LEC or the dependence with lattice artifacts such as finite size or spurious meson masses. We will give more details below, specially regarding the EM LEC which will play an important role in our present work. An exception to the LEC estimates are the contact LEC h_i and H_i , which are needed for renormalization but cannot be directly measured. The physical quantities depending on them are therefore ambiguous, which comes from the definition of the condensates in QCD perturbation theory, requiring subtractions to converge [2]. It is therefore phenomenologically convenient to define suitable combinations which are independent of the h_i, H_i . We will bear this in mind throughout this work, providing such combinations when isospin-breaking is included.

We will analyze in one-loop ChPT (next to leading order) the quark condensates, which for a given flavour q_i can be written at that order as:

$$\langle \bar{q}_i q_i \rangle = - \left\langle \frac{\partial \mathcal{L}_{eff}}{\partial m_i} \right\rangle. \quad (9)$$

The above equation is nothing but the functional derivative with respect to the i -th component of the scalar current, particularized to the values of the physical quark masses, according to the external source method [2, 3]. Therefore, we will be interested only in the terms of the fourth-order lagrangian containing at least one power of the quark masses. These are the operators given in eqns.(A1) and (A2) for $SU(2)$ and $SU(3)$ respectively. Thus, the LEC that enter in our calculation are $l_3, h_1, h_3, k_5, k_6, k_7$ in $SU(2)$, and $L_6, L_8, H_2, K_7, K_8, K_9, K_{10}$ in $SU(3)$. Besides, up to NLO, only tree level diagrams from the fourth-order lagrangian can contribute to the condensates, so that in practice it is enough to set $U = \mathbb{1}$ in (A1)-(A2) for getting those tree-level contributions from (9).

A. Masses and low-energy constants

For most of the numerical values of the different low-energy constants and parameters in the $SU(3)$ case, we will follow [26], where fits to K_{l4} experimental data are performed in terms of $\mathcal{O}(p^6)$ ChPT expressions, including the isospin mass difference $m_u/m_d \neq 1$ and EM corrections to the meson masses, extending a previous work [27] where isospin breaking was not considered. Those fits have been improved in a recent work [28], which takes into account new phenomenological and lattice results. We will however stick to the values of [26], since our main interest is to compare the isospin-breaking condensates with the two sources included and to estimate the effect of the EM LEC. In the new fits [28] isospin breaking is included only to correct for the charged kaon mass and the condensate values are not provided. For a review of different estimates of the quark masses and condensates see also [29] and [30]. In addition, in [31] a recent update of lattice results for low-energy parameters can be found, including LEC and the quark condensate. We will use the central values of the main fit in [26]. The value of $m_s/\hat{m} = 24$ [29, 30] is used as an input in [26], as well as $L_6^r = 0$, as follows e.g. from OZI rule or large- N_c arguments [3]. The more recent fits [28] consider an updated value of $m_s/\hat{m} = 27.8$ in accordance with recent determinations [31] and a nonzero value of L_6^r is obtained as an output. The suppression of L_6^r has been questioned in connection with a reduction of the light quark condensate when the number of flavours is increased [32], within the framework of generalized ChPT. In that context, the chiral power counting is modified due to the smallness of the condensate. Here, we will adhere to the standard

ChPT picture, where the condensate and the GOR-like relations are dominated by the leading order [33], sustained by the recent lattice LEC estimates [31]. The values of $F = 87.1$ MeV, $2B_0\hat{m} = 0.0136$ GeV², $m_u/m_d = 0.46$ and $L_8(\mu = 770$ MeV) = 0.62×10^{-3} are outputs from the main fit in [26]. With those values we get from (6) $\varepsilon = 0.014$ and from (7) the tree level masses of π^0, K^0, η .

To calculate the tree level charged meson masses, we need also the value of the C constant, which can be inferred also from the results in [26] since the EM correction is numerically very small in the charged kaon mass with respect to the pure QCD contribution. This allows to extract the tree level charged kaon mass directly from the expressions for $M_{K^\pm}/M_{K^\pm, QCD}$ in [26], approximating $M_{K^\pm, QCD}$ by the full physical mass. From there we extract the value of C by subtracting the tree level QCD part in (7) calculated with the above given quark masses. In this way we get $C = 5.84 \times 10^7$ MeV⁴, which is very close to the values obtained simply from the charged-neutral pion mass difference in (7) setting the masses and F to their physical values [6] or from resonance saturation arguments [4]. From that C value we obtain the tree-level charged pion mass, using again (7). Nevertheless, to the order we are calculating we could have used as well the physical meson masses and decay constants instead of the tree level ones, since formally the difference is hidden in higher orders. The main reason why we choose the values in [26] is to compare directly with their numerical quark condensates and estimate the importance of the K_i^r corrections (see section IV A for details). The constant H_2 will also appear explicitly in quark condensates. Since it cannot be fixed with meson experimental data, when needed we will estimate it from scalar resonance saturation arguments as $H_2^r = 2L_8^r$ [16, 26], although we will comment below more about the H_2^r dependence of the results and provide physical quantities which are independent of the contact terms.

Regarding the EM LEC, the $SU(3)$ K_i^r have been estimated in the literature under different theoretical schemes. Resonance saturation was used in [35], large- N_c and NJL models in [34], complemented with QCD perturbative information in [36] and a sum-rule approach combined with low-lying resonance saturation has been followed in [37, 38]. The works [34, 36–38] have in common the use of perturbative QCD methods for the short-distance part of the LEC and different model approaches for the long-distance part. This procedure implies that the LEC estimated in that way depend (roughly logarithmically) in general on the QCD renormalization scale, which we call μ_0 to distinguish it from the low-energy scale μ , as well as on the gauge parameter. A closely related problem is that the separation of the strong ($e = 0$) and EM contributions for a given physical quantity is in principle ambiguous [34, 37, 39, 40]. The origin of this ambiguity [40] is that QCD scaling quantities such as quark masses contain also EM contributions through the Renormalization Group (RG) evolution in the full QCD+EM theory. Thus, a particular prescription for disentangling those contributions must be provided. In addition, when matching such quantities between the low-energy sector and the underlying theory, the choice of a given prescription will necessarily affect the scale and gauge dependence of the EM LEC. These theoretical uncertainties, as well numerical ones, make those theoretical EM LEC estimates not fully compatible among them. For these reasons, in many works analyzing EM corrections, the LEC are simply assumed to lie within “natural” values $|K_i^r|, |k_i^r| \lesssim \frac{1}{16\pi^2} \simeq 6.3 \times 10^{-3}$ at the scale $\mu \sim M_\rho$ [4, 6]. The above theoretical issues will be addressed in more detail in sections III and IV D. The LEC dependencies on the QCD scale and on the gauge parameter do not affect directly our results, being only relevant when comparing them with approaches where those LEC are obtained by matching the underlying theory. In that context we will see that the LEC combinations that we will deal with are gauge independent and lie within the stability range where the dependence on μ_0 is smooth and the matching makes sense [34, 36]. The theoretical errors quoted e.g. in [34] account for the uncertainty related to the μ_0 dependence.

As for the $SU(2)$ k_i^r , no direct estimate is available to our knowledge, although one can relate them to the K_i^r by performing formally a $1/m_s$ expansion in a given physical quantity calculated in $SU(3)$ and comparing to the corresponding $SU(2)$ expression, similarly as the $l_i \leftrightarrow L_i$ conversion given in [3]. This has been done partially for some combinations of the LEC, namely, those appearing in the neutral pion mass [41], in pion scattering [9, 41] and in the pionium lifetime [41, 42]. More recently, a full matching of the EM $SU(2)$ and $SU(3)$ LEC at the lagrangian level has been performed in [43] using functional integral methods in the chiral limit. In the present work, we will provide a complementary analysis. Namely, in section IV we will obtain the matching relations between the LEC involved in the quark condensates, including both $m_u - m_d$ and EM contributions. Those relations will be consistent with the results in [43] and phenomenologically useful when dealing with approximate LEC determinations where isospin and mass corrections may be entangled.

