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MULTIVALUED SYSTEMS IN RELIABILITY: A CASE STUDY

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Abstract: This paper deals with general systems reliability, where the system under consideration has a multicriteria nature (that is, the state of the system takes values according various different criteria), being each component and each criterion represented by linearly ordered sets. Practical advantages of such mathematical model are discussed and some properties are developed in order to be applied in a particular manufacturing system.

Keywords: Manufacturing Processes, Reliability Structure Function, Reliability Bounds.

INTRODUCTION

Since performance of most real systems are usually described according not only one criterion, developing reliability models for such multicriteria nature systems should be considered really as a need when dealing with real problems. For example, since very often production systems are subject to different kinds of failures, and quality in such cases can be rarely evaluated just with the ratio of "perfect" items.

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In general, we assume that our system can be modeled in terms of a mapping $\phi: L^n \to L^k$ -called "multivalued L-structure function", MLSF- establishing the state of the system depending on the states of its n components, through a single value set L, being a complete and linearly ordered. If $L=\{0,1\}$ and $k=1$, our system is a classical binary system: if L is a finite linearly ordered set and $k=1$, we are dealing with multistate reliability theory (cfr. El-Neweihi *et al.*, 1978, and Griffith ,1980,): if the representation into a real interval L is possible, our structure function is said to represent to a "continuum system" (cfr. Baxter, 1984; Block & Savits, 1984; Montero et al., 1990).

Of course, though all theoretical results in this paper assume that all components and criteria take values in the same complete and linearly ordered set L, it will be clear that any result can be easily translated to systems where the state spaces vary from component to component or from criterion to criterion (the essential property is just the order relationship in L). For example, many common real systems are described through a mixture of discrete and continuous components or criteria.

MINIMAL PATHS AND CUTS FOR COHERENT MLSF

As usual, we shall assume that our MLSF $\phi: L^n \to L^k$ is "coherent" in the sense of being monotonic ($\phi(x) \geq \phi(y)$ if $x \geq y$ holds) and with no irrelevant components: being ϕ monotonic, component i is called "irrelevant" if $\phi(1_i, x) = \phi(0_i, x)$ $\forall x \in L^n$, being $0 = \inf L$ and $1 = \sup L$, where (y_i, x) represents the vector $(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n)$ and "$\geq$" represents simultaneously the usual product order in L^n or L^k ($x \geq y$ holds if $x_i \geq y_i$ $\forall i$, $x > y$ holds if $x \geq y$ but not $y \geq x$ hold, and $x >> y$ will denote $x_i > y_i$ $\forall i$).

Moreover, since we are going to assume the existence of a probability measure $\mathbb{P}$ on the Borel σ-field $\mathcal{B}^n$ generated by the order topology in the initial space L^n, such MLSF should also be assumed -with no loose of practical generality- measurable with respect to the Borel σ-fields $\mathcal{B}^n$ and $\mathcal{B}^k$ associated to the natural order topologies in L^n and L^k.

Following Montero et al., 1992, we shall consider the problem of bounding reliability function (the mapping $R: L^k \to [0,1]$ measuring the probability of each path level set, $R(s) = \mathbb{P}\{x \in L^n / \phi(x) \geq s\}$ $\forall s \in L^k$) based upon general definitions of minimal path and minimal cuts of the associated single-evaluated systems, but also a global equivalent approach will be also considered, leading to the same reliability bounds. In any case, basic concepts given here generalize in fact most particular definitions in the literature (those that can be found in the context of binary systems, multistate systems or continuum systems).

Definition 1.- Let us consider a monotonic MLSF ϕ. A vector $p \in L^n$ is a minimal path at level $s \in L_0^k$ if

i) $\exists (y^m)_{m \in \mathbb{N}}$ / *lim* $y^m = p$ and $\phi(y^m) \geq s$ $\forall m$

ii) $\forall x < p$ $\exists j$ / $\phi_j(x) < s_j$

(by convenience, if $\exists j / \phi_j(x) < s_j$ holds for each element $x \in L^n$, we

take p with $p_i = \sup L$ $\forall i$ as the minimal path at each level s).

Definition 2.- A vector $c \in L^n$ is a minimal cut at level $s \in L_0^k$ of a monotonic MLSF ϕ if

i) $\exists (y^m)_{m \in \mathbb{N}}$ / *lim* $y^m = c$ and $\phi(y^m) \leq s$ $\forall m$

ii) $\forall x > c$ $\exists j$ / $\phi_j(x) > s_j$

(also by convenience, c with $c_i = \inf L$ $\forall i$ will be the minimal cut at level s if $\exists j / \phi_j(x) > s_j$ is assured for each element $x \in L^n$).

