A FORMAL APPROACH TO THE CONCEPT OF STRUCTURE FUNCTION

Javier Montero

Dept. Statistics & O.R.

Faculty of Mathematics

Complutense University

Madrid 28040 (Spain)

Abstract: This paper deals with the concept of structure function, basic in reliability theory. The main purpose is to develop some properties of general structure functions, defined as mappings $\phi:L^n\to S$, being L and S complete lattices. Such structure function will be assumed to be measurable with respect the Borel σ-field generated by the order topology, and a fixed measure reveals how the system is observed and distinguished from others . In this context it is discussed the idea of duality, together with the concept of coherency, and some results and examples are shown.

Keywords: Fuzzy Coherent Systems, Generalized Structure Function, Reliability Theory.

1.- INTRODUCTION

It is well known that Classical Reliability Theory has been built for binary systems with binary components, where only two states are allowed: perfect functioning and complete failure. Such systems are then modelled by an structure function $\phi:\{0,1\}^n\to\{0,1\}$, which assigns the performance level of the system -"0" if it is failed, "1" if it is functioning- to each profile of states for its n components -"0_i" if the component i is failed, "1_i" if component i is functioning- (see, e.g., Barlow & Proschan, 1975). A first generalization tries to develop a theory of finite multistate systems, where a finite and linearly ordered set of performance levels is assumed (see, e.g., El-Neweihi *et al.*, 1978, and Griffith, 1980). A second generalization have been introduced by some authors (see, e.g., Baxter, 1984 & 1986, and Block & Savits, 1984) by considering a continuum of different states between both extreme performance levels, just taking a real interval as the space of states for the system and its components (for example, the unit interval [0,1]). Such "continuum systems" have been also called "fuzzy systems" in order to avoid any confusion with topological continuity (Montero, 1988a), and their associated structure functions will be given by mappings $\phi:[0,1]^n\to[0,1]$. A third generalization has been recently introduced by the author by considering a general complete lattice as the basic structure of states, with an associate measure defined on the Borel σ-field generated by the order topology (Montero, 1986b). In this paper we shall discuss the concept of dual structure function when the space of states of the system has a multidimensional nature ($S=L^k$).

2.- GENERAL STRUCTURE FUNCTIONS

We shall assume the existence of a basic space of performance levels given by a complete partially ordered set $(L,\geq)$. As usual, $a>b$ means that $a\geq b$ holds but not $b\geq a$, and then the level of performance a is strictly higher than the level b. Therefore, L is also compact and there exists alowest performance level $0=\inf L$ (complete failure) and a highest performance level $1=\sup L$ (perfect functioning).

Definition 1.- Any mapping $\phi:L^n\to L^k$ will be said to be a multievaluated L-structure function (MLSF).

Hence, the variable ϕ indicates the state of the system, which is determined by the states of its n components. The usual product lattice is then associated to L^n and L^k ($x\leq y$ holds if and only if $x_i\leq y_i$ $\forall i$; $x<y$ holds if and only if $x\leq y$ but not $y\leq x$). Moreover, since L is compact, such product spaces are assured to be also compact.

But as pointed out by Barlow and Proschan (1975), a system would be quite unusual (or perhaps poorly defined) if improving the performance of a component causes the system to deteriorate its performance level. Moreover, non informative components can be eliminated. Thus we can restrict our study to "coherent" MLSF, that is, MLSF being monotonic ($\phi(x)\geq\phi(y)$ when $x\geq y$) and with no irrelevant components (there is no component i such that $\phi(y_i,x)=\phi(z_i,x)$ $\forall x\in L^n$ $\forall y_i,z_i\in L$ holds, being $(y_i,x)=(x_1,\ldots,x_{i-1},y_i,x_{i+1},\ldots,x_n)$; obviously, a component i will be irrelevant in a monotonic system if and only if $\phi(0_i,x)=\phi(1_i,x)$ $\forall x\in L^n$).

But it is clear that any realistic structure function assumes the existence of an appropriate measure on the initial space, which reveals which states are really observable, or how the structure function is really recognized. Hence, two MLSF ϕ_1 and ϕ_2 defined on the same measure space $(L^n,\sigma(\mathcal{B}^n),\mu)$ are indistinguishable if they are identical almost everywhere. If we consider the Borel σ-fields $\sigma(\mathcal{B}^n)$ and $\sigma(\mathcal{B}^k)$ generated by the product order topology in L, then we can assume, without loss of practical generality, that our MLSF is measurable (that is, $\phi^{-1}(B)\in\sigma(\mathcal{B}^n)$ $\forall B\in\sigma(\mathcal{B}^k)$). Hence, given a fixed measure space with L a complete lattice, an observable structure function (OSF) Φ will be an equivalence class of measurable MLSF being indistinguishable between them, and it will be said "coherent" if there exists $\phi\in\Phi$ being monotonic, and there is no irrelevant component for any given $\phi\in\Phi$.

