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Inverse amplitude method and Adler zeros

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The inverse amplitude method is a powerful unitarization technique to enlarge the energy applicability region of effective Lagrangians. It has been widely used to describe resonances in hadronic physics, combined with chiral perturbation theory, as well as in the strongly interacting symmetry breaking sector. In this work we show how it can be slightly modified to also account for the subthreshold region, incorporating correctly the Adler zeros required by chiral symmetry and eliminating spurious poles. These improvements produce negligible effects on the physical region.

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I. INTRODUCTION

Effective field theories provide a systematic and model independent approach to systems whose symmetries and low-energy degrees of freedom are known but whose description in terms of an underlying fundamental quantum field theory is out of reach. The two cases of interest for this work are, on the one hand, chiral perturbation theory (ChPT) [1] which describes effectively the low-energy dynamics of hadrons, inaccessible to perturbative QCD calculations in terms of quarks and gluons; and, on the other hand, the effective description of the strongly interacting electroweak symmetry breaking sector (SISBS) [2], whose underlying fundamental theory remains unknown.

Both cases have in common the existence of a spontaneous symmetry breaking of a global chiral $SU(N)_L \times SU(N)_R$ group down to an $SU(N)_{L+R}$ group. The Goldstone theorem implies the presence of $N^2 - 1$ massless Goldstone bosons (GB) in the particle spectrum. These GB thus become the relevant degrees of freedom of the system below a chiral scale Λ_χ, where an effective chiral Lagrangian can be built in terms of just those GB as the most general derivative expansion consistent with the known symmetries. Note that there is a well-defined power counting so that divergences generated in loop diagrams that use vertices up to a given order can be renormalized into the coefficients of higher order terms. In this sense these effective theories are renormalizable order by order.

In addition, in both examples, there are small explicit symmetry breaking terms. For QCD the relatively small masses of the lightest three quarks provide a mass $M \ll \Lambda_\chi$ to the GB, identified with the pions, kaons, and etas, so that the effective approach becomes, in practice, a derivative and mass expansion. For the SISBS, there is a local $SU(2)_L \times U(1)$ symmetry whose gauge bosons couple to the "would-be GB" that, in a unitary gauge, disappear from the spectrum, giving rise to gauge boson longitudinal components that thus acquire a mass M_V. In this way the $SU(2)_L \times U(1)$ gauge symmetry is spontaneously, but not explicitly, broken to the electromagnetic group $U(1)_{\text{EM}}$. In this case, in addition to the derivative expansion, one expands also in terms of electroweak coupling constants g and g'. The so-called equivalence theorem (ET) [3] states that at high energies (in R_ξ gauges, and to leading order in momenta over M_V and g and g') amplitudes involving longitudinal gauge bosons can be calculated as if they were GB, which, being pseudoscalars, are much easier to handle. Although this is a high-energy limit, there is a generalization to the effective Lagrangian formalism [4] that, for practical purposes, allows us to identify, up to the difference in scales, the formalisms of $SU(2)$ ChPT and the SISBS, and therefore, from now on we will be referring to ChPT, but keeping in mind that our results have a straightforward translation to the SISBS.

Both cases above are examples of strongly interacting systems whose most salient feature is the saturation of unitarity and the associated resonant states, which lie beyond the reach of perturbative energy expansions. Thus, it may seem that the use of effective Lagrangians is limited to energies below those resonances, whose effects are encoded in the values of higher order effective coupling constants. However, since unitarity fixes the imaginary part of *inverse* partial waves in the elastic region, the effective Lagrangian approach is also useful in the resonant region, for instance, used inside a dispersion relation, in order to obtain the rest of the amplitude. These techniques are known as unitarization methods, and reproduce simultaneously the low-energy expansion and the lightest resonances without including them explicitly in the Lagrangian. The great advantage is that such resonances and their properties are generated without prejudices about their nature or their existence. Also, since the Lagrangian symmetries and some features of the effective constants can be directly related to the underlying theory, like QCD, one can study the properties of these states based on more fundamental grounds. One of the most extensively used unitarization techniques is the inverse amplitude method (IAM) [5–8], which uses the fully renormalized effective chiral expansion, without any further approximation and without introducing any other spurious parameter, but just the effective constants up to a given order. Within hadronic physics, it generates the well-known vector resonances and the more controversial scalars using parameters consistent with one-loop ChPT, allowing one to establish their different nature in terms of their dependence on the number of colors. The IAM has also been extended to two loops [7,9], the finite temperature formalism [10,11], and to the pion-nucleon sector [12]. Within the SISBS [13], it provides the prediction of the general resonance spectrum and how well it could be detected at the CERN LHC.

However, it is known [7,14] that the IAM fails to reproduce correctly the Adler zeros that appear in the subthreshold region of some partial waves as a consequence of chiral symmetry [15]. Furthermore, it generates spurious poles, or "ghosts," thus questioning its reliability in that region, and also casting some doubts about the robustness of the results in the physical region if such structures were properly accounted for.

The aim of this paper is to show that a very simple modification of the IAM can correctly take into account those zeros and ghosts. This modification corresponds to terms that had been neglected in the original dispersive derivation of the IAM since they contribute to higher orders in the chiral expansion. Actually, we will check explicitly that such a procedure is justified in the physical region, where these modifications yield negligible contributions, thus showing the robustness of the standard IAM. However, apart from improving the IAM consistency, these terms are essential in the subthreshold region which is relevant to study the effect of chiral symmetry restoration on resonances [11], or their dependence on quark masses [16].

In the next section we will thus revisit the standard IAM derivation from dispersive theory, where Adler zeros are neglected, paying special attention to the role of those zeros in the subthreshold region. In Sec. III we will present a naive way of extending the IAM amplitude, without using dispersion relations, that solves the caveats in that region. Section IV will show a dispersive derivation of a more general modified amplitude, for the case of equal masses (e.g. $\pi\pi$ scattering). The case of unequal masses, like in πK scattering, deserves a separate discussion, for the reasons explained in Sec. V. Finally, in Sec. VI we will present some numerical results for the modified amplitudes.

II. THE INVERSE AMPLITUDE METHOD

A. Dispersive derivation

The *one-channel* IAM [5–7,9] can be obtained by using ChPT up to a given order inside a dispersion relation. To simplify the discussion, let us first consider pion-pion scattering, partial wave amplitudes of definite isospin I and angular momentum J, although for brevity we will simply call them t, whose analytic structure in the s plane is shown in Fig. 1. The physical right-hand cut comes from unitarity and starts at threshold $s_{\rm th}$, while the left-hand cut comes from the t, u channels.

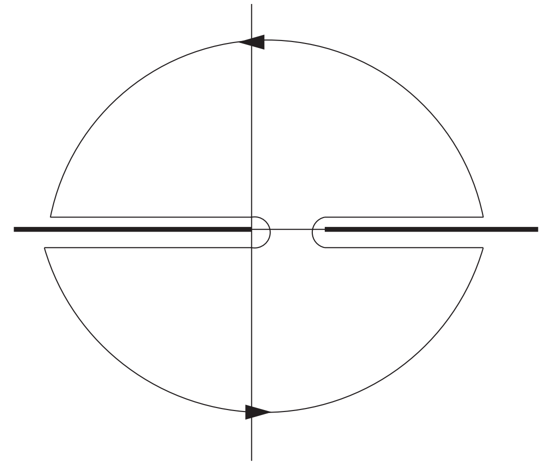

FIG. 1. Analytic structure of pion-pion scattering partial waves and the integration contour used to obtain their dispersion relations.

The inverse of $t(s)$ has the same analytic structure, except for the possible presence of poles corresponding to zeros of $t(s)$. In particular, chiral symmetry requires the existence of the so-called Adler zeros below threshold in scalar partial waves [1,15], the case of interest in this work, which we denote by s_A. Hence it is then possible to write the following dispersion relation for the inverse amplitude:

$$\frac{1}{t(s)} = \frac{1}{t(z_0)} + \frac{s - z_0}{\pi} \int_{s_{th}}^{\infty} dz \frac{\mathrm{Im}1/t(z)}{(z - s)(z - z_0)} + LC(1/t) + PC(1/t). \quad (1)$$

Here and in the following, we will simply write z instead of $z + i\epsilon$ with $\epsilon > 0$ for the imaginary parts inside the cut integrals. Note that we have explicitly written the integral over the right-hand cut (or physical cut, extending from threshold $s_{\rm th}$ to infinity), but we have shortened by LC the equivalent expression for the left cut (from $-\infty$ to 0) and the pole contribution. We could proceed in the same way with other cuts, if present, as in the πK case. In addition we have made one subtraction to ensure convergence, at a point $z_0 \neq s_A, s_2$.