III. TWO-FLAVOUR QUARK CONDENSATES AND BOUNDS FOR THE EM LEC

We start by giving the explicit one-loop ChPT expressions for the quark condensates in $SU(2)$ with all the isospin-breaking corrections included, which we derive from (9):

$$\langle \bar{q}q \rangle \equiv \langle \bar{u}u + \bar{d}d \rangle = -2F^2 B_0 \left[1 - \mu_{\pi^0} - 2\mu_{\pi^\pm} + 2\frac{M_{\pi^0}^2}{F^2} (l_3^r(\mu) + h_1^r(\mu)) + e^2 \mathcal{K}_2^r(\mu) + \mathcal{O}(p^4) \right], \quad (10)$$

$$\langle \bar{u}u - \bar{d}d \rangle = 4B_0^2(m_d - m_u)h_3 - \frac{8}{3}F^2 B_0 e^2 k_7 + \mathcal{O}(p^2), \quad (11)$$

where:

$$\mathcal{K}_2^r(\mu) = \frac{4}{9} [5(k_5^r(\mu) + k_6^r(\mu)) + k_7], \quad (12)$$

and throughout this work we will follow the same notation as in [2, 3]:

$$\mu_i = \frac{M_i^2}{32\pi^2 F^2} \log \frac{M_i^2}{\mu^2} \quad ; \quad \nu_i = \frac{1}{32\pi^2} \left[1 + \log \frac{M_i^2}{\mu^2} \right]. \quad (13)$$

The μ_i arise from the finite part of the one-loop tadpole-like contribution $G_i(x=0)$, with G the free meson propagator [2]. The renormalization conditions for the LEC involved in (10)-(11) can be found in [2] and [6] and we give them in Appendix A. With that LEC renormalization, one can check that the condensates in (10)-(11) are finite and scale-independent, which is a nontrivial consistency check. Recall that h_3 and k_7 do not need to be renormalized. The condensates still depend on the h_1 and h_3 contact LEC, which, as explained above, yield an ambiguity in the determination of the condensates. The result (10) for $e \neq 0$ and $m_u \neq m_d$ reduces for $e = 0$ to the expressions given in [2]. The condensate difference (11) is given in [2] for $e = 0$, which we agree with, and in [6] for $e \neq 0$, which we also agree with, except for the relative sign between the two terms, which should be a minus in their eq.(3.12) [44].

Let us now argue on how the EM corrections to the condensates may lead to constraints for the EM LEC. Those corrections come directly from the coupling of the EM field to the quarks in the QCD action, which break chiral symmetry. Actually, to understand better the origin of the different sources involved in chiral symmetry and isospin breaking, it is useful to keep the charges of the u and d flavours arbitrary and to separate the isoscalar and isovector contributions of the charge matrix in $SU(2)$:

$$Q = \frac{e_u + e_d}{2} \mathbb{1} + \frac{e_u - e_d}{2} \tau_3, \quad (14)$$

with $\tau_3 = \text{diag}(1, -1)$ corresponding to the third isospin component. The EM part of the QCD lagrangian $\bar{q}\gamma^\mu A_\mu Qq$ breaks explicitly chiral symmetry $SU_L(2) \times SU_R(2)$ if $e_u \neq e_d$, through the isovector part in (14). The isovector also breaks the isospin symmetry $SU_V(2)$ ($L=R$) except for transformations in the third direction, which corresponds to electric charge conservation. On the other hand, the mass term $\bar{q}\mathcal{M}q$ breaks chiral symmetry for any nonzero value of the quark masses, preserving isospin symmetry if $m_u = m_d$. Altogether, the conclusion is that the QCD lagrangian is chiral invariant only if $e_u = e_d$ and $m_u = m_d = 0$. Thus, chiral symmetry is explicitly broken even if $e_u = e_d$, as long as any of the quark masses $m_q \neq 0$, or equivalently, in the presence of an external scalar source, as needed to derive the condensates. If $e_u = e_d$ and $m_u = m_d \neq 0$, chiral symmetry is broken but isospin symmetry is conserved.

Now, let us remind how this charge and mass symmetry breaking pattern translates into the low-energy sector. The leading order $\mathcal{L}_{p^2+e^2}$ in (2) contains separate combinations of the charge and mass terms, both sharing the QCD pattern. Thus, the charge contribution proportional to C in (2) can be decomposed according to (14), giving a constant term proportional to $(e_u + e_d)^2$ independent of masses and fields, plus the term $C [(e_u - e_d)^2/4] \text{tr} [\tau_3 U \tau_3 U^\dagger]$, which contributes directly to the pion EM mass difference in (4). Therefore, in the second order lagrangian all the EM chiral symmetry breaking terms are proportional to $(e_u - e_d)^2$. This is no longer true for the fourth order lagrangian in (A1), for which the symmetries of the theory allow for crossed mass-charge terms, like those proportional to k_5, k_6 and k_7 . Those crossed terms break chiral symmetry even for $e_u = e_d$ for any nonzero quark mass, the strength of chiral breaking being proportional both to the quark charge e and to the quark mass \hat{m} . Consequently, they contribute to $\langle \bar{q}q \rangle = -2\langle \partial\mathcal{L}/\partial(m_u + m_d) \rangle$, the expectation value of the $SU_V(2)$ singlet behaving as an order parameter for chiral symmetry breaking. Setting $U = \mathbb{1}$ in the lagrangian gives a piece proportional to $k_5 + k_6$ yielding $(e_u + e_d)^2$ and $(e_u - e_d)^2$ contributions to $\langle \bar{q}q \rangle$ and another one proportional to k_7 contributing both to $\langle \bar{q}q \rangle$ and to $\langle \bar{u}u - \bar{d}d \rangle = -2\langle \partial\mathcal{L}/\partial(m_u - m_d) \rangle$, the isotriplet order parameter of isospin breaking:

$$\begin{aligned} \mathcal{L}_{e^2 p^2}^{\langle \bar{q}q \rangle} &= 2F^2 B_0 \{ (k_5 + k_6) [(e_u + e_d)^2 + (e_u - e_d)^2] (m_u + m_d) \\ &\quad + 2k_7 [(e_u + e_d)^2 (m_u + m_d) + (e_u - e_d)(e_u + e_d)(m_u - m_d)] \} + \dots \end{aligned}$$

where the dots indicate terms not contributing to the condensates at this order.

Thus, we see how the two sources of isospin breaking show up in the order parameter (11), which does not receive pion loop contributions in $SU(2)$. The latter is the explicit confirmation that isospin symmetry is not spontaneously broken in QCD [19], since all the contributions to this order parameter vanish for $m_u = m_d$ and $e = 0$.

On the other hand, the quark condensate $\langle \bar{q}q \rangle$ behaves as an order parameter for chiral symmetry and therefore measures the different sources of symmetry breaking: spontaneous and explicit. Thus, it is naturally expected that its absolute value increases when a new symmetry-breaking source, as the EM contribution, is switched on. This increasing behaviour is what we denote as ferromagnetic, in analogy with the behaviour of the magnetization in a ferromagnetic material under an external magnetic field.

There is no a priori formal argument to ensure the ferromagnetic-like nature of the QCD low-energy vacuum. We can nevertheless learn from the response of the system to the light quark mass \hat{m} , which is the actual counterpart of the magnetic field in a ferromagnet, since it breaks the chiral symmetry explicitly by coupling to the order parameter. For the mass, this ferromagnetic behaviour is actually followed by the $e = 0$ condensates in the standard ChPT framework, both to $\mathcal{O}(p^4)$ and to $\mathcal{O}(p^6)$ with the LEC in [27], although assuming L_6 suppression and the dependence on contact terms still introducing a source of ambiguity. With the recent fit giving nonzero L_6^r and a new value also for L_8^r [28] we get also that $\langle \bar{u}u + \bar{d}d \rangle$ increases with light quark masses, from our $\mathcal{O}(p^4)$ expressions. The same ferromagnetic effect implies the increasing of the critical temperature of chiral restoration when increasing the pion mass, confirmed by ChPT calculations [45] and lattice simulations [46]. Finally, lattice results for the condensate [31] reveal a systematic increase of its absolute value with respect to direct estimates [29], reflecting again the same behaviour, since the pion masses used in the lattice remain above the physical values. The EM symmetry-breaking is of different nature as the mass, the former coming from vector-like interactions while the later being of scalar type. However, as we have discussed in the previous paragraphs, their symmetry breaking effects on certain observables are similar. Thus, the isovector part in (14) increases the masses of the charged mesons, according to (4) and (7), while the isovector and isoscalar both mix with the mass and contribute to $\langle \bar{q}q \rangle$. There are other arguments pointing in the same direction when EM interactions are switched on. At finite temperature, the EM pure thermal corrections to the condensate also increase its absolute value for any temperature [20]. On the other hand, the condensate increases under the influence of an external magnetic field eH , which can be also understood as the reduction of the free energy $\epsilon \sim m_q \langle \bar{q}q \rangle < 0$ (to leading order) needed to compensate for the EM energy increasing $\Delta\epsilon_{EM} \sim (eH)^2/2 > 0$ [47].

Our purpose here will be to explore the consequences of that EM ferromagnetic behaviour to leading order. If the vacuum response is ferromagnetic, certain bounds for the EM LEC involved should be satisfied. We will derive those bounds and show that they are independent of the low-energy scale and thus can be checked in terms of physical quantities. Next we will check that the bounds are satisfied for the different estimates available for the EM LEC, with more detail for the $SU(3)$ case in section IV D, where we also discuss the gauge independence of our results. This will provide a consistency check for the ferromagnetic behaviour.