3.- DUAL OBSERVABLE MULTIEVALUATED L STRUCTURE FUNCTIONS

It must be noted that the set of performance levels is usually defined according to a high internal duality which assures the existence of a one-to-one mapping $\delta:(L,\geq)\to(L,\leq)$ being order-preserving ($\delta(a)\leq\delta(b)$ if and only if $a\geq b$).

Definition 2.- Let us consider a MLSF ϕ. Supposed a one-to-one mapping $\delta:L\to L$ being order preserving, its δ-dual structure function will be given by $\phi^{\delta}(x)=\Delta^{-1}[\phi(\Delta(x))]$, where $\Delta:(L^m,\sigma(\mathcal{B}^m))\to(L^m,\sigma(\mathcal{B}^m))$ is the associated one-to-one mapping such that $\Delta_i(x)=\delta(x_i)$ $\forall i$ $\forall x\in L^m$ for any arbitrary dimension m.

Obviously, $(\phi^{\delta})^{\delta}=\phi$ is not assured for an arbitrary isomorphism δ in L. Moreover, such isomorphism is not unique in general, but it must be noted that dual structure can be unique in some important cases:

Theorem 1.- Let us consider a "parallel" structure such that $\phi(x)=sup(x_1,x_2,\ldots,x_n)$ $\forall x\in L^n$. Then its δ-dual structure function is the "series" structure $\phi(x)=inf(x_1,x_2,\ldots,x_n)$ $\forall x\in L^n$, and conversely, for any given isomorphism δ in L.

Theorem 2.- Let us consider a linearly ordered set L, and a σ-finite measure μ associated to $(L^n,\sigma(\mathcal{B}^n))$. Let us assume that the isomorphism δ in L must be measure preserving, in the sense that $\mu(B)=\mu(\Delta(B))$ $\forall B\in\sigma(\mathcal{B}^n)$. Then, if it exists, the associated $\Delta:(L^n,\sigma(\mathcal{B}^n))\to(L^n,\sigma(\mathcal{B}^n))$ is unique almost everywhere.

Theorem 3.- Let us consider a closed interval L of the compact real line $\mathbb{R}\cup\{-\infty,\infty\}$, and let us assume that it is defined a proper product probability measure in the L^n, being absolutely continuous with density function strictly possitive in L^n. Then it is defined a unique measure-preserving isomorphism.

It is easy to prove that, in any case, ϕ^{δ} is a coherent MLSF whenever ϕ is a coherent MLFS. But when we try to translate the concept of duality and this closure theorem into the context of observable MLSF, we find that the δ-dual set $\Phi^{\delta}=\{\phi^{\delta}\,/\,\phi\in\Phi\}$ of a coherent observable MLSF Φ is not assured to be an equivalence class of measurable and indistinguishable MLSF. The desired closure theorem will appear when a measure preserving isomorphism δ can be defined:

Theorem 6.- Let δ be a measure-preserving isomorphism defined on our general space of states $(L^n,\sigma(\mathcal{B}^n),\mu)$, and let Φ be a coherent observable MLSF. Then Φ^{δ} is also a coherent observable MLSF.

4.- FINAL COMMENTS

As in classical reliability theory, the concept of dual structure function will be useful in analyzing systems of components subject to two kinds of failure, as in most safety systems. For example, if we consider a control flood of two independent sluices in a river, we find that each one can respond at a level x_L to a command to close and at a level y_L to a command to open, being L a linearly ordered set and δ a natural order-preserving isomorphism between the degrees of response to each command. If we want to close the flood, it is clear that the system can be represented by a series structure function depending on the values x_L ; but if we want to open the flood, the system will be represented by a parallel structure function depending on the values y_L . Then it is very useful that both failure to close and failure to open can be analyzed using the same structure function (one is the dual structure of the other).

REFERENCES

Barlow, R.E. & Proschan, F. (1975) *Statistical Theory of Reliability and Life Testing*. Holt-Rinehart-Winston, New York.

Baxter, L.A. (1984) *Continuum structures I*. Journal of Applied Probability 19, 391-402.

Baxter, L.A. (1986) *Continuum structures II*. Mathematical Proceedings of the Cambridge Philosophical Society 99, 331-338.

Block, H.W. & Savits, T.H. (1984) *Continuous multistate structure functions*. Operations Research 32, 703-714.

El-Newihi, E.; Proschan, F.& Sethuraman, J. (1978) *Multistate coherent systems*. Journal of Applied Probability 15, 675-688.

Griffith, W.S. (1980) *Multistate reliability models*. Journal of Applied Probability 17, 735-744.

Montero, J. (1988a) *Coherent fuzzy systems*. Kybernetes 17, 28-33.

Montero, J. (1988b) *Observable structure functions*. First Spanish Polish Meeting. Granada.

[illegible] Professor [illegible]

I am pleased to inform you that your excellent paper you presented to our B. Baden Conference was recently published in our new book „Advances in Support Systems Research." (see the enclosure)

Looking forward to the pleasure of seeing you this year.

Cordialement Georg Lwl