We now recall that unitarity, for physical values of s in the elastic region, implies

$$\mathrm{Im}\, t(s) = \sigma(s)|t(s)|^2 \Rightarrow \mathrm{Im}\frac{1}{t(s)} = -\sigma(s), \quad (2)$$

where $\sigma(s) = 2p_{CM}/\sqrt{s}$. Let us remark that since $\mathrm{Im}1/t = -\sigma$ we know exactly the integrand over the elastic cut.

In contrast, ChPT amplitudes are obtained as a series expansion $t(s) = t_2(s) + t_4(s) + \ldots$ where $t_2(s) = O(p^2)$, $t_4(s) = O(p^4)$, and p stands for the pion mass or momentum. Therefore elastic unitarity is not satisfied exactly, but only order by order as follows:

$$\mathrm{Im}\, t_2(s) = 0, \qquad \mathrm{Im} t_4(s) = \sigma(s)|t_2(s)|^2, \ldots. \quad (3)$$

Let us also note that $t_2(s)$ is a pure polynomial and has no cuts, and we can thus write a trivial dispersion relation for $1/t_2(s)$ that reads

$$\frac{1}{t_2(s)} = \frac{1}{t_2(z_0)} + PC(1/t_2), \tag{4}$$

where now the pole contribution is due to s_2, the Adler zero of t_2. In addition, except for the poles, the function $t_4(s)/t_2^2(s)$ has the same analytic cut structure of $1/t$, and, using Eqs. (2) and (3), over the physical cut we find

$$\mathrm{Im}\,\frac{t_4(s)}{t_2^2(s)} = \sigma(s) = -\mathrm{Im}\frac{1}{t(s)}. \tag{5}$$

We can therefore write another dispersion relation similar to that of $1/t(s)$, but for $t_4(s)/t_2^2(s)$,

$$\frac{t_4(s)}{t_2(s)^2} = \frac{t_4(z_0)}{t_2(z_0)^2} + \frac{s-z_0}{\pi}\int_{s_{\rm th}}^{\infty} dz \frac{\mathrm{Im} t_4(z)/t_2(z)^2}{(z-s)(z-z_0)} + LC(t_4/t_2^2) + PC(t_4/t_2^2), \tag{6}$$

where the pole contribution, once again, is due to the Adler zero of t_2.

We are now going to relate the dispersion relation for $1/t(s)$ with that for $t_4(s)/t_2^2(s)$. As we already commented, $\mathrm{Im}1/t(s) = -\mathrm{Im}t_4/t_2^2(s)$ on the right cut, and therefore the integrals over the physical cuts for $1/t(s)$ and $t_4/t_2^2(s)$ *are exactly opposite to each other*. In addition, using ChPT we find that $LC(1/t) \simeq -LC(t_4/t_2^2)$, which is a well-justified approximation, since, due to the subtraction, the integrand of LC is weighted at low energies, precisely where ChPT applies. Finally, we have to evaluate the subtraction constant in Eq. (1), and this can only be done as long as z_0 is in the low-energy region, where it is perfectly justified to use ChPT to find $1/t(z_0) \simeq 1/t_2(z_0) - t_4(z_0)/t_2(z_0)^2$. However, note that this expansion is a very bad approximation for z_0 near s_2 or s_A, where t_2 and t vanish. Therefore, we only know how to relate those dispersion relations for subtraction points z_0 in the low-energy region, but sufficiently far from the Adler zeros. In Sec. VI we will check that the results have very little sensitivity to the choice of z_0 as long as it lies in this region. When this is the case, using Eqs. (1), (4), and (6) we can write $1/t$ as

$$\frac{1}{t(s)} \simeq \frac{1}{t_2(s)} - \frac{t_4(s)}{t_2(s)^2} - PC(1/t_2) + PC(t_4/t_2^2) + PC(1/t). \tag{7}$$

The standard dispersive derivation of the IAM [6,7] simply neglected the sum of pole contributions to arrive at

$$t^{\rm IAM}(s) \simeq \frac{t_2^2(s)}{t_2(s) - t_4(s)}, \tag{8}$$

thus providing an elastic amplitude that satisfies unitarity and has the correct low-energy expansion of ChPT up to the order we have used. When such amplitude is chirally expanded to $O(p^4)$, this implies [7] that the total pole contribution to $t(s)$, even without its explicit calculation, has to be $O(p^6)$. In Secs. III, IV, and V we will calculate it explicitly to arrive at the modified IAM.

B. IAM properties and its naive derivation

Incidentally, we can recast Eq. (2) as

$$t(s) = \frac{1}{\mathrm{Re}t^{-1}(s) - i\sigma(s)}, \tag{9}$$

and thus it seems that the IAM can also be derived in a much simpler way by replacing $\mathrm{Re}t^{-1}$ by its $O(p^4)$ ChPT expansion $\mathrm{Re}t^{-1} = (t_2 - \mathrm{Re}t_4)/t_2^2$. This is the way unitarization methods are usually presented, although it makes no use of the strong analytic constraints of amplitudes, which are indeed absent in Eq. (9). Furthermore, the criticism is immediately raised that the ChPT expansion cannot be used at high energies.

However, note that the dispersive one-channel IAM derivation in the previous section imposes analyticity in the form of two dispersive integrals and makes use of the ChPT expansion (up to one loop in this case) for the subtraction constants and the left cut. The use of ChPT for the subtraction constants is well justified, since it is used at $s = z_0$ in the low-energy region. Since the integral extends to infinity, ChPT may seem a worse approximation for the left cut or possible inelastic cuts; however, the subtraction suppresses the high-energy region that contributes little, as explained above. Furthermore, when the IAM is used for physical values of s above the physical threshold, the left cut is damped by an additional $1/(z-s)$ factor, and not only the high-energy part, but its whole contribution is rather small. There are no model dependent assumptions, but just *approximations to a given order*, and therefore the approach provides an elastic amplitude that satisfies unitarity and has the correct ChPT expansion up to that given order. It is also straightforward to extend it to higher orders [7,9].

Moreover, Eq. (2) is only valid on the physical cut, whereas the dispersive derivation allows us to consider the amplitude in the complex plane and, in particular, to look for poles of the associated resonances. Actually, already ten years ago [7] the poles for the $\rho(770)$, $K^*(892)$, and most interestingly, the controversial σ [also called $f_0(600)$] were generated without any model dependent assumptions.

Obviously, and contrary to wide belief, the IAM *contains a left cut* and *respects crossing symmetry* up to, of course, the order in the ChPT expansion that has been used. The confusion may come from the fact that the IAM has also been applied in a coupled channel formalism, for which *there is still no dispersive derivation*, and has been frequently used by approximating further the amplitudes neglecting tadpole and left cut terms [17]. But, strictly speaking, that would not be the full IAM, which definitely has a left cut.

C. IAM caveats: Adler zeros and ghosts

To end this section, we recall that in the dispersive derivation the sum of pole terms (PC) in Eq. (7) is neglected, since it yields a higher order contribution [7]. However, this leads to a couple of problems related to the presence of the Adler zeros below threshold in the scalar waves. Let us first note that, despite we have used in the IAM the ChPT expansion up to next to leading order (NLO), due to the $t_2(s)^2$ factor in the numerator, it vanishes at s_2, which is only the leading order (LO) chiral approximation to the exact Adler zero s_A. In addition, it is a double zero instead of a single zero. This, as we will see, leads to the appearance of a spurious pole on the real axis close to the Adler zero. As a consequence, the predictions of the standard IAM below threshold and, in particular, around the Adler zero position are not reliable.