An important comment is that we will discuss the ferromagnetic-like condition on the EM correction to $\langle \bar{q}q \rangle$ and, as explained above, the splitting of the $e = 0$ and $e \neq 0$ parts in QCD+EM is ambiguous [39, 40]. This does not affect the low-energy representation of the condensates, which can be written in terms of physical quantities such as meson masses and decay constants. However, we will test our bounds with the EM LEC estimates obtained by matching low-energy results with the underlying theory [34–37]. Therefore, we have to be consistent with the prescription for charge splitting followed in those works. This amounts to the direct separation of the $e = 0$ part, which still may contain residual charge and μ_0 QCD scale dependence through running parameters. The consequence is that the EM LEC thus defined are in general μ_0 -dependent, as discussed in [40]. Therefore, those estimates are reliable only if there is a stability range where the dependence on μ_0 is smooth and lies within the theoretical errors [34]. Actually, such stability range criteria are met for the LEC involved in our analysis (see section IV D). Within that range, our identification of the e^2 dependent part in $\langle \bar{q}q \rangle$ is consistent with the splitting scheme followed in those works. Actually, in that scheme F_0 is μ_0 -independent and the e^2 -dependent part of the μ_0 running of $B_0 m_{u,d}$ is the same as that of $m_{u,d}$ in perturbative QCD+EM [40]. Hence, it is consistent to assume that the ChPT leading order of $\langle \bar{q}q \rangle = -2B_0 F_0^2 + \dots$ does not introduce any residual e^2 dependence when performing the charge splitting in the low-energy expression. Using a different splitting prescription would lead in general to different bounds and a different definition of the EM LEC. For instance, an alternative splitting procedure is introduced in [40] by matching running parameters of the e^2 theory with those of the $e^2 = 0$ one at a given matching scale μ_1 . In that way, the EM part can be chosen as μ_0 independent but it depends on the matching scale μ_1 . We will not consider that splitting here, since there are no available theoretical estimates for the LEC defined with that scheme. The scale dependence for the LEC in either scheme is roughly expected to be logarithmic.

Having the above considerations in mind, and going back to the case of physical quark charges, we separate the EM corrections to the condensate through the ratio:

$$\begin{aligned}
\frac{\langle \bar{q}q \rangle^{e \neq 0}}{\langle \bar{q}q \rangle^{e=0}} &= 1 + 2[\mu_{\pi^0} - \mu_{\pi^\pm}] + e^2 \mathcal{K}_2^r(\mu) + \mathcal{O}(p^4) \\
&= 1 + e^2 \mathcal{K}_2^r(\mu) - \frac{4Ce^2}{F^4} \nu_{\pi^0} + \mathcal{O}(\delta_\pi^2) + \mathcal{O}(p^4),
\end{aligned} \tag{15}$$

with ν_i defined in (13) and where we have expanded in $\delta_\pi \equiv (M_{\pi^\pm}^2 - M_{\pi^0}^2)/M_{\pi^0}^2 \simeq 0.1$, as $\mu_{\pi^\pm} - \mu_{\pi^0} = \delta_\pi M_{\pi^0}^2 (\nu_{\pi^0}/F^2) + \mathcal{O}(\delta_\pi^2)$, which is numerically reliable and can be performed in addition to the chiral expansion, in order to simplify the previous expression.

We note that in $SU(2)$ and to this order, the ratio (15) is not only finite and scale-independent but it is also independent of the not-EM LEC, including the contact h_1, h_3 , and therefore free of ambiguities related to the condensate definition. In fact, this ratio is also independent of B_0 , unlike the individual quark condensates, which have only physical meaning and give rise to observables when multiplied by the appropriate quark masses, since $m_i B_0 \sim M_i^2$. In $SU(2)$, the above ratio does not depend either on the mass difference $m_d - m_u$, i.e., it depends only on the sum \hat{m} and its deviations from unity are therefore purely of EM origin. All these properties make the ratio (15) a suitable quantity to isolate the EM effects on the condensate. Thus, the ferromagnetic-like nature of the chiral order parameter $\langle \bar{q}q \rangle$, within its low-energy representation, would require that this ratio is greater than one, or equivalently to this order, $\partial \langle \bar{q}q \rangle / \partial e^2 \geq 1$. That condition leads to the following lower bound for the combination of EM LEC involved to this order, neglecting the $\mathcal{O}(\delta_\pi^2)$ in (15) which changes very little the numerical results:

$$5[k_5^r(\mu) + k_6^r(\mu)] + k_7^r \geq \frac{9C}{F^4} \nu_{\pi^0}. \tag{16}$$

We remark that the bound (16) is independent of the low-energy scale μ at which the LEC on the left-hand side are evaluated as long as the same scale is used on the right-hand side. Thus, it provides a well-defined low-energy prediction, expressed in terms of meson masses. The LEC on the left-hand side could be estimated by fitting low-energy processes or theoretically from the underlying theory, with all the related subtleties commented above.

The condition (16) and the corresponding ones for $SU(3)$ that will be derived in section IV D are obtained as a necessary condition that the LEC should satisfy if the QCD physical vacuum is ferromagnetic. This positivity condition on the quark condensate probes the vacuum by taking the mass derivatives (9) through the external source method so that the quark masses have to be kept different from zero and in that way the explicit symmetry breaking corrections are revealed in the condensate. If one is interested in the chiral limit, it must be taken only *after* differentiation, i.e., directly in eq.(15). In that case, it is not justified to perform the additional δ_π expansion in the charge because the $e = 0$ masses vanish and one would be left only with the μ_{π^\pm} contribution in the r.h.s. of (15), now with $M_{\pi^\pm}^2 = 2Ce^2/F^2$. That would actually give a larger negative value for the lower bound, coming from the smallness of the charged part of the pion mass (see details below) so that the bound in the chiral limit is less predictive. The fact that our bounds depend on quark masses is similar to other bounds on LEC obtained from QCD inequalities [48].

Nevertheless, the main physical interest is to test this bound for physical masses, using different estimates of the LEC in the literature. Thus, as a rough estimate, setting $\mu = M_\rho \simeq 770$ MeV and with the physical pion masses $M_{\pi^\pm} \simeq 139.57$ MeV, $M_{\pi^0} \simeq 134.97$ MeV, the bound (16) gives $5[k_5^r(M_\rho) + k_6^r(M_\rho)] + k_7^r \geq -6.32(5.62) \times 10^{-2}$ taking $F = 87.1(92.4)$ MeV. This is a bit more restrictive than the “natural” lower bound -6.93×10^{-2} for the above LEC combination, obtained by setting all of them to $-1/(16\pi^2)$. The chiral limit gives -0.17 (with the value of C discussed in section II A and $F = 87.1$ MeV) i.e., much less restrictive, as commented above. More detailed numerical analysis will be done for $SU(3)$ in section IV D.

On the other hand, the maximum value for the ratio (15) for the k_i^r within “natural” values is obtained by setting the three of them to $k_i^r(\mu = M_\rho) = 1/(16\pi^2)$, giving $\frac{\langle \bar{q}q \rangle^{e \neq 0}}{\langle \bar{q}q \rangle^{e=0}} = 1.0054$, which gives an idea of the size of this correction. We remark that the term proportional to ν_π on the ratio (15) comes directly from the dependence of the pion masses on e^2 , so that it parametrizes the corrections in the condensate coming from any source of pion mass increasing, not only the EM one. Therefore, the same result can be used in order to provide a rough estimate of lattice errors in the condensate due to including heavier pion masses as lattice artifacts. In some lattice algorithms like the staggered fermion one, the situation is very similar to the mass differences induced by the charge terms. In that formalism, the finite lattice spacing induces terms [49] that break explicitly the so called taste symmetry (four different quark species or “tastes” are introduced for every quark flavour) leaving a residual $U(1)$ symmetry, pretty much in the same way as the charge term in (2). As a rough estimate, we can then replace $2Ce^2/F^4$ by the corresponding δ_π from the lattice, obtained as the difference between the mass of the lightest lattice meson and the true pion mass. For a lattice pion mass of about 300 MeV, the ν_π term in (15) gives a correction of about 6%, which for a condensate value of (250 MeV)³ represents about (5 MeV)³, which is within the order of magnitude quoted in [31]. Nevertheless, it should be

taken into account that the staggered ChPT [49] has a much richer structure than the EM terms considered here and in particular there will be other operators contributing to the condensates at tree level, multiplied by the pertinent low-energy constants. If those constants are of natural size, we expect the size of the corrections to the condensate to remain within the range quoted above.

