ANOTHER VIEW TO ARROW'S THEOREM

J. Montero (*)
Faculty of Mathematics
Complutense University
Madrid 28040 (Spain)

Abstract: Arrow's theorem shows how difficult are *rational* aggregations of individual opinions, if they are given by *rational* binary preference relations. Many authors have proposed different approaches in order to avoid his impossibility result, relaxing some particular requirement. In this paper, by introducing the concept of *rationality* as a fuzzy property, it is postulated that the main reason for such a negative result is just its *aristotelian* conception. Hence, it is shown that *non complete irrational* aggregations are always possible, in such a way that classical Arrow's theorem should be understood just as an impossibility of getting *complete rational* aggregations.

Keywords: aggregation rules, group decision making, fuzzy opinions.

1. INTRODUCTION

When dealing with the problem of aggregating individual opinions in a social group, the reference to the work of Prof. Arrow [1] is really a must. As it is well known, he considered an arbitrary finite and non-void group D of human beings, expressing their preferences about a finite and non-empty set X of feasible alternatives in order to reach to some kind of social preferences. It was assumed a situation with at least two individuals and at least three alternatives, being all the opinions -individual and consensus opinions- always given by a linear order, that is, a complete, reflexive and transitive *crisp* binary relation defined on such a set X of alternatives. Then it was proved the impossibility of aggregation rules verifying some *ethical* conditions which seemed to be desirable at a first sight. Many impossibility theorems have followed to his work. For example, it can be shown that there is *no social welfare function* assigning a social preference ordering to each profile of individual preference orderings, if the following conditions are assumed: *unrestricted domain* (each individual is free to choose his/her personal preference ordering), *non-negative response* (if some changes are made by individuals improving their opinions about some alternative, it does not cause a final change against such alternative), *independence of irrelevant alternatives* (there is no influence between disjoint subsets of alternatives), *citizen sovereign* (if people agrees about a particular ordering, it should represent the social consensus) and *non-dictatorship* (there is no individual imposing systematically his/her opinion).

Much effort has been devoted in the past in order to avoid such a discouraging result, looking for some possibility theorems. Two approaches have been mainly tried: by assuming some restriction on the domain, arguing that the most important difficulty in practice is due to the variety of individual opinions (for example, by considering some condition based on Black's *single-peakedness* [2]); or by relaxing the concept of consensus, arguing that the real objective in practice is just

(*) Research supported by Dirección General de Investigación Científica y Técnica (National Grant number PB88-0137)

some useful decision-oriented information (for example, by considering Sen's *social decision functions*). Some positive results have been obtained under both approaches, but even Sen himself (see [3]) recognizes that Arrow's negative philosophy still remains.

Every society is faced with the problem of amalgamating individual opinions into a consensus, and therefore it represents a key point for the development of any group of persons. Though it can be understood that there is no general methodology for aggregating crisp individual preferences, in practice consensus are usually reached, perhaps through a dynamic process, or just suppressing -perhaps against the usual democratic principles- *discordant* opinions or *discordant* individuals. Our thesis here is that the main difficulty under Arrow's focus is not due to how any *ethical* condition is formally written or how the social opinion should be expressed, but how the idea of *rationality* is understood: the underlying Aristotelian concept of *rationality* based upon any *crisp* transitivity or *crisp* acyclity, which provokes to think that if something is not absolutely rational, then it is absolutely irrational. In this sense, as pointed out in Montero [4], Lukasievicz's censure to sciences based on using Aristotelian logic is fully applicable in our context.

2.- A MEASURE OF RATIONALITY FOR FUZZY PREFERENCE RELATIONS

Let us assume from now on that each opinion is given by a fuzzy preference relation, that is, a mapping $\mu: X\times X \to [0,1]$ in such a way that the value $\mu(x,y)$ represents the intensity of preference of alternative x over alternative y. We shall also assume that such a fuzzy preference relation is complete, in the sense that $\mu(x,y)+\mu(y,x)\geq 1$ $\forall x,y$; hence, the values

$$\mu_I(x,y)=\mu(x,y)+\mu(y,x)-1$$

$$\mu_S(x,y)=\mu(x,y)-\mu_I(x,y)$$

can be understood, respectively, as the degree of indifference (xIy) between each pair of alternatives {x,y} and the degree of strict preference of alternative x over alternative y (x>y). It could be also assumed the condition $\mu(x,x)=1$ $\forall x$, but such a rationality condition will be in fact -as it will be shown later- just a particular case under our approach.

Our proposal is just to measure rationality of a given preference relation by considering all possible chains $x_1-x_2-\ldots-x_k-x_1$ of k distinct alternatives and weighting in a natural way their acyclic paths. For example, if we take only the alternative x, there is only one acyclic path -xIx, which can be weighted by $\mu_I(x,x)$-, and if we take two alternatives {x,y}, there are three acyclic paths -x>y<x, y>x<y and xIyIx, which can be weighted respectively by $\mu_S^2(x,y)$, $\mu_S^2(y,x)$ and $\mu_I^2(x,y)$-. In general, a path of k distinct alternatives will be non-acyclic if and only if $x_1\geq x_2\geq\ldots\geq x_k\geq x_1$ with some strict preference or $x_1\leq x_2\leq\ldots\leq x_k\leq x_1$ with some strict preference, and the associated path weight will be the product of the weights of all the preferences contained in such a path. Based on these path weights, in Montero [4] it was proposed to consider *rationality* -that is, *acyclity*- as a fuzzy property

$$A: \mathcal{R}(X)\to[0,1]$$

being $\mathcal{R}(X)$ the family of all complete fuzzy preference relations on X, with $A(\mu)$ the minimum sum of weights of acyclic paths in all possible chains (see Montero [4] for a formal definition).