In fact, note that, since the interval $s \in (0, s_{\rm th})$ lies within the convergence region of the chiral expansion and t_2 changes sign at $s = s_2$, $t_2 - t_4$ turns out to be, for the cases of interest here, a continuous, monotonically increasing (or decreasing) function in $(0, s_{\rm th})$ that changes sign from $s = 0$ to $s = s_{\rm th}$. Therefore, there is one single point $\tilde{s}$ where the denominator of (8) vanishes, which, as long as $t_4(s_2) \neq 0$, produces the spurious pole below

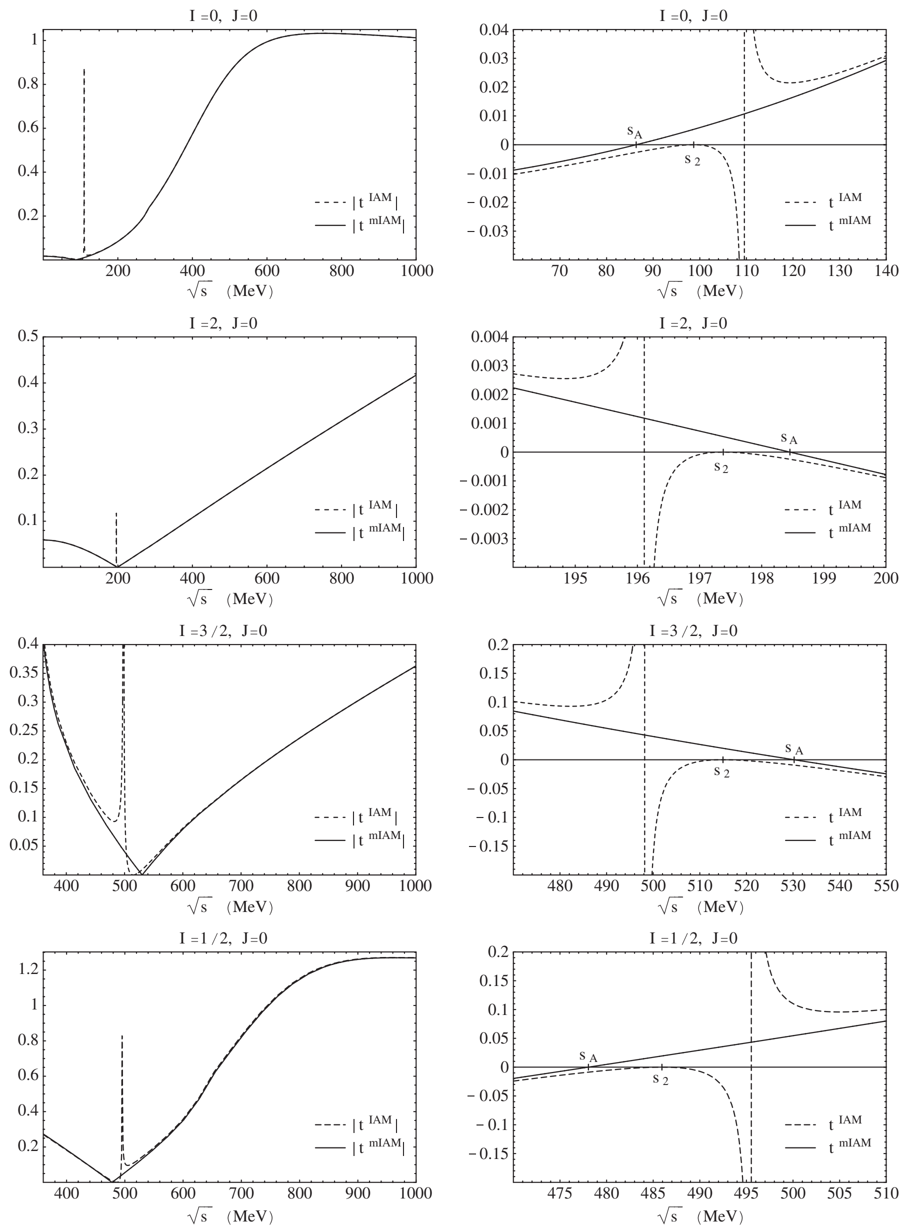

FIG. 2. Comparison between the IAM and the mIAM for different isospin I partial waves of $\pi\pi$ and πK in the scalar $J = 0$ channel. The left column covers the region from the left cut up to 1 GeV. The only significant differences between both methods occurs in the region around the Adler zeros of each partial wave, which is shown in detail in the right column.

threshold discussed above, i.e., a nonobserved $\pi\pi$ bound state.

For instance, let us consider $I = J = 0$ $\pi\pi$ scattering (for the values of the low-energy constants quoted in Sec. VI). This is an attractive channel, so that $t_2 - t_4$ is positive at threshold. Since that function is negative at $s = 0$ and at $s = s_2$, $\tilde{s} > s_2$ in that case, as shown in the upper right panel of Fig. 2, we find $\tilde{s} \simeq (110\ \text{MeV})^2$ and $s_2 \simeq (99\ \text{MeV})^2$. In the $I = 2$, $J = 0$ channel, which is repulsive, $t_2 - t_4$ is positive at $s = 0$ and negative at $s = s_2$ $[t_4(s_2) > 0]$ so that $\tilde{s} < s_2$. In that channel, $\tilde{s} \simeq (196\ \text{MeV})^2$ and $s_2 \simeq (198\ \text{MeV})^2$ as seen also in Fig. 2.

Let us also point out that, together with the first Riemann sheet spurious pole just discussed, the IAM has a companion pole in the second Riemann sheet below threshold. For instance, the second-sheet IAM for $\pi\pi$ scattering reads

$$t(s)^{\text{IAM,IIsheet}} = \frac{t_2(s)^2}{t_2(s) - t_4(s) - 2\tilde{\sigma}(s)t_2^2(s)} \qquad (10)$$

where $\tilde{\sigma}(s) = i\sigma(s - i0^+) = \sqrt{4m_\pi^2/s - 1}$ for $0 < s < 4m_\pi^2$. Thus, if we are dealing with an attractive channel, like the 00 one, the denominator of (10) is positive near threshold (dominated by $t_2 > 0$) and diverges to minus infinity as $s \to 0^+$, so that it must have at least one zero, which again generates a pole if $t_4(s_2) \neq 0$. Since $-\tilde{\sigma}t_2^2$ in Eq. (10) is also an increasing function, there will be only one such pole.

In conclusion, the existence of the perturbative Adler zeros and the fact that the IAM amplitude does not reproduce them well leads to the presence of spurious poles. A similar conclusion had been noticed in [18]. In the next sections, we present, first, a very simple construction of a modified IAM, along the lines of the previously discussed naive derivation of the standard IAM obtained without using dispersion relations, which solves these problems, and next we show two dispersive derivations of the modified IAM. One of them is subtracted at arbitrary z_0 (within the range of validity of our approximations) and the other one at the Adler zero. We will show how the modified IAM obtained naively corresponds to a particular limit of the first dispersive approach and comes out directly in the second. Finally, we will also show that the differences between the modified IAM formulas are negligible numerically, and that, while fixing the above-mentioned problems, the modified IAM does not yield any significant modification over the standard IAM in the physical region and to the resonance poles. Therefore, the results obtained so far in the literature with the IAM remain valid.

III. MODIFIED IAM: NAIVE DERIVATION

First of all, we will set some notation: as before, we denote by s_A the Adler zero of the "complete" partial wave, namely, $t(s_A) = 0$. In addition, we will also use the approximations to the Adler zero at LO, s_2, and NLO, $s_2 + s_4$. Thus, $t_2(s_2) = 0$ and $t_2(s_2 + s_4) + t_4(s_2 + s_4) = 0$.

In this section we present a naive, and intuitive, derivation leading to a partial wave definition which does not have the Adler zero related problems discussed above. The derivation follows closely [11], where this problem was addressed in the context of real pion scattering poles arising from medium effects such as temperature and density. In that paper, there was not a formal proof based on dispersion relations, such as the one we will present here later, and it was limited to pion-pion scattering.