IV. THREE FLAVOUR QUARK CONDENSATES.

A. Results for light and strange condensates.

In the $SU(3)$ case, we derive to one loop in ChPT the light condensate $\langle \bar{q}q \rangle_l = \langle \bar{u}u + \bar{d}d \rangle$ and the strange one $\langle \bar{s}s \rangle$, taking into account both $m_u - m_d$ and $e \neq 0$ corrections. Apart from the kaon and eta loops, an important distinctive feature in this case is the appearance of the $\pi^0\eta$ mixing term with the tree-level mixing angle ε defined in (6) which is one of the sources of isospin-breaking corrections. The results we obtain for the condensates with all the corrections included are the following:

$$\begin{aligned} \langle \bar{q}q \rangle_l^{SU(3)} \equiv \langle \bar{u}u + \bar{d}d \rangle^{SU(3)} &= -2F^2 B_0 \left\{ 1 + \frac{8B_0}{F^2} [\hat{m} (2L_8^r(\mu) + H_2^r(\mu)) + 4(2\hat{m} + m_s)L_6^r(\mu)] + e^2 \mathcal{K}_{3+}^r(\mu) \right. \\ &\quad \left. - \frac{1}{3} (3 - \sin^2 \varepsilon) \mu_{\pi^0} - 2\mu_{\pi^\pm} - \mu_{K^0} - \mu_{K^\pm} - \frac{1}{3} (1 + \sin^2 \varepsilon) \mu_\eta + \mathcal{O}(p^4) \right\} \end{aligned} \quad (17)$$

$$\begin{aligned} \langle \bar{u}u - \bar{d}d \rangle^{SU(3)} &= 2F^2 B_0 \left\{ \frac{4B_0}{F^2} (m_d - m_u) (2L_8^r(\mu) + H_2^r(\mu)) - e^2 \mathcal{K}_{3-}^r(\mu) \right. \\ &\quad \left. + \frac{\sin 2\varepsilon}{\sqrt{3}} [\mu_{\pi^0} - \mu_\eta] + \mu_{K^\pm} - \mu_{K^0} \right\} + \mathcal{O}(p^2) \end{aligned} \quad (18)$$

$$\begin{aligned} \langle \bar{s}s \rangle &= -F^2 B_0 \left\{ 1 + \frac{8B_0}{F^2} [m_s (2L_8^r(\mu) + H_2^r(\mu)) + 4(2\hat{m} + m_s)L_6^r(\mu)] + e^2 \mathcal{K}_s^r(\mu) \right. \\ &\quad \left. - \frac{4}{3} [\mu_{\pi^0} \sin^2 \varepsilon + \mu_\eta \cos^2 \varepsilon] - 2[\mu_{K^\pm} + \mu_{K^0}] + \mathcal{O}(p^4) \right\} \end{aligned} \quad (19)$$

where we use the notation (13) and:

$$\begin{aligned} \mathcal{K}_{3+}^r(\mu) &= \frac{4}{9} [6(K_7 + K_8^r(\mu)) + 5(K_9^r(\mu) + K_{10}^r(\mu))], \\ \mathcal{K}_{3-}^r(\mu) &= \frac{4}{3} [K_9^r(\mu) + K_{10}^r(\mu)], \\ \mathcal{K}_s^r(\mu) &= \frac{8}{9} [3(K_7 + K_8^r(\mu)) + K_9^r(\mu) + K_{10}^r(\mu)]. \end{aligned} \quad (20)$$

Note that in some of the above terms we have preferred, for simplicity, to leave the results in terms of quark instead of meson masses. An important difference between the $SU(2)$ and $SU(3)$ cases is that now there are loop corrections in $\langle \bar{u}u - \bar{d}d \rangle$, where eta and pion loops enter through the mixing angle and kaon ones through the charged-neutral kaon mass difference. We have checked that the results are finite and scale-independent with the renormalization of the LEC given in Appendix A and that they agree with [3] for $e = 0$. Some unpublished results related to the $SU(2)$ and $SU(3)$ isospin-breaking condensates can also be found in [50].

Numerical results for the condensates to this order can be found in [26]. As explained in section II A, the effect of the K_i^r constants (20) in the condensates is not fully considered in that work, where the EM contributions are included through the corrections of Dashen's theorem [34], so that only the K_i^r combinations entering mass renormalization appear. We will use then our results with all corrections included to estimate the range of sensitivity to the K_i^r of the condensates, analyzing the possible relevance for the fit in [26]. Our results are displayed in Table I. As discussed in section II A, we take the same input values $L_6^r = 0$, $m_s/\hat{m} = 24$ as in [26] as well as the assumption $H_2^r = 2L_8^r$, and the output values of $B_0 m_{u,d,s}$, m_u/m_d , F , L_8^r from their main fit. In the second and third columns of Table I, we give the results with all the EM K_i^r fixed to their minimum and maximum "natural" values. Since the K_i^r appear all with positive sign in (20), the absolute values of the condensates obtained in this way are, respectively, lower and upper bounds within the natural range. We compare with the results quoted in [26] to the same $\mathcal{O}(p^4)$ order (fourth column) for their main fit and we also show for comparison the results in the isospin limit $e = 0$, $m_u = m_d$ (fifth column). Our results agree reasonably with [26], although we note that the values in that work lie outside the natural range

for the individual condensates. The largest relative corrections are about 2% for the light condensates and about 4% for the strange one. These isospin-breaking corrections are significant given the precision of the condensates quoted in [26]. On the other hand, the corrections lie within the error range quoted by lattice analysis [31]. In turn, note the bad ChPT convergence properties of the strange condensate, somehow expected since $\langle \bar{s}s \rangle$ is much more sensitive to the strange quark mass m_s than the light condensate [27] and therefore the large strange explicit chiral symmetry breaking m_s is responsible in this case for the spoiling of the ChPT series, based on perturbative mass corrections. For the vacuum asymmetry $\frac{\langle \bar{d}d \rangle}{\langle \bar{u}u \rangle} - 1$, the natural values band cover the result in [26], although the numerical discrepancies in that case are relatively larger, between 15% and 24% for the lower and upper limit of the K_i^r respectively. Recall that this quantity vanishes to leading order in ChPT, according to (18), so that we expect it to be more sensitive to the K_i^r correction, which in this case comes mostly from the combination $K_9^r + K_{10}^r$. Nevertheless, it is worth noting that the results [26] imply $\langle \bar{d}d \rangle / \langle \bar{u}u \rangle > 1$ and $\langle \bar{s}s \rangle / \langle \bar{u}u \rangle > 1$, both in disagreement with many sum rule estimates of the condensate ratios [29]. Not surprisingly, we have the same discrepancy, since we use the same ChPT approach and the same numerical constants, except for the K_i^r corrections. The discrepancy in the relatively large value of $\langle \bar{s}s \rangle / \langle \bar{u}u \rangle$ comes possibly from the bad convergence of the ChPT series for the strange condensate, which in addition is very sensitive to the choice of H_2^r [26]. The light condensates converge much better and although the sign of $\langle \bar{d}d \rangle / \langle \bar{u}u \rangle - 1$ is under debate, its magnitude is very small. In the latter case, our present calculation may become useful since the $K_9^r + K_{10}^r$ contribution may change the sign of the vacuum asymmetry, although its precise value to fit a given prediction for $\langle \bar{d}d \rangle / \langle \bar{u}u \rangle - 1$ would still be subject to the H_2^r value. For this reason, it is important to make predictions for quantities which are independent of this ambiguity, as we have done in section III and as we will do in section IV B, where the sum rule for condensate ratios will allow to make a more reliable estimate of the vacuum asymmetry including both sources of isospin breaking. Finally, we comment on the numerical differences by considering the more recent low-energy fits in [28]. Still keeping $H_2^r = 2L_8^r$, these new values for the low-energy parameters increase considerably the total and strange condensates, which to $\mathcal{O}(p^4)$ give $\langle \bar{q}q \rangle / (2B_0 F^2) \simeq -2.15$ and $\langle \bar{s}s \rangle / (B_0 F^2) = -2.79$. These higher values are mostly due to the much smaller $F = 65$ MeV, obtained in the main fit of [28] to accommodate a rather high L_4^r also with a large error $L_4^r = (0.75 \pm 0.75) \times 10^{-3}$ (an output result in [28]). With the previous value $F = 87.1$ MeV but keeping the rest of LEC and masses as in [28] we get $\langle \bar{q}q \rangle / (2B_0 F^2) \simeq -1.63$ and $\langle \bar{s}s \rangle / (B_0 F^2) = -1.99$. The EM corrections remain of the same size and therefore their relative effect is somewhat smaller. As commented before, $m_u \neq m_d$ isospin breaking is not implemented in those new fits and EM corrections are included only in kaon masses.

| | $K_{7-10}^r = -\frac{1}{16\pi^2}$ | $K_{7-10}^r = \frac{1}{16\pi^2}$ | [26] value $\mathcal{O}(p^4)$ | Isospin limit |
|---|-----------------------------------|----------------------------------|-------------------------------|---------------|
| $-\langle \bar{u}u \rangle_0 / (B_0 F^2)$ | 1.278 | 1.292 | 1.271 | 1.290 |
| $-\langle \bar{d}d \rangle_0 / (B_0 F^2)$ | 1.297 | 1.305 | 1.284 | 1.290 |
| $-\langle \bar{s}s \rangle_0 / (B_0 F^2)$ | 1.899 | 1.907 | 1.964 | 1.904 |
| $\frac{\langle \bar{d}d \rangle}{\langle \bar{u}u \rangle} - 1$ | 0.015 | 0.010 | 0.013 | 0 |

TABLE I: Results for quark condensates. We compare with the values of [26] to $\mathcal{O}(p^4)$ using the same set of low-energy parameters as in the main fit of that work, except the K_i^r , which we consider at their lower (second column) and upper (third column) “natural” values. We also quote the values in the isospin limit to the same chiral order.