For example, if $X=\{x,y,z\}$ with $\mu(x,x)=\mu(y,y)=\mu(z,z)=1$ and $\mu(x,y)=0.5$, $\mu(y,x)=0.8$, $\mu(y,z)=0.7$, $\mu(z,y)=0.7$, $\mu(z,x)=0.8$, and $\mu(x,z)=0.6$, then the degree of acyclity of such a complete fuzzy preference relation μ will be

$$A(\mu) = \min\{\ 1,\ 1,\ 1;$$
$$;0.2^2+0.3^2+0.5^2,\ 0.3^2+0.4^2+0.3^2,\ 0.4^2+0.4^2+0.2^2;$$

$$;1-\{\ (0.2+0.3)*(0.3+0.4)*(0.4+0.4)+$$
$$+(0.3+0.5)*(0.4+0.3)*(0.4+0.3)-$$
$$-2*0.3*0.4*0.4)\ \} =$$
$$= \min\{\ 1,\ 1,\ 1;\ 0.38,\ 0.34,\ 0.36;\ 0.48\} = 0.34$$

where we can find -in this order- the sum of acyclic weights for the three chains with only one alternative (in this case we find what we know as *crisp* reflexivity), the sum of acyclic weights for the three chains with two alternatives and the acyclity for the chain with three alternatives (there is only one here, and it has been evaluated through its non-acyclic paths). In this example we have found that the lowest acyclity lies on the comparison between alternatives y and z.

Finally it must be pointed out that some sensitivity analysis should be developed in order to know how big can be the *structural* acyclity, that is, that irrationality only due to the size of the problem: the bigger number of comparisons needed, the lowest acyclity is expected. For example, in Montero [4] it was shown that in very long chains we could expect acyclities lower than 0.5.

3.- A POSSIBILITY THEOREM

Following the welfare-oriented approach of Arrow, and assuming that preference intensities are allowed, in the sense that they are represented by fuzzy preference relations being complete, for each individual and the group itself, then we can think as a first step that a social welfare function should be a mapping

$$S:\ \mathcal{R}(X)^n \rightarrow \mathcal{R}(X)$$

where n is the number of individual opinions to be aggregated (*card*(D)=n), imposing some *ethical* conditions translated from the classical *crisp* context: non-negative response, independence of irrelevant alternatives, citizen sovereign and

non-dictatorship, together with some *unrestricted* domain. It will not be really important for our objective now how the first four conditions are in fact formalized (here we adopt the proposals given in Montero [4]); the key point for us will be how that *unrestricted domain* condition should be understood: since it is clear that it is very difficult for any individual to define a *crisp* binary preference relation being acyclic, the minimum condition could be something like *individuals should be absolutely irrational*, and then it is coherent to look for an aggregated opinion *not being absolutely irrational*. In this way, our social welfare function will be a mapping

$$S:\ \mathcal{F}(X)^n \rightarrow \mathcal{F}(X)$$

being

$$\mathcal{F}(X) = \{\mu\in\mathcal{R}(X)\ /\ A(\mu)>0\}$$

and satisfying the above four ethical conditions. The main result -though weak indeed because it merely implies the absence of absolute irrationality, no matter how close the opinions are to absolute rationality- can be summarized in the following theorem:

Theorem.- It is assured the existence of social welfare functions $S:\ \mathcal{F}(X)^n \rightarrow \mathcal{F}(X)$ satisfying the ethical conditions as formalized in Montero [4].

Therefore, we can assure the existence of *democratic* aggregation rules for individual preferences in such a way that $A(\sigma)>0$ always holds for the social fuzzy preference σ (analogous results can be obtained with other alternative families of *ethical* conditions). Hence, it makes sense to look for the best -in some way- aggregation rule.

One of these *democratic* rules is that obtained by the mean, $\mu(x,y)=\sum_{i=1}^{n}\mu_i(x,y)/n$ (see Montero [5] for a discussion -in a more general context- on some *ethical* and *democratic* properties, together with some conditions for its *stability* against manipulation].

REFERENCES

[1] K.J. Arrow (1951, 1964), *Social Choice and Individual Values*. Wiley, New York.

[2] D. Black (1958), *The Theory of Committees and Elections*, Cambridge University Press, Cambridge.

[3] A.K. Sen (1970), *Collective Choice and Social Welfare*, Holden-Day, San Francisco.

[4] J. Montero (1987), *Arrow's theorem under fuzzy rationality*. Behavioral Science 32, 267-273.

[5] J. Montero (1990), *Some fuzzy models in group decision making*. Polish Operational and Systems Research Society Meeting, Warsaw.