From the discussion in the previous sections, it is clear that if we modify the inverse amplitude as $1/t^{\text{IAM}}(s) \to 1/t^{\text{IAM}}(s) + A(s)/t_2^2$ with $A(s)$ an analytic function at least off the real axis, real for real s, the unitarity and analytic properties of the amplitude remain unaltered. The modified IAM (from now on called mIAM), then reads

$$t^{\text{mIAM}}(s) = \frac{t_2^2(s)}{t_2(s) - t_4(s) + A^{\text{mIAM}}(s)}. \qquad (11)$$

Consider now the case of $\pi\pi$ scattering, where we have simply $t_2(s) = t_2'(s_2)(s - s_2)$ with $t_2'(s_2)$ constant. Then the Laurent expansion around $s = s_2$ of the standard IAM reads

$$\frac{1}{t^{\text{IAM}}(s)} = -\frac{t_4(s_2)}{t_2'(s_2)^2(s - s_2)^2} + \frac{t_2'(s_2) - t_4'(s_2)}{t_2'(s_2)^2(s - s_2)} + O(s - s_2)^0. \qquad (12)$$

The idea is that if we want the amplitude to have *only* an Adler zero of order one at $s = s_A$, we must subtract from $1/t^{\text{IAM}}$ the above double and single pole contributions at $s = s_2$ and add a single pole at s_A, i.e.,

$$\frac{1}{t^{\text{mIAM}}(s)} = \frac{1}{t^{\text{IAM}}(s)} + \frac{t_4(s_2)}{t_2'(s_2)^2(s - s_2)^2} - \frac{t_2'(s_2) - t_4'(s_2)}{t_2'(s_2)^2(s - s_2)} + \frac{c}{s - s_A}, \qquad (13)$$

where c is a so far undetermined constant that, as we will show now, can be fixed by demanding that the mIAM formula matches the perturbative ChPT series to fourth order, namely, $A = O(p^6)$.

In practice, it is simpler to keep track of the different chiral powers by counting the powers of f^{-2}, where f is the pion decay constant. Thus, since $t_2'(s_2) = O(f^{-2})$, $t_4 = O(f^{-4})$, and $s_4 = O(f^{-2})$, expanding the expression $t_2(s_2 + s_4) + t_4(s_2 + s_4) = 0$ around s_2 we find

$$s_4 = -t_4(s_2)/t_2'(s_2) + O(f^{-4}). \qquad (14)$$

Using this in Eq. (13) with $s_A = s_2 + s_4 + O(f^{-4})$, and requiring that Eq. (11) matches the chiral expansion at low energies, we find the first two orders of the chiral expansion for c:

TABLE I. σ and κ pole positions calculated with the IAM, the mIAM, and the z_0IAM with $z_0 = s_{\rm th}$.

Method	σ pole	κ pole
IAM	$443.71 - i217.58$	$724.2 - i216.2$
mIAM	$443.68 - i217.56$	$725.3 - i216.3$
z_0IAM, $z_0 = s_{\rm th}$	$443.82 - i216.99$	$727.7 - i210.0$

$$c = \frac{1}{t_2'(s_2)} - \frac{t_4'(s_2)}{t_2'(s_2)^2} + O(f^{-2}), \tag{15}$$

which leads to

$$A^{\rm mIAM}(s) = t_4(s_2) - \frac{(s_2 - s_A)(s - s_2)}{s - s_A}[t_2'(s_2) - t_4'(s_2)]. \tag{16}$$

Therefore, the mIAM in Eq. (11) with $A^{\rm mIAM}(s)$ in Eq. (16) and s_A approximated by its chiral expansion given above matches the chiral expansion of the amplitude up to fourth order and has the Adler zero at the same position and with the same order as the perturbative amplitude. Furthermore, we have solved, in turn, the spurious pole problem. In fact, since $A^{\rm mIAM}(s_2) = t_4(s_2)$, the denominator of Eq. (11) vanishes at $s = s_2$. But, for $s \neq s_2$ we have shown that $A^{\rm mIAM}(s) = O(f^{-6})$, and therefore our previous argument about the monotonous behavior of the denominator still holds so that the denominator vanishes *only* at $s = s_2$, where there is no pole contribution. The same holds for the spurious second-sheet poles. In Fig. 2 we show the mIAM amplitude in the $I = J = 0$ channel, and we observe that is not singular below threshold and remains close to the standard IAM result away from the Adler zero region. The same situation takes place in the $I = 2$, $J = 0$ channel. Finally, we have checked that, as expected from our previous arguments, the $f_0(600)$ or σ pole remains at the same place either using the second Riemann sheet extensions of the mIAM or the IAM (see Table I).

In the next sections, we will check how a modified IAM can also be obtained by considering explicitly the pole contributions in the dispersive derivation, thus ensuring the correct analytic properties of the amplitude. We will also show that the modified formula obtained with the naive derivation in this section can also be obtained as a particular case.

IV. MODIFIED IAM: DISPERSIVE DERIVATION FOR EQUAL MASSES

A. Pole contribution to the standard derivation

The derivation of the modified IAM from dispersion theory follows that in Sec. II up to Eq. (7), but keeping the pole contributions, which, by evaluating the corresponding residues, read

$$PC(1/t_2) = \frac{1}{t_2'(s_2)}\left(\frac{1}{s - s_2} - \frac{1}{z_0 - s_2}\right), \tag{17}$$

$$PC(t_4/t_2^2) = \frac{t_4(s_2)}{t_2'(s_2)^2}\left(\frac{1}{(s - s_2)^2} - \frac{1}{(z_0 - s_2)^2}\right) + \frac{t_4'(s_2)}{t_2'(s_2)^2}\left(\frac{1}{s - s_2} - \frac{1}{z_0 - s_2}\right), \tag{18}$$

$$PC(1/t) = \frac{1}{t'(s_A)}\left(\frac{1}{s - s_A} - \frac{1}{z_0 - s_A}\right), \tag{19}$$

where we have assumed a *single* zero in $t(s_A)$, as the presence of the $1/(s - s_A)$ factor in $PC(1/t)$ shows. As we have discussed above, and as is detailed in the Appendix, by expanding chirally $-PC(1/t_2) + PC(t_4/t_2^2) + PC(1/t)$, the poles contribute to $t(s)$ at higher order, and that is why they were customarily neglected. However, these pole contributions contain the terms needed to have the Adler zero in the correct position. In addition, by taking the limit $s \to s_2$, $(-PC(1/t_2) + PC(t_4/t_2^2))$ tends to the term needed to cancel the double zero of the IAM in s_2 [see Eq. (12)], and the spurious pole will also disappear.

In summary, the modified IAM obtained from dispersive relations subtracted at z_0, that we will denote by z_0IAM, can be written again as

$$t^{z_0{\rm IAM}}(s) \simeq \frac{t_2^2(s)}{t_2(s) - t_4(s) + A^{z_0{\rm IAM}}(s)}, \tag{20}$$

where now

$$A^{z_0{\rm IAM}}(s) = A^{\rm mIAM}(s) - \frac{t_2(s)^2}{t_2(z_0)^2}A^{\rm mIAM}(z_0). \tag{21}$$

Of course, the position of the Adler zero for the total amplitude is not known, but since we have been working with ChPT to one loop, we can impose the Adler zero to be located in its one-loop position; namely, in the above formulas, we have to replace $s_A \to s_2 + s_4$ which is obtained from the equation $t_2(s_2 + s_4) + t_4(s_2 + s_4) = 0$ as explained in Sec. III. Thus, to obtain (21) we have made use of $t_2(s) = t_2'(s_2)(s - s_2)$ and chirally expanded $1/t'(s_A) \simeq 1/t_2'(s_2) - t_4'(s_2)/t_2'(s_2)^2$, which is perfectly justified near s_A. Note that we can use the chiral expansions around s_A and s_2 because we no longer expand the inverse of the amplitudes but rather that of their derivatives, as they appear in the residues of the pole contributions.

Note that, once again, the factor $A^{\rm mIAM}(s)$ that appeared in the previous section is present, but now there is an additional and very similar piece that carries a z_0 dependence, which occurs due to our truncation of the ChPT series when approximating the subtraction constants and pole contributions. This additional term in $A^{z_0{\rm IAM}}$ is also $O(f^{-6})$, but, as we will see below, as long as z_0 lies within the range where our approximations remain valid, it is numerically small not only in the physical region but also below threshold. Therefore, this term can be dropped without spoiling the right chiral behavior of the amplitude in the subthreshold region, obtaining again the mIAM in Sec. III, which, for the moment, we have justified only numerically.