B. Sum rule corrections

As noted in [3], for $m_u \neq m_d$ one can combine the isospin-breaking condensates into a sum rule relating the isospin asymmetry $\langle \bar{d}d \rangle / \langle \bar{u}u \rangle$ with the strange one $\langle \bar{s}s \rangle / \langle \bar{u}u \rangle$. Such relation is phenomenologically interesting because it does not include contact terms and hence is suitable for numerical estimates on the size of the isospin-breaking corrections. Our purpose in this section is to discuss the EM $e \neq 0$ contribution to that sum rule. To leading order in $m_u - m_d$ and e^2 we find:

$$\Delta_{SR} \equiv \frac{\langle \bar{d}d \rangle}{\langle \bar{u}u \rangle} - 1 + \frac{m_d - m_u}{m_s - \hat{m}} \left[1 - \frac{\langle \bar{s}s \rangle}{\langle \bar{u}u \rangle} \right] = \frac{m_u - m_d}{m_s - \hat{m}} \frac{1}{16\pi^2 F^2} \left[M_K^2 - M_\pi^2 + M_\pi^2 \log \frac{M_\pi^2}{M_K^2} \right] + e^2 \left[\frac{C}{8\pi^2 F^4} \left(1 + \log \frac{M_K^2}{\mu^2} \right) - \frac{8}{3} (K_9^r(\mu) + K_{10}^r(\mu)) \right]. \quad (21)$$

The last term, proportional to e^2 is scale-independent and is the charge correction to the result in [3]. With the numerical set we have been using, the $m_d - m_u$ term in the right hand side gives -3.3×10^{-3} , whereas the e^2 term gives -3.37×10^{-3} with $K_9^r(M_\rho) + K_{10}^r(M_\rho) = 1/(8\pi^2)$ and -9.4×10^{-4} with $K_9^r(M_\rho) + K_{10}^r(M_\rho) = 2.7 \times 10^{-3}$, the central value given in [34]. Therefore, the charge term above is of the same order as the pure QCD isospin correction and must be included when estimating the relative size of condensates through this sum rule. In fact, using the values quoted in [29] $m_u/m_d = 0.55$, $m_s/m_d = 18.9$ and $\langle \bar{s}s \rangle / \langle \bar{u}u \rangle = 0.66$, we get from (21) with physical pion and kaon masses:

$$-0.015 < \frac{\langle \bar{d}d \rangle}{\langle \bar{u}u \rangle} - 1 < -0.009,$$

where the lower (upper) bound corresponds to the natural value $K_9^r + K_{10}^r = +(-)1/(8\pi^2)$, while the value without considering the charge correction is -0.012 and the value quoted in [29] collecting various estimates in the literature is -0.009 . The inclusion of the charge corrections may then help to reconcile this sum rule with the different condensate estimates available. In fact, through this sum rule we see that ChPT is also compatible with the asymmetries $\langle \bar{d}d \rangle / \langle \bar{u}u \rangle$ and $\langle \bar{s}s \rangle / \langle \bar{u}u \rangle$ being both smaller than one (see our comments in section IV A). Note that the ferromagnetic-like arguments used in sections III and IV D cannot be applied to $\langle \bar{u}u - \bar{d}d \rangle$, which does not behave as an order parameter under chiral transformations, since it is not invariant under $SU_V(2)$. Finally, we recall that estimates based on the sum rule (21) are more precise than the ones we have made directly from the condensates in section IV A, since this sum rule is free of the H_2^r ambiguity.

C. Matching of low-energy constants

Our aim in this section is to explore the consequences of including the two sources of isospin breaking for the matching of the LEC involved in the condensates. For that purpose, we perform a $1/m_s$ expansion in the $SU(3)$ sum and difference condensates given in (17)-(18). Matching the $\mathcal{O}(1)$ and $\mathcal{O}(\log m_s)$ terms with the corresponding $SU(2)$ expressions in (10)-(11) yields the following relations between the LEC, for the sum and difference of condensates respectively:

$$2M_{\pi^0}^2 [l_3^r(\mu) + h_1^r(\mu)] + e^2 F^2 \mathcal{K}_2^r(\mu) = 2M_{\pi^0}^2 \left[16L_6^r(\mu) + 4L_8^r(\mu) + 2H_2^r(\mu) - \frac{\nu_\eta}{18} - \frac{\nu_{K^0}}{2} \right] + e^2 F^2 \left[\mathcal{K}_{3+}^r(\mu) - \frac{2C}{F^4} \nu_{K^0} \right], \quad (22)$$

$$B_0(m_d - m_u)h_3 - \frac{2e^2 F^2}{3} k_7 = B_0(m_d - m_u) \left[4L_8^r(\mu) + 2H_2^r(\mu) - \frac{\nu_\eta}{3} - \frac{\nu_{K^0}}{2} + \frac{1}{96\pi^2} \right] - \frac{2e^2 F^2}{3} \left[K_9^r(\mu) + K_{10}^r(\mu) - \frac{3C}{2F^4} \nu_{K^0} \right]. \quad (23)$$

In the above expressions, we have displayed the $SU(2)$ contribution on the left hand side and the $SU(3)$ ones on the right hand side, with $\mathcal{K}_2^r(\mu)$ and $\mathcal{K}_{3+}^r(\mu)$ given in (12) and (20). Note that the $1/m_s$ expansion has been implemented also in the tree level relations (7), so that $M_{\pi^0}^2 = (m_u + m_d)B_0 + \mathcal{O}(1/m_s)$, $M_{K^0}^2 = B_0 m_s + \mathcal{O}(1)$ and $M_\eta^2 = 4B_0 m_s/3 + \mathcal{O}(1)$. It is important to point out that the pion mass charge difference is not negligible in the $1/m_s$ expansion, and for that reason we keep M_{π^0} in (22). For kaons, it is justified to consider the charge contribution negligible against the dominant m_s term, so that at this order M_{K^\pm} and M_{K^0} are not distinguishable.

In the sum matching relation (22), the isospin corrections are not very significant. The mass difference $m_u - m_d$ does not appear in the neutral and kaon masses to leading order in $1/m_s$ and the charge correction, although being of the same chiral order as the $M_{\pi^0}^2$ term, numerically $e^2 F^2 / M_{\pi^0}^2 \simeq C e^2 / (F^2 M_{\pi^0}^2) \simeq 0.05$. However, in the difference matching (23), the $m_u - m_d$ corrections contribute on the same footing as the EM ones and are numerically comparable.

The above matching relations can be used directly for the approximated LEC (estimated theoretically or fitted to data) and for physical masses, since the difference with the tree level masses and LEC is hidden in higher orders. On the other hand, for the tree-level LEC, i.e. the ChPT $\mathcal{O}(p^4)$ lagrangian parameters, since they are formally independent of the light quark masses, we can just take the chiral limit $m_u = m_d = 0$ in the above expressions (22)-(23) and read off the corresponding matching of the e^2 contributions. Using the latter again in (22)-(23) gives

then independent relations between the tree-level LEC involved at $e^2 = 0$ and the EM ones. Doing so, the EM and not-EM LEC combinations decouple and the results are compatible with those obtained in [3] for $e = 0$ and in [43] for $e \neq 0$ (setting $m_u = m_d = 0$ from the very beginning):

$$\begin{aligned}
l_3^r(\mu) + h_1^r(\mu) &= 16L_6^r(\mu) + 4L_8^r(\mu) + 2H_2^r(\mu) - \frac{\nu_\eta}{18} - \frac{\nu_{K^0}}{2} \\
h_3 &= 4L_8^r(\mu) + 2H_2^r(\mu) - \frac{\nu_\eta}{3} - \frac{\nu_{K^0}}{2} + \frac{1}{96\pi^2} \\
5(k_5^r(\mu) + k_6^r(\mu)) &= 6(K_7 + K_8^r(\mu)) + 4(K_9^r(\mu) + K_{10}^r(\mu)) - \frac{3C}{F^4}\nu_{K^0} \\
k_7 &= K_9^r(\mu) + K_{10}^r(\mu) - \frac{3C}{2F^4}\nu_{K^0}
\end{aligned} \tag{24}$$

where the ν functions are evaluated exactly in the chiral limit, i.e, for $M_{K^0}^2 = B_0 m_s$ and $M_\eta^2 = 4B_0 m_s/3$, the first and third equation coming from (22) and the second and fourth from (23).