Furthermore, it is tempting to take the $z_0 \to \infty$ limit in Eq. (21) and recover the mIAM of the previous section by noting that the second term in Eq. (21) vanishes in that limit. However, this is just a formal justification of our naive derivation since we required the subtraction point z_0 to remain in the low-energy applicability region of ChPT. Nevertheless, we will see in the next section that there is an alternative dispersive derivation of the mIAM, which is somewhat different from the standard dispersive derivation since it requires subtractions at the Adler zero of each function.

B. Subtraction at the Adler zeros

The way to derive Eq. (16) from dispersive theory is to make the subtractions at the place where we already have a pole. Note, however, that $1/t$ has its pole in the Adler zero at s_A whereas t_4/t_2^2 has its pole at s_2, so that we have to write

$$\frac{1}{t(s)} = -\frac{s-s_A}{\pi}\int_{RC} dz \frac{\sigma(z)}{(z-s_A)(z-s)} + LC(1/t) + PC(1/t), \tag{22}$$

$$\frac{t_4(s)}{t_2(s)^2} = \frac{s-s_2}{\pi}\int_{RC} dz \frac{\sigma(z)}{(z-s_2)(z-s)} + LC(t_4/t_2^2) + PC(t_4/t_2^2), \tag{23}$$

where, for brevity, we have already used the elastic unitarity condition Eq. (5). As usual, we will approximate $LC(1/t) \simeq -LC(t_4/t_2^2)$, since that is the result of the chiral expansion at low energies where the integral is weighted. In the above relations, the pole contributions $PC(1/t)$ and $PC(t_4/t_2^2)$ now come from a double and a triple pole, respectively, and read

$$PC(1/t) = \frac{1}{t'(s_A)(s-s_A)} - \frac{t''(s_A)}{2t'(s_A)^2}, \tag{24}$$

$$PC(t_4/t_2^2) = \frac{t_4(s_2)}{t_2'(s_2)^2(s-s_2)^2} + \frac{t_4'(s_2)}{t_2'(s_2)^2(s-s_2)} + \frac{t_4''(s_2)}{2t_2'(s_2)^2}. \tag{25}$$

In addition, it is important to remark that these pole contributions diverge either on s_2 or s_A, so that for $s \simeq s_2$ or $s \simeq s_A$ they are, by far, the dominant contributions, since at the same time the right and left cut terms tend to zero and therefore become negligible.

Outside that region, the other terms become relevant and the difference with our previous derivations is that now we also approximate the $1/t$ integral term over the right cut by using $(s-s_A)/(z-s_A) \simeq (s-s_2)/(z-s_2)$, which is its LO chiral expansion. This is a remarkably good approximation for the dispersion relation as long as z is sufficiently far from s_2 and s_A, which is indeed the case for the right cut integral. Of course, the $1/t$ right cut term should vanish at s_A and now it does not, but, as we have just commented, the pole contribution diverges precisely at s_A and thus is largely dominant over the integral, which therefore can be completely neglected.

Once again, we simply add Eqs. (22) and (23) to obtain the mIAM equation

$$\frac{1}{t^{\text{mIAM}}(s)} = -\frac{t_4(s)}{t_2(s)^2} + \frac{t_4(s_2)}{t_2'(s_2)^2(s-s_2)^2} + \frac{t_4'(s_2)}{t_2'(s_2)^2(s-s_2)} + \frac{1}{t'(s_A)(s-s_A)}, \tag{26}$$

where, in the pole contributions, we have used that

$$-\frac{t''(s_A)}{2t'(s_A)^2} + \frac{t_4''(s_2)}{2t_2'(s_2)^2} = O(f^{-2}), \tag{27}$$

which, once again, can be safely neglected since it correspond to the chiral expansion at very low energies.

Finally, if we evaluate perturbatively

$$\frac{1}{t'(s_A)} \simeq \frac{1}{t_2'(s_2)} - \frac{t_4'(s_2)}{t_2'(s_2)^2} + O(f^{-2}), \tag{28}$$

and add 0 to Eq. (26) written as $0 = 1/t_2(s) - 1/(t_2'(s_2)(s-s_2))$, we reobtain Eq. (11) with $A(s)$ given by Eq. (16); i.e., we recover our naive derivation of the modified IAM by choosing the subtraction points at the Adler zeros.

Note, however, that in contrast to the derivation where we used an arbitrary z_0, now all subtractions have been performed in the very low-energy region so that chiral expansions for pole terms are well justified within ChPT. The price to pay is that in the right cut integral terms we have approximated s_A by s_2, which is irrelevant in the Adler zero region since the pole contributions dominate there, and a remarkably good approximation in the physical and resonance regions. Of course, we have still used exact elastic unitarity in the *integrands* over the physical cut, which ensures that the modified IAM satisfies exact elastic unitarity.

V. MODIFIED IAM: UNEQUAL MASSES.

When dealing with unequal masses, as in the πK scattering case that we will use for reference, in addition to the left and right cuts, there is also a circular cut centered at $s = 0$ with radius $\sqrt{m_K^2 - m_\pi^2}$ that contributes to the dispersion relation. This circular cut lies in the low-energy region within the applicability range of ChPT, and thus, as we do with the left cut, we will approximate the inverse amplitude by its ChPT series to fourth order. Taking into account that $t_2(s)$ has no cuts, we will have $CC(1/t) = -CC(t_4/t_2^2)$, where CC stands for the circular cut contribution. This is the same approximation used before for the left cut, and therefore we obtain the same IAM dispersive derivation. Hence, there is still the same problem with Adler zeros and spurious poles.

The solution given in previous sections works similarly well for the $I = 3/2$ and $J = 0$ partial wave. However, for the $I = 1/2$, $J = 0$ πK scattering channel, complications arise due to the form of the LO partial wave $t_2(s)$ which has *two* zeros instead of one.

$$t_2(s) = -\frac{5(s - s_{2+})(s - s_{2-})}{128\pi f s}, \tag{29}$$

with $s_{2\pm} = \frac{1}{5}(m_K^2 + m_\pi^2 \pm 2\sqrt{4m_K^4 - 7m_K^2 m_\pi^2 + 4m_\pi^4})$ whose values are $s_{2+} = 0.24\ \mathrm{GeV}^2$, and $s_{2-} = -0.13\ \mathrm{GeV}^2$. In particular, this means that, contrary to the previous cases, $t''(s_{2\pm}) \neq 0$, which complicates the derivation of a modified IAM. Nevertheless, once again we have found, for this special case of the $I = 1/2$, $J = 0$ channel, a naive and two dispersive derivations, that we detail next.

A. Naive derivation, $I = 1/2$, $J = 0$ channel

Following Sec. III, we define $1/t^{\mathrm{mIAM}} = 1/t^{\mathrm{IAM}} + A^{\mathrm{mIAM}}(s)/t_2^2(s)$ with $A^{\mathrm{mIAM}}(s)$ an analytic function, at least outside the real axis, and real for real s to preserve the unitarity and analytic properties of the original amplitude. Next we expand $1/t^{\mathrm{IAM}}$ in Laurent series around $s = s_{2+}$, taking into account that now $t_2''(s) \neq 0$, obtaining

$$\frac{1}{t^{\mathrm{IAM}}(s)} = -\frac{t_4(s_2)}{t_2'(s_{2+})^2(s - s_{2+})^2} + \frac{t_2'(s_{2+}) - t_4'(s_{2+})}{t_2'(s_{2+})^2(s - s_{2+})} + \frac{t_4(s_{2+})t_2''(s_{2+})}{t_2'(s_{2+})^3(s - s_{2+})} + O(s - s_{2+})^0. \tag{30}$$

As in Sec. III we subtract the pole at s_{2+} and add a pole at s_A to the inverse amplitude, and the constant in the s_A pole term is calculated perturbatively in order to match the chiral expansion at low energies. Note that, following our previous arguments, we do not need to subtract the s_{2-} pole, since $\mathrm{Im} t_4 \neq 0$ on the LC, so that no spurious pole will appear in that region. Proceeding then as in Sec. III, we now get

$$A^{\mathrm{mIAM}}(s) = \frac{t_2(s)^2}{t_2'(s_{2+})^2}\left[\frac{t_4(s_{2+})}{(s - s_{2+})^2} - \frac{(s_{2+} - s_A)}{(s - s_{2+})(s - s_A)} \times \left(t_2'(s_{2+}) - t_4'(s_{2+}) + \frac{t_4(s_{2+})t_2''(s_{2+})}{t_2'(s_{2+})}\right)\right]. \tag{31}$$

Then, Eq. (11) with $A^{\mathrm{mIAM}}(s)$ above unitarizes the $I = 1/2$, $J = 0$ channel and has an Adler zero of the correct chiral order and has no spurious pole. Let us remark that the above $A^{\mathrm{mIAM}}(s)$ is a generalization also valid for the equal mass case, since it is reduced to the $A^{\mathrm{mIAM}}(s)$ we already obtained in Eq. (16) when $t_2''(s_{2+}) = 0$.