Our first conclusion is then that to this order of approximation, the formal matching of the condensates is consistent with the matching relations previously obtained. In other words, mass and charge terms can be separately matched. This would be no longer true at higher orders where for instance $e^2(m_u - m_d)$ contributions may appear.

Although the relations (22) and (23) reduce to (24) in the chiral limit for the tree-level LEC, it is better justified to use the original expressions (22)-(23) when dealing with physical meson masses and when the LEC are obtained either from phenomenological or theoretical analysis. The LEC obtained in that way are approximations to the lagrangian values and consequently they depend on mass scales characteristic of the approximation method used. For instance, the LEC obtained by phenomenological fits are sensitive to variations both in \hat{n} and in $m_u - m_d$ [26], in resonance saturation approaches they depend on vector meson masses [37, 38] which themselves depend on quark masses and in the NJL model some LEC like K_{10}^r depend on the scale where the quark masses are renormalized [34]. We do not expect large differences between using the general matching relation (22) or the first and third equations in (24), since the latter can be understood also as the $e = 0$ limit of the former and we have seen that this is numerically a good approximation. However, that is not so clear for (22) where the two isospin-breaking contributions are of the same order, both in the chiral expansion and numerically.

Finally, we can use the previous matching relations to estimate numerically the $SU(2)$ condensates in (10)-(11) without having to appeal to the values of the $SU(2)$ LEC. Doing so we obtain $-\langle \bar{u}u + \bar{d}d \rangle^{SU(2)}/B_0 F^2 \simeq (2.16, 2.18)$ and $\langle \bar{u}u - \bar{d}d \rangle^{SU(2)}/B_0 F^2 \simeq (0.014, 0.02)$ where we indicate in brackets the natural range of the EM LEC, to be compared to $-\langle \bar{u}u + \bar{d}d \rangle^{SU(3)}/B_0 F^2 \simeq (2.58, 2.6)$ and $\langle \bar{u}u - \bar{d}d \rangle^{SU(3)}/B_0 F^2 \simeq (0.013, 0.018)$ from Table I. The larger difference in $\langle \bar{u}u + \bar{d}d \rangle$ comes from the $\mathcal{O}(m_s)$ and $\mathcal{O}(m_s \log m_s)$ terms in the $1/m_s$ expansion, which were separated when doing the matching and which are absent in the condensate difference. In fact, the numerical contribution of those terms to $-\langle \bar{u}u + \bar{d}d \rangle^{SU(3)}/B_0 F^2$ is about 0.41, which explains perfectly the numerical differences and confirms the idea that in standard ChPT the light condensates calculated either in the $SU(2)$ or in the $SU(3)$ cases give almost the same answer near the chiral limit. This may be not the case in other scenarios of chiral symmetry breaking [32].

D. EM corrections and $SU(3)$ LEC bounds

We have seen in the $SU(2)$ case that the EM ratio given in (15) is a relevant physical quantity allowing to establish a constraint for the EM LEC based on explicit chiral symmetry breaking. The same argument applied to the $SU(3)$ case leads also to a constraint on the EM LEC obtained from the full condensate $\langle \bar{q}q \rangle = \langle \bar{u}u + \bar{d}d + \bar{s}s \rangle$, which behaves as an order parameter, being an isosinglet under $SU_V(3)$. In addition, we can still consider the light condensate $\langle \bar{q}q \rangle_l$ as the order parameter of chiral transformations of the $SU(2)$ subgroup, which in principle will lead to a different constraint. In fact, the latter is nothing but the constraint obtained in the $SU(2)$ case (16), once the equivalence between the LEC obtained in section IV C is used. As for the full condensate, it should be kept in mind that the large violations of chiral symmetry due to the strange quark mass may spoil our simple description of small explicit breaking. As commented above, this reflects in the large NLO contributions to the strange condensate, which in the standard ChPT framework depends strongly on m_s , unlike the light condensate. Therefore, the bounds of the LEC obtained for the full condensate are less trustable, since neglecting higher orders, say of $\mathcal{O}(e^2 m_s)$ is not so well justified for $\langle \bar{s}s \rangle$. Proceeding then as in section III, where the same prescription of charge splitting when comparing with QCD approaches is understood, we calculate the ratios:

$$\left[\frac{\langle \bar{q}q \rangle_l^{e \neq 0}}{\langle \bar{q}q \rangle_l^{e=0}} \right]^{SU(3)} = 1 + \frac{4e^2}{9} [6(K_7 + K_8^r(\mu)) + 5(K_9^r(\mu) + K_{10}^r(\mu))] - \frac{2Ce^2}{F^4} [2\nu_{\pi^\pm} + \nu_{K^\pm}] + \mathcal{O}(\delta_\pi^2, \delta_K^2) + \mathcal{O}(p^4) \quad (25)$$

$$\left[\frac{\langle \bar{q}q \rangle^{e \neq 0}}{\langle \bar{q}q \rangle^{e=0}} \right]^{SU(3)} = 1 + \frac{8}{9}e^2 [2(K_9^r(\mu) + K_{10}^r(\mu)) + 3(K_7 + K_8^r(\mu))] - \frac{8Ce^2}{3F^4} [\nu_{\pi^\pm} + \nu_{K^\pm}] + \mathcal{O}(\delta_\pi^2, \delta_K^2) + \mathcal{O}(p^4), \quad (26)$$

where, as in the $SU(2)$ case, we have expanded in δ_π and also in $\delta_K = (M_{K^\pm}^2 - M_{K^\pm, e=0}^2)/M_{K^\pm}^2 \simeq 0.008$, which allows to express the results in terms of the full π^\pm and K^\pm masses. Otherwise we should take into account that now $M_{\pi^\pm}(e=0) \neq M_{\pi^0}$, unlike the $SU(2)$ case, and $M_{K^\pm}(e=0) \neq M_{K^0}$, by terms of order $m_d - m_u$. This is important when using this result for numerical estimates, since, as discussed before, the separation of the $e=0$ correction to the masses is formally not unique. As in $SU(2)$, the ratios (25)-(26) are finite and independent of the scale μ , of B_0 and of the not-EM LEC, so they are free of contact ambiguities.

As in section III, we want to explore the consequences of the ferromagnetic nature of the physical QCD vacuum under explicit chiral symmetry breaking for the EM LEC. Here also the charge-mass crossed terms in the fourth order lagrangian (A2) give explicit breaking contributions to the quark condensate coming now from the isoscalar, isovector and strangeness part of the charge matrix. For physical quark charges, demanding that the ratios (25)-(26) are greater than one we obtain the following EM bounds, to leading order in the chiral expansion and in δ_π, δ_K :

$$\langle \bar{q}q \rangle_l \rightarrow 6(K_7 + K_8^r(\mu)) + 5(K_9^r(\mu) + K_{10}^r(\mu)) \geq \frac{9C}{2F^4} (2\nu_{\pi^\pm} + \nu_{K^\pm}) \quad (27)$$

$$\langle \bar{q}q \rangle \rightarrow 2(K_7 + K_8^r(\mu)) + 3(K_9^r(\mu) + K_{10}^r(\mu)) \geq \frac{3C}{F^4} (\nu_{\pi^\pm} + \nu_{K^\pm}) \quad (28)$$

We remark that these constraints are independent of the low-energy scale μ . It is also clear that the light bound (27) is nothing but the one obtained in the $SU(2)$ case (16) once the equivalence between the LEC given in the third equation of (24) is used. In Figure 1 we have plotted these two constraints in the $(K_7 + K_8^r) - (K_9^r + K_{10}^r)$ plane at $\mu = M_\rho$ and within the natural region. We have used the same numerical values for the tree level LO masses, C and F as in previous sections. We observe that the bound on the full condensate is more restrictive than the light one in that range. However, as we have commented above, it is also less trustable, due to the large distortion of the chiral invariant vacuum due to the strange mass. Both bounds give also a more restrictive condition than just the natural size.