Apart from this naive formal derivation we can follow a dispersive approach, detailed next, in which we make use of analyticity and the ChPT series is only used within its applicability region.

B. Dispersive derivation at z_0, $I = 1/2$, $J = 0$ channel

The derivation follows exactly that of Sec. II A, but now, due to the additional zero, s_{2-}, of t_2, which lies on the negative axis, we cannot simply write, as usually done, $\int_{LC} \mathrm{Disc} t_4/t_2^2 = 2i \int_{LC} \mathrm{Im} t_4/t_2^2$, nor $\int_{LC} \mathrm{Disc} 1/t_2 = 0$, where $\mathrm{Disc} f = f(x + i\epsilon) - f(x - i\epsilon)$ with real x. This zero also modifies the form of the dispersion relation for $1/t_2$, that now reads

$$\frac{1}{t_2(s)} = \frac{1}{t_2(z_0)} + PC_+(1/t_2) + PC_-(1/t_2),$$
$$PC_\pm(1/t_2) = \frac{1}{t_2'(s_{2\pm})}\left(\frac{1}{s - s_{2\pm}} - \frac{1}{z_0 - s_{2\pm}}\right). \tag{32}$$

Following the same steps as in Sec. II A, we approximate $1/t$ on the left cut by its chiral expansion, but now taking into account that, as mentioned above, $\int_{LC} dz \mathrm{Disc}(1/t_2(z)) \neq 0$, i.e.

$$LC(1/t) = \frac{s - z_0}{2\pi i}\int_{LC} dz \frac{\mathrm{Disc} 1/t(z)}{(z - s)(z - z_0)} \simeq \frac{s - z_0}{2\pi i}\int_{LC} dz \frac{\mathrm{Disc}(1/t_2(z) - t_4(z)/t_2(z)^2)}{(z - s)(z - z_0)}, \tag{33}$$

where we now get $-LC(t_4/t_2^2)$ as before, and a new term coming from $1/t_2$,

$$\frac{s - z_0}{2\pi i}\int_{LC} dz \frac{\mathrm{Disc} 1/t_2(z)}{(z - s)(z - z_0)} = \frac{s - z_0}{2\pi i}\int_{LC} dz \frac{(1/t_2(z + i\epsilon) - 1/t_2(z - i\epsilon))}{(z - s)(z - z_0)} = -(s - z_0)\int_{LC} dz \frac{(z - s_{2-})/t_2(z)}{(z - s)(z - z_0)}\delta(z - s_{2-}) = \frac{1}{t_2'(s_{2-})}\left(\frac{1}{s - s_{2-}} - \frac{1}{z_0 - s_{2-}}\right) \tag{34}$$

which is equal to $PC_-(1/t_2)$, so they will cancel in the expression for $1/t$. Thus we again obtain Eq. (7), but now the explicit expression for $PC(t_4/t_2^2)$ at $s = s_{2+}$ is different from Eq. (18) because $t_2''(s) \neq 0$,

$$PC(t_4/t_2^2) = \frac{t_4(s_{2+})}{t_2'(s_{2+})^2}\left(\frac{1}{(s-s_{2+})^2} - \frac{1}{(z_0-s_{2+})^2}\right) + \left(\frac{t_4'(s_{2+})}{t_2'(s_{2+})^2} - \frac{t_4(s_{2+})t_2''(s_{2+})}{t_2'(s_{2+})^3}\right)\left(\frac{1}{s-s_{2+}} - \frac{1}{z_0-s_{2+}}\right). \quad (35)$$

Finally, we get

$$t^{z_0\mathrm{IAM}}(s) = \frac{t_2(s)^2}{t_2(s) - t_4(s) + A^{z_0\mathrm{IAM}}(s)}, \quad (36)$$

with

$$A^{z_0\mathrm{IAM}}(s) = t_2^2(s)(-PC_+(1/t_2) + PC(t_4/t_2^2) + PC(1/t)), \quad (37)$$

where, again, we evaluate $1/t'(s_A)$ in $PC(1/t)$ perturbatively, obtaining for $A(s)$

$$A^{z_0\mathrm{IAM}}(s) = A^{\mathrm{mIAM}}(s) - \frac{t_2(s)^2}{t_2(z_0)^2}A^{\mathrm{mIAM}}(z_0). \quad (38)$$

Formally, this looks the same as Eq. (21) but now $A^{\mathrm{mIAM}}(s)$ is of a more general form. Indeed, we obtain again $A^{\mathrm{mIAM}}(s)$ plus a term depending on z_0, which is also $O(f^{-6})$. Nevertheless we will see that, as long as z_0 remains in the range of validity of the approximations, the latter is numerically small not only in the physical region but also in the subthreshold region, so that it can be neglected to obtain again the result from the naive derivation, thus justifying numerically the mIAM and the IAM.

However, a derivation of the mIAM that differs from the standard one, since subtractions are made at the Adler zeros, is also possible for the unequal mass case, and we detail it next.

C. Dispersive derivation subtracting at the Adler zeros, $I = 1/2$, $J = 0$ channel

The derivation follows closely that in Sec. IV B, but now we will have extra terms (not negligible in the chiral expansion) coming from Disc$1/t_2$ in the left cut when s_A is expanded around s_2. In addition, the expressions for the pole contributions are more complicated because $t_2''(s) \neq 0$.

We again expand the left cut to NLO:

$$\begin{aligned} LC(1/t) &\simeq \frac{s - s_{2+} - s_4}{2\pi i}\int_{LC} dz \frac{\mathrm{Disc}(1/t_2(z) - t_4(z)/t_2(z)^2)}{(z-s)(z-s_{2+})}\left(1 + \frac{s_4}{z - s_{2+}}\right) \\ &\simeq -LC(t_4/t_2^2) + \frac{s - s_{2+}}{t'(s_{2-})(s-s_{2-})(s_{2-}-s_{2+})} - \frac{t_4(s_{2+})}{t_2'(s_{2+})t_2'(s_{2-})(s_{2+}-s_{2-})^2}, \end{aligned} \quad (39)$$

where we obtain $-LC(t_4/t_2^2)$ plus two extra terms. The pole contributions now read

$$PC(1/t) = \frac{1}{t'(s_A)(s-s_A)} - \frac{t''(s_A)}{2t'(s_A)^2}, \quad (40)$$

$$\begin{aligned} PC(t_4/t_2^2) = {}& \frac{t_4(s_{2+})}{t_2'(s_{2+})^2(s-s_{2+})^2} + \left(\frac{t_4'(s_{2+})}{t_2'(s_{2+})^2} - \frac{t_4(s_{2+})t_2''(s_{2+})}{t_2'(s_{2+})^3}\right)\frac{1}{s-s_{2+}} + \frac{t_4''(s_{2+})}{2t_2'(s_{2+})^2} - \frac{t_4'(s_{2+})t_2''(s_{2+})}{t_2'(s_{2+})^3} + \frac{3t_4(s_{2+})t_2''(s_{2+})^2}{4t_2'(s_{2+})^4} \\ & - \frac{t_4(s_{2+})t_2'''(s_{2+})}{3t_2'(s_{2+})^3}, \end{aligned} \quad (41)$$

where all terms in $PC(1/t)$ will be evaluated perturbatively excepting $1/(s - s_A)$, which is the one that gives the needed pole at s_A to the inverse amplitude. Then, adding the dispersion relations for $1/t$ and t_4/t_2^2, and using that

$$\begin{aligned} \frac{1}{t_2(s)} = {}& \frac{1}{t_2'(s_{2+})(s-s_{2+})} + \frac{s-s_{2+}}{t_2'(s_{2-})(s-s_{2-})(s_{2-}-s_{2+})} \\ & - \frac{t_2''(s_{2+})}{2t_2'(s_{2+})^2}, \end{aligned} \quad (42)$$

we thus arrive at A^{mIAM} in Eq. (31) plus $O(f^{-6})$ terms that can be safely neglected since they all correspond to the chiral expansion at very low energies.

VI. NUMERICAL RESULTS: COMPARISON BETWEEN DIFFERENT APPROACHES AND z_0 SENSITIVITY

In this section we compare numerically the IAM with the modified methods we have derived. The precise values of the ChPT low-energy constants are not relevant for what we want to show here. Just for illustration, we take for SU(2) ChPT the typical values of $l_3^r = 0.82 \times 10^{-3}$, $l_4^r = 6.2 \times 10^{-3}$ from the second reference in [1], at a renormalization scale $\mu = 770$ MeV. The values $l_1^r = -3.7 \times 10^{-3}$, $l_2^r = 5.0 \times 10^{-3}$ have been obtained from an IAM fit to $\pi\pi$ scattering data. In particular, we have used the same data sets used in the IAM fits of the second reference in [8], but we have updated the controversial $f_0(600)$ channel by using the choice of data explained in [19] consistent with forward dispersion relations and Roy equations as shown in [20]. For πK scattering we have chosen the central values of the $SU(3)$ ChPT constants of the IAM I set in the last reference of [8].

We show in Fig. 2 several plots comparing the IAM and mIAM results for the modulus of different partial waves of $\pi\pi$ and πK elastic scattering in the scalar channel. We see that both methods are indistinguishable in the physical region, shown in the left column, and only differ in the Adler zero region which is shown in detail in the right column. Note that the IAM has a spurious pole and a double zero in that region, whereas the mIAM does not have such a pole and its zero is simple.

We have also calculated the σ and κ pole positions with the IAM and the mIAM, obtaining the same pole position within $\sim$1 MeV, as shown in Table I.

In the derivation of the z_0IAM, we have an arbitrary subtraction point $z_0 \neq s_2, s_A$, but we only know how to calculate the amplitude at z_0 if we can use the chiral expansion $1/t(z_0) \simeq 1/t_2(z_0) - t_4(z_0)/t_2(z_0)^2$, which is only valid if z_0 lies in the low-energy region. Also, due to t_2 having a zero at s_2, the above expansion is a very bad approximation if z_0 is near s_2. We can estimate how close z_0 could be to s_A and s_2 by looking at the expansion

$$\frac{1}{t(z_0)} \simeq \frac{1}{t_2(z_0)} - \frac{t_4(z_0)}{t_2(z_0)^2} \simeq \frac{1}{t_2(z_0)}\left(1 + \frac{s_4}{(z_0 - s_2)} + \cdots\right), \tag{43}$$

where we have only made explicit the $s_4/(z_0 - s_2)$ pole term. Hence, for our approximations to remain valid, we have to make sure that the ratio $s_4/(z_0 - s_2)$ is small enough. Thus, we show in Fig. 3 the contour plots in the energy and z_0 plane of the relative difference between the mIAM and the z_0IAM,

$$\Delta = \frac{|t^{\text{mIAM}}(s) - t^{z_0\text{IAM}}(s)|}{\frac{1}{2}|t^{\text{mIAM}}(s) + t^{z_0\text{IAM}}(s)|}, \tag{44}$$

as a function of the energy and z_0. We show two contour lines corresponding to $\Delta = 10\%$ and 5%. We see that, as long as the choice of z_0 is sufficiently far from s_A and s_2 (the white lines in the plots) the result of the z_0IAM differs little from the mIAM. In particular, we have checked that, in order to obtain a relative difference less than 10% and 5%, in the worst case, which is the $I = 2$, $J = 0$ wave, we have to be sure that our choice of subtraction point z_0 makes the ratio of $|s_4/(z_0 - s_2)| < 3\%$ and 2%, respec-

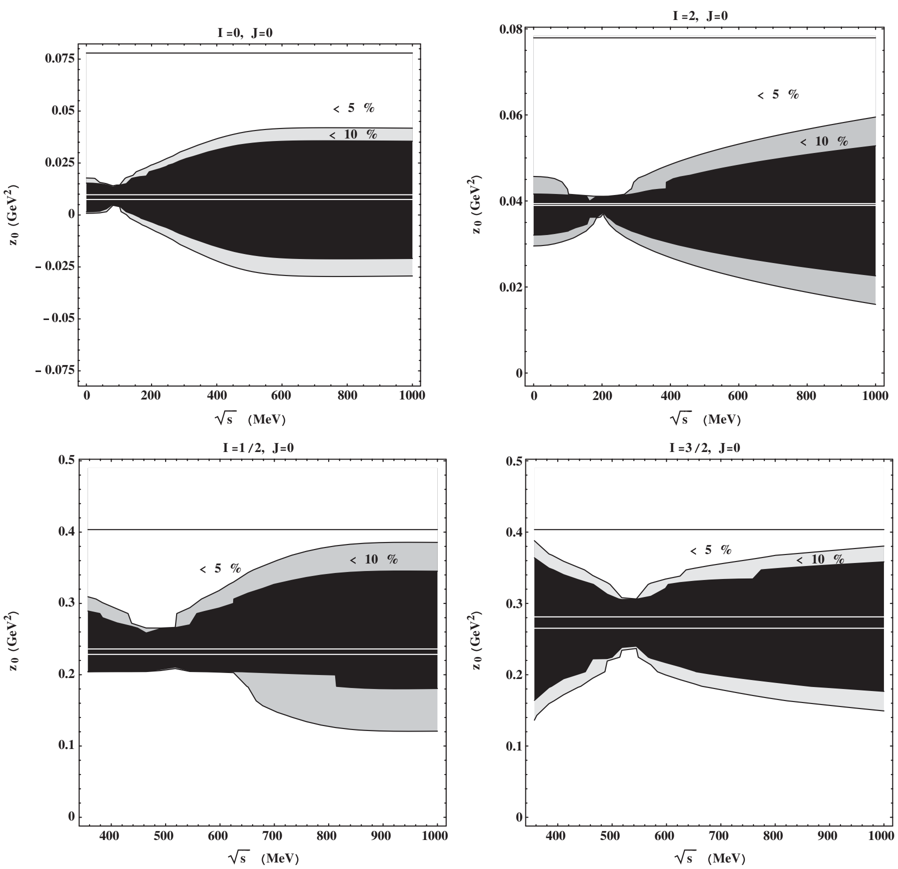

FIG. 3. Contour plots of the relative differences between the mIAM and the z_0IAM in the (s, z_0) plane. We see that, as long as z_0 lies in the low-energy region but sufficiently far from the Adler zeros s_A and s_2 (white lines), the differences become small for all partial waves. Note that for $z_0 = s_{\text{th}}$ (black line) the differences for all cases are less than 5%.

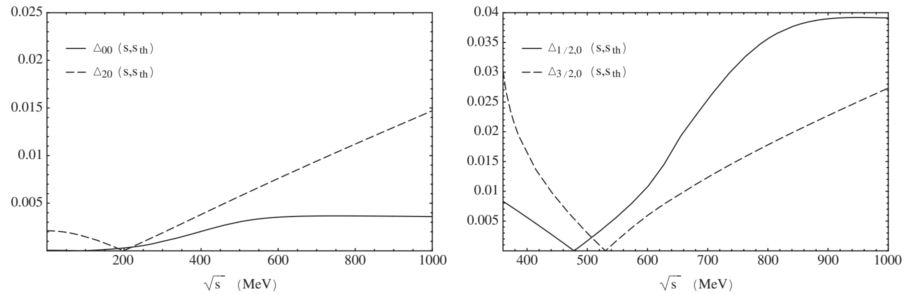

FIG. 4. Relative differences Δ, with $z_0 = s_{\rm th}$, for $\pi\pi$ (left panel) and πK scattering (right panel). Note that the differences are less than 1.5% and 4%, respectively.

tively. Let us recall that an error of 5% is a very precise result in this context, since it is much less than the uncertainties (mostly of systematic origin) of the existing data on meson-meson scattering. For instance, just the isospin violation effects, that are usually not taken into account by these experiments, can be estimated at the level of 2% or 3%, and these are added to further statistical and very big systematic uncertainties.