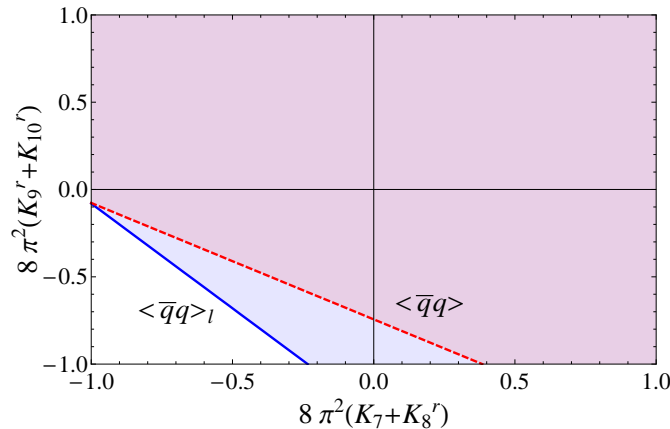


FIG. 1: Regions in the LEC space constrained by the bounds on the light condensate (27), above the full blue line, and the full one (28), above the dashed red line. The LEC are renormalized at $\mu = M_\rho$ and are plotted within the natural range.

Let us now check these bounds against some estimates of the K_i^r in the literature. We start with the sum rule approach for $K_{7\dots 10}^r$ in [37]. In that work, $K_7 = K_8^r(M_\rho) = 0$, but what is more relevant for us is that the combination

$K_9^r + K_{10}^r$ at any scale is gauge-independent despite being both K_9^r , K_{10}^r dependent on the gauge parameter ξ , as one can readily check from the explicit expressions given in [37] (c.f. their eqns. (94) and (95)). This is an interesting consistency check of our present bounds (27) and (28), which are gauge independent in addition to their μ low-energy scale independency commented previously, supporting their validity and predictive character. Numerically, the constant K_9^r could not be estimated in [37] due to the slow convergence of the integrals involved, but they provide a numerical estimate for $K_{10}^r(M_\rho) = 5.2 \times 10^{-3}$ at $\mu_0 = 0.7$ GeV and $\xi = 1$, for which we get $K_9^r(M_\rho) \geq -0.021$ from (27) and $K_9^r(M_\rho) \geq -0.015$ from (28). See our comments about the μ_0 scale dependence in section III.

In [35] resonance saturation gives $K_{7..10}^r(M_\rho) = 0$, which is compatible with our present bound. This is apparently incompatible with a previous value for $K_8^r(M_\rho) = (-4 \pm 1.7) \times 10^{-3}$ obtained in [4]. The possible reasons to explain this difference were discussed in [35]. That value for K_8^r is compatible with our bounds as long as $6K_7 + 5(K_9^r + K_{10}^r) \geq -0.05$ from (27) and $2K_7 + 3(K_9^r + K_{10}^r) \geq -0.02$ from (28).

In [34], based on large N_c and the NJL-model, the LEC estimates give $K_7 = 0$, $K_8(M_\rho) = (-0.8 \pm 2.0) \times 10^{-3}$ (K_7 and K_8 are $\mathcal{O}(1/N_c)$ suppressed) and $K_9^r(M_\rho) + K_{10}^r(M_\rho) = (2.7 \pm 1.0) \times 10^{-3}$, all of them at $\mu_0 = 0.7$ GeV. These values are also compatible with both bounds (27) and (28). We note that in [36], where the short-distance contributions are evaluated as in [34], the explicit expressions given for the LEC show again that K_7, K_8^r and $K_9^r + K_{10}^r$ are gauge independent. Furthermore, K_9^r and K_{10}^r , dominant for large N_c in that approach, show a rather large stability range in the μ_0 scale around $\mu_0 = 0.7$ GeV [34, 36]. Since $K_9^r + K_{10}^r$ is the only combination surviving for large N_c in our bounds, the comparison with those works is robust concerning the gauge and QCD scale dependence.

V. CONCLUSIONS

In this work we have carried out an analysis of strong and electromagnetic isospin-breaking corrections to the quark condensates in standard one-loop ChPT, providing their explicit expressions and studying some of their main phenomenological consequences for two and three light flavours.

Our results have allowed us to analyze the sensitivity of recent isospin-breaking numerical analysis of the condensates to considering all the EM LEC involved. The effect of those LEC is smaller for individual condensates than for the vacuum asymmetry, where they show up already in the leading order. These corrections lie within the error range quoted in lattice analysis. Our analysis can also be used to estimate corrections to the quark condensate coming from lattice artificially large meson masses.

We have shown that if EM explicit chiral symmetry breaking induces a ferromagnetic-like response of the physical QCD vacuum, as in the case of quark masses, one obtains useful constraints as lower bounds for certain combinations of the EM LEC, both in the two and three flavour sectors. We have explored the consequences of this behaviour for the ratios of $e \neq 0$ to $e = 0$ light and total quark condensates, which are free of contact-term ambiguities, and for a given convention of charge separation. The large ChPT corrections to the strange condensate make the constraints on the full condensate less reliable. In this context, we have discussed the different sources for EM explicit chiral symmetry breaking and isospin breaking terms, by considering formally arbitrary quark charges. Thus, there are chiral symmetry breaking terms proportional to the sum of charges squared, coming from crossed charge-mass contributions in the effective action, which show up in the vacuum expectation value. In accordance with the external source method, we keep the quark masses different from zero to account correctly for all the explicit symmetry breaking sources. The chiral limit can be taken at the end of the calculation. The bounds obtained are explicitly independent of the low-energy scale μ , providing then a complete and model-independent prediction at low energies. However, when this low-energy representation is compared with theoretical estimates based on QCD, one has to take into account that due to the convention used in the charge separation, the estimated LEC depend on the QCD renormalization scale μ_0 , as well as being gauge dependent. Our bounds are numerically compatible with those estimates, based on sum rules, resonance saturation and QCD-like models, within the stability range of μ_0 where those approaches are reliable. Furthermore, the LEC combinations appearing in our bounds are gauge independent. We believe that our results can be useful in view of the few estimates of the EM LEC in the literature.

We have found that the EM correction to the sum rule relating condensate ratios is of the same order as the previously calculated $e = 0$ one, and therefore must be taken into account when using this sum rule to estimate the relative size of quark condensates. We have actually showed that using the complete sum rule, which is also free of contact terms, yields a ChPT model-independent prediction for the vacuum asymmetry compatible with the results quoted in the literature.

Finally, we have performed a matching between the $SU(2)$ and $SU(3)$ condensates, including all isospin-breaking terms. Matching the sum and difference of light condensates gives rise to matching relations between the LEC involved, where EM and not-EM LEC enter on the same footing in the chiral expansion. These matching relations may be useful when working with physical masses and LEC estimated by different approximation methods. In the case of the sum, the charge contribution is numerically small with respect to the pion mass one, but in the difference

the two sources of isospin-breaking are comparable. Taking the chiral limit, EM and not-EM constants decouple and the matching conditions are compatible with previous works for the LEC in the lagrangian, which are defined in this limit.

Appendix A: Fourth order lagrangians and renormalization of the LEC

We collect here some results available in the literature and needed in the main text. To calculate the quark condensates to NLO one needs the $\mathcal{L}_{p^4+p^2e^2+e^4}$ lagrangians to absorb the divergences coming from loops with vertices from $\mathcal{L}_{p^2+e^2}$. We denote by a superscript $\bar{q}q$ the relevant terms in the lagrangian, which are those containing the quark mass matrix. For $SU(2)$ they are [5, 6]:

$$\begin{aligned}\mathcal{L}_{p^4}^{\bar{q}q} &= \frac{l_3}{16} \text{tr}[\chi(U + U^\dagger)]^2 + \frac{1}{4}(h_1 + h_3) \text{tr}[\chi^2] + \frac{1}{2}(h_1 - h_3) \det(\chi), \\ \mathcal{L}_{p^2e^2}^{\bar{q}q} &= F^2 \left(k_5 \text{tr}[\chi(U + U^\dagger)] \text{tr}[Q^2] + k_6 \text{tr}[\chi(U + U^\dagger)] \text{tr}[QUQU^\dagger] + k_7 \text{tr}[(\chi U^\dagger + U\chi)Q + (\chi U + U^\dagger\chi)Q] \text{tr}[Q] \right),\end{aligned}\tag{A1}$$

where $\chi = 2B_0\mathcal{M}$, whereas for $SU(3)$ [4]:

$$\begin{aligned}\mathcal{L}_{p^4}^{\bar{q}q} &= L_6 \text{tr}[\chi(U + U^\dagger)]^2 + L_8 \text{tr}[\chi U \chi U + \chi U^\dagger \chi U^\dagger] + H_2 \text{tr}[\chi^2] \\ \mathcal{L}_{p^2e^2}^{\bar{q}q} &= F^2 \left(K_7 \text{tr}[\chi(U + U^\dagger)] \text{tr}[Q^2] + K_8 \text{tr}[\chi(U + U^\dagger)] \text{tr}[QUQU^\dagger] + K_9 \text{tr}[(\chi U + U^\dagger\chi + \chi U^\dagger + U\chi) Q^2] + \right. \\ &\quad \left. K_{10} \text{tr}[(\chi U + U^\dagger\chi) QU^\dagger QU + (\chi U^\dagger + U\chi) QUQU^\dagger] \right),\end{aligned}\tag{A2}$$

and $\mathcal{L}_{e^4}^{\bar{q}q} = 0$ for both cases.