In summary, the choice of subtraction constant has little relevance for the z_0IAM, as long as it lies in a place where the NLO chiral expansion of the inverse amplitude is a good approximation to the inverse itself. In such case, the z_0IAM results are very close to those of the mIAM, and therefore to those of the standard IAM in the physical region. This provides a strong check of the stability and robustness of the standard NLO IAM results.

We have also shown in the plots of Fig. 3 a line at $z_0 = s_{\rm th}$, which is a very natural choice of subtraction point, since the subtraction constants can then be written in terms of threshold parameters, which are well studied in the literature. We see that, with this choice of z_0, in all cases we have a relative difference Δ which is less than 5%. Actually, it is even smaller, as seen in Fig. 4, where we plot in detail the results for Δ subtracting at $z_0 = s_{\rm th}$, and we see that for $\pi\pi$ we have a relative difference less than 1.5% for all energies and less than 4% for the πK case.

VII. DISCUSSION AND SUMMARY

In this work we have shown that it is possible to modify slightly the one-channel IAM so that, also in the scalar channel, it provides a reliable description of the unitarized partial wave amplitudes below threshold. In particular, we have shown that it is possible to obtain a modified IAM that has the Adler zeros located in the same place as the effective chiral expansion up to the desired order, that these zeros are single, and that the spurious poles below threshold that occur in the standard IAM are no longer present.

The IAM has been most frequently used at NLO, where the chiral expansion of a partial wave is written as $t(s) = t_2(s) + t_4(s) + \ldots$. For such a case the simplest modification to the IAM that we have found can be written as follows:

$$t^{\rm mIAM}(s) = \frac{t_2^2(s)}{t_2(s) - t_4(s) + A^{\rm mIAM}(s)}, \qquad (45)$$

where

$$A^{\rm mIAM}(s) = \frac{t_2(s)^2}{t_2'(s_2)^2}\left[\frac{t_4(s_2)}{(s-s_2)^2} - \frac{(s_2 - s_A)}{(s-s_2)(s-s_A)} \times \left(t_2'(s_2) - t_4'(s_2) + \frac{t_4(s_2)t_2''(s_2)}{t_2'(s_2)}\right)\right], \qquad (46)$$

and in order to have the Adler zero exactly on its NLO position, we set $s_A \to s_2 + s_4$ where $s_2 = O(m^2)$ and $s_4 \simeq O(m^4/f^2)$ are the Adler zeros at LO and its NLO correction, respectively. That is, they are obtained from $t_2(s_2) = 0$ and $t_2(s_2 + s_4) + t_4(s_2 + s_4) = 0$. In general, s_4 should be calculated numerically. The above formula is valid for the elastic scattering of both equal and unequal meson masses. However, in most of the cases, like $\pi\pi$ scattering, the formula simplifies further since $t_2''(s_2) = 0$ and $t_2(s) = t_2'(s_2)(s - s_2)$. Note also that the $A^{\rm mIAM}(s)$ piece counts as next to next to leading order (NNLO) in the chiral expansion, and for that reason it was neglected in the standard IAM derivation, which is recovered by setting $A^{\rm mIAM}(s) \to 0$ and remains valid in the physical region, since there $A^{\rm mIAM}(s)$ is indeed negligible.

In Sec. III we have given a "naive" formal derivation of $A^{\rm mIAM}(s)$ by adding, in a rather *ad hoc* way, the pieces needed to fix the Adler zero and spurious pole problems without spoiling unitarity and the chiral symmetry expansion. However, in Secs. IV and V we have shown that the mIAM formulas above can be derived by using the analytic properties of amplitudes in the form of dispersion relations and imposing elastic unitarity on the right cut. The use of the chiral effective expansion is well justified to calculate the subtraction constants and pole contributions to the dispersion relations, and is also used to approximate contributions from other cuts. In particular, the integral over the left cut is calculated to NLO, which is a good approximation in the low-energy region that dominates the integral. Therefore, there are no model dependencies but just approximations within the effective theory up to a given order. This allows for a straightforward and systematic extension of the elastic IAM and modified IAM to higher orders.

A usual criticism to unitarization methods is their arbitrariness, but we have shown here that the IAM, modified or not, is not just unitarizing, but also imposing a stringent analytic structure on the amplitudes, something that leaves little room for such arbitrariness: the choice of the subtraction points. However, if the chiral effective expansion is to be used to calculate the amplitude at the subtraction points or pole positions, the subtraction points should lie, first of all, in the low-energy region. But, in addition, since a dispersion relation is written for the inverse amplitude, that subtraction point should also lie far from the Adler zero. In this work, we have explicitly shown that, as long as those two constraints are satisfied, the choice of subtraction point has a very small numerical effect on the resulting amplitude. Moreover, we have shown that the results of the standard IAM in the physical axis and resonance region in the complex plane remain unchanged when using the modified IAM and are extremely stable under different choices of subtraction points.

In summary, we have presented a slightly modified inverse amplitude method for the elastic case, that has the Adler zeros in the correct position and of the correct order and no spurious poles in that region. We have shown that the results already obtained with the standard IAM are robust in the physical region, where it can still be used safely, but the new modifications allow for the study of the subthreshold region that is of interest in problems like thermal restoration of chiral symmetry [11] or the quark mass dependence [16] of resonances in meson-meson scattering.

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APPENDIX: CHIRAL COUNTING OF POLE CONTRIBUTIONS

Next we will check by explicit calculation that the total pole contribution counts as $O(f^{-6})$ in the amplitude. We will do it for the more general case where $t_2''(s) \neq 0$. In particular, we only need to expand $PC(1/t)$. First note that

$$\begin{aligned} t'(s_A) &= t_2'(s_2 + s_4 + \cdots) + t_4'(s_2 + s_4 + \cdots) + O(f^{-6}) \\ &= t_2'(s_2) + t_2''(s_2)s_4 + t_4'(s_2) + O(f^{-6}). \end{aligned} \tag{A1}$$

With this

$$\frac{1}{t'(s_A)} = \frac{1}{t_2'(s_2)} - \frac{t_4'(s_2)}{t_2'(s_2)^2} + \frac{t_2''(s_2)t_4(s_2)}{t_2'(s_2)^3} + O(f^{-2}). \tag{A2}$$

We also have to expand $1/(s - s_A)$:

$$\begin{aligned} \frac{1}{s - s_A} &= \frac{1}{s - s_2} + \frac{s_4}{(s - s_2)^2} + O(f^{-4}) \\ &= \frac{1}{s - s_2} - \frac{t_4(s_2)}{t_2'(s_2)(s - s_2)^2} + O(f^{-4}), \end{aligned} \tag{A3}$$

and similarly for $1/(z_0 - s_A)$, where we have taken into account that $t_2'(s_2)s_4 = -t_4(s_2) + O(f^{-6})$. Hence, we find that

$$\begin{aligned} PC(1/t) &= \frac{1}{t'(s_A)}\left(\frac{1}{s - s_A} - \frac{1}{s - s_2}\right) \\ &= \frac{1}{t_2'(s_2)}\left(\frac{1}{s - s_2} + \frac{1}{z_0 - s_2}\right) \\ &\quad - \frac{t_4(s_2)}{t_2'(s_2)^2}\left(\frac{1}{(s - s_2)^2} - \frac{1}{(z_0 - s_2)^2}\right) \\ &\quad - \left(\frac{t_4'(s_2)}{t_2'(s_2)^2} - \frac{t_2''(s_2)t_4(s_2)}{t_2'(s_2)^3}\right)\left(\frac{1}{s - s_2} - \frac{1}{z_0 - s_2}\right) \\ &\quad + O(f^{-2}) = PC(1/t_2) - PC(t_4/t_2^2) + O(f^{-2}), \end{aligned} \tag{A4}$$

so the total pole contribution $PC(1/t) - PC(1/t_2) + PC(t_4/t_2^2) = O(f^{-2})$, which yields an $O(f^{-6})$ contribution to $t(s)$.

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