In order to renormalize the meson loops it is necessary to separate the low energy constants appearing in the NNLO lagrangian in finite and divergent parts. The renormalization of the LEC involved in the calculation of the $SU(2)$ condensates is given by [2, 5, 6]:

$$\begin{aligned}l_i &= l_i^r(\mu) + \gamma_i \lambda, \\ h_i &= h_i^r(\mu) + \delta_i \lambda, \\ k_i &= k_i^r(\mu) + \sigma_i \lambda,\end{aligned}$$

with $\gamma_3 = -\frac{1}{2}$, $\delta_1 = 2$, $\delta_3 = 0$, and $\sigma_5 = -\frac{1}{4} - \frac{1}{5}Z$, $\sigma_6 = \frac{1}{4} + 2Z$ and $\sigma_7 = 0$, for physical quark charges $e_u = 2e/3$, $e_d = -e/3$, being $Z := \frac{C}{F^4}$. The part that diverges in $d = 4$ dimensions is isolated from the loop integrals and is expressed as

$$\lambda = \frac{\mu^{d-4}}{16\pi^2} \left(\frac{1}{d-4} - \frac{1}{2} [\log 4\pi + \Gamma'(1) + 1] \right),$$

where $-\Gamma'(1)$ is the Euler constant. As for the $SU(3)$ ones, we have [3, 4]

$$\begin{aligned}L_i &= L_i^r(\mu) + \Gamma_i \lambda, \\ H_i &= H_i^r(\mu) + \Delta_i \lambda, \\ K_i &= K_i^r(\mu) + \Sigma_i \lambda,\end{aligned}$$

with $\Gamma_6 = \frac{11}{144}$, $\Gamma_8 = \frac{5}{48}$, $\Delta_2 = \frac{5}{24}$, and $\Sigma_7 = 0$, $\Sigma_8 = Z$, $\Sigma_9 = -\frac{1}{4}$, $\Sigma_{10} = \frac{1}{4} + \frac{3}{2}Z$.

Acknowledgments

We are grateful to J.R.Peláez and E.Ruiz Arriola for useful comments. R.T.A would like to thank Buenaventura Andrés López for invaluable advice. Work partially supported by the Spanish research contracts FPA2008-00592,

-
- [1] S. Weinberg, *Physica* **A96**, 327 (1979).
- [2] J. Gasser and H. Leutwyler, *Annals Phys.* **158**, (1984) 142.
- [3] J. Gasser and H. Leutwyler, *Nucl. Phys. B* **250**, 465 (1985).
- [4] R. Urech, *Nucl. Phys. B* **433**, 234 (1995).
- [5] U. G. Meissner, G. Muller and S. Steininger, *Phys. Lett. B* **406**, 154 (1997) [Erratum-ibid. *B* **407**, 454 (1997)].
- [6] M. Knecht and R. Urech, *Nucl. Phys. B* **519**, 329 (1998).
- [7] A. Nehme and P. Talavera, *Phys. Rev. D* **65**, 054023 (2002).
- [8] B. Kubis and U. G. Meissner, *Nucl. Phys. A* **699**, 709 (2002); *Phys. Lett. B* **529**, 69 (2002).
- [9] M. Knecht and A. Nehme, *Phys. Lett. B* **532**, 55 (2002).
- [10] J. Schweizer, *Eur. Phys. J. C* **36**, 483 (2004).
- [11] G. Ecker, G. Muller, H. Neufeld and A. Pich, *Phys. Lett. B* **477**, 88 (2000).
- [12] C. Hanhart, B. Kubis and J. R. Pelaez, *Phys. Rev. D* **76**, 074028 (2007).
- [13] A. Nehme, *Phys. Rev. D* **69**, 094012 (2004). *Phys. Rev. D* **70**, 094025 (2004).
- [14] G. Colangelo, J. Gasser and A. Rusetsky, *Eur. Phys. J. C* **59**, 777 (2009).
- [15] A. Rusetsky, *PoS C* **D09**, 071 (2009) [arXiv:0910.5151 [hep-ph]].
- [16] G. Ecker, J. Gasser, A. Pich and E. de Rafael, *Nucl. Phys. B* **321**, 311 (1989).
- [17] H. Neufeld and H. Rupertsberger, *Z. Phys. C* **71**, 131 (1996).
- [18] J. Schweizer, *JHEP* **0302**, 007 (2003).
- [19] C. Vafa and E. Witten, *Nucl. Phys. B* **234**, 173 (1984).
- [20] A. Gómez Nicola and R. Torres Andrés, *Phys. Rev. D* **83**, 076005 (2011).
- [21] M. Gell-Mann, R. J. Oakes and B. Renner, *Phys. Rev.* **175**, 2195 (1968).
- [22] C. Ditsche, B. Kubis and U. G. Meissner, *Eur. Phys. J. C* **60**, 83 (2009).
- [23] M. Gell-Mann, Caltech Report CTSL-20 (1961). S. Okubo, *Prog. Theor. Phys* **27** (1962) 949.
- [24] R. F. Dashen, *Phys. Rev.* **183**, 1245 (1969).
- [25] T. Das, G. S. Guralnik, V. S. Mathur, F. E. Low and J. E. Young, *Phys. Rev. Lett.* **18**, 759 (1967).
- [26] G. Amoros, J. Bijnens and P. Talavera, *Nucl. Phys. B* **602**, 87 (2001).
- [27] G. Amoros, J. Bijnens and P. Talavera, *Nucl. Phys. B* **585** (2000) 293 [Erratum-ibid. *B* **598** (2001) 665].
- [28] J. Bijnens and I. Jemos, arXiv:1103.5945 [hep-ph].
- [29] S. Narison in “QCD as a theory of hadrons: from partons to confinement”, Cambridge University Press 2004, arXiv:hep-ph/0202200.
- [30] B. L. Ioffe, *Prog. Part. Nucl. Phys.* **56**, 232 (2006).
- [31] G. Colangelo *et al.*, arXiv:1011.4408 [hep-lat].
- [32] S. Descotes-Genon and J. Stern, *Phys. Lett. B* **488**, 274 (2000). B. Moussallam, *Eur. Phys. J. C* **14**, 111 (2000); *JHEP* **0008**, 005 (2000).
- [33] G. Colangelo, J. Gasser and H. Leutwyler, *Phys. Rev. Lett.* **86**, 5008 (2001).
- [34] J. Bijnens and J. Prades, *Nucl. Phys. B* **490**, 239 (1997).
- [35] R. Baur and R. Urech, *Nucl. Phys. B* **499**, 319 (1997).
- [36] A. Pinzke, arXiv:hep-ph/0406107.
- [37] B. Moussallam, *Nucl. Phys. B* **504**, 381 (1997).
- [38] B. Ananthanarayan and B. Moussallam, *JHEP* **0406**, 047 (2004).
- [39] J. Bijnens, *Phys. Lett. B* **306**, 343 (1993).
- [40] J. Gasser, A. Rusetsky and I. Scimemi, *Eur. Phys. J. C* **32**, 97 (2003).
- [41] J. Gasser, V. E. Lyubovitskij, A. Rusetsky and A. Gall, *Phys. Rev. D* **64**, 016008 (2001).
- [42] H. Jallouli and H. Sazdjian, *Phys. Rev. D* **58**, 014011 (1998) [Erratum-ibid. *D* **58**, 099901 (1998)].
- [43] C. Haefeli, M. A. Ivanov and M. Schmid, *Eur. Phys. J. C* **53**, 549 (2008).
- [44] M. Knecht, private communication.
- [45] P. Gerber and H. Leutwyler, *Nucl. Phys. B* **321**, 387 (1989).
- [46] M. Cheng *et al.*, *Phys. Rev. D* **81**, 054504 (2010).
- [47] I. A. Shushpanov and A. V. Smilga, *Phys. Lett. B* **402**, 351 (1997). N. O. Agasian, *Phys. Lett. B* **488**, 39 (2000). T. D. Cohen, D. A. McGady and E. S. Werbos, *Phys. Rev. C* **76**, 055201 (2007).
- [48] J. Comellas, J. I. Latorre and J. Taron, *Phys. Lett. B* **360**, 109 (1995).
- [49] C. Aubin and C. Bernard, *Phys. Rev. D* **68**, 034014 (2003).
- [50] A. Nehme, *La Brisure d’Isospin dans les Interactions Meson-Meson à Basse Energie*, PhD thesis, CPT Marseille, July 2002.