Definition 3.- A vector $f \in L^n$ is a minimal felling at level $s \in L_0^k$ of a monotonic MLSF ϕ if

i) $\exists (y^m)_{m \in \mathbb{N}}$ such that *lim* $y^m = f$ & $\forall m$ $\exists j$ / $\phi_j(y^m) \leq s_j$

ii) $\phi(x) >> s$ $\forall x > f$

(as above, $f_i = \inf L$ $\forall i$ if $\phi_j(x) > s_j$ $\forall j$ always holds).

RELIABILITY BOUNDS

Let us consider here a fixed measurable multivalued usual structure function, modeled by a monotonic measurable MLSF $\phi: L^n \to L^k$ defined over a probability space $(L^n, \mathcal{B}^n, \mathbb{P})$, and the associated reliability function R.

Theorem 1.- Let ϕ be a monotonic MLSF and let $s \in L_0^k$ be fixed, with $s_j > \inf L_0$ $\forall j$. Then

$$1 - \mathbb{P}\{x / x_i \leq \sup f_i \ \forall i\} \leq R(s) \leq \mathbb{P}\{x / x_i \geq \inf p_i \ \forall i\}$$

where supreme and infimum are taken in the set of minimal fellings f at any level $\alpha << s$, and in the set of minimal paths p at level s.

Theorem 2.- Let ϕ a monotonic MLSF and let $s \in L_0^k$ be fixed, with $s_j > \inf L_0$ $\forall j$. Then

$$\sup P\{x / x >> p\} \le R(s) \le 1 - \sup P\{x / x << f\}$$

where supremes are taken, respectively, in the set of minimal fellings f at any level $\alpha << s$ and in the set of minimal paths p at level s.

Theorem 3.- Let ϕ a monotonic MLSF and let $s \in L_0^k$ be fixed, with $s_j > \inf L_0$ $\forall j$. Then

$$1 - P\{x / x_i \le \sup f_i \ \forall i\} + \sup P\{x / p_i < x_i \le \sup f_i \ \forall i\} \le R(s) \le$$

$$\le P\{x / x_i \ge \inf p_i \ \forall i\} - \sup P\{x / \inf p_i < x_i < f_i \ \forall i\}$$

where supremes and infimum are taken in the set of minimal fellings f at any level $\alpha << s$ and in the set of minimal paths p at level s.

The above reliability bounds can also be obtained by considering the associated single valued usual systems $\phi_j : L^n \to L$ $(j=1,2,\ldots,k)$ and their minimal paths and minimal cuts:

Theorem 1'.- Let ϕ be a monotonic MLSF, and let $s \in L_0^k$ be fixed, with $s_j > \inf L_0$ for some j. Let us consider the points $q, r \in L^n$ such that

$$q_i = \max_{j=1,\ldots,k} \{\inf p_i^j \,/\, p^j \in \mathcal{P}_j(s_j)\} \qquad \forall i=1,2,\ldots,n$$

$$r_i = \max_{j=1,\ldots,k} \{\sup c_i^j \,/\, c^j \in \mathcal{C}_j(s_j)\} \qquad \forall i=1,2,\ldots,n$$

where $\mathcal{P}_j(s_j)$ represents the set of minimal paths at level s_j for ϕ_j, and $\mathcal{C}_j(s_j)$ the set of minimal cuts/fellings at any level $\alpha_j < s_j$ for ϕ_j. Then

$$1 - P\{x / x \le r\} \le R(s) \le P\{x / x \ge q\}$$

Theorem 2'.- Let ϕ be a monotonic MLSF, and let $s \in L_0^k$ be fixed, with $s_j > \inf L_0$ for some j. Let $\{p^j\}_{j=1}^k$ denote an arbitrary family of minimal paths $p^j \in \mathcal{P}_j(s_j)$ $(j=1,\ldots,k)$, and let c^j denote an arbitrary minimal cut/felling within $\mathcal{C}_j(s_j)$. Then

$$\sup P\{x \,/\, x_i > \max_{j=1,k} p_i^j \ \forall i\} \le$$

$$\le R(s) \le 1 - \sup P\{x \,/\, x << c^j\}$$

Theorem 3'.- Let ϕ be a monotonic MLSF and let $q, r \in L^n$ and $p^j, c^j \in L^n$ be given according to theorems 1' and 2'. Then

$$1 - P\{x / x \le r\} + \sup P\{x / \max_{j=1,k} p_i^j < x_i \le r_i \ \forall i\} \le$$

$$\le R(s) \le P\{x / x \ge q\} - \sup P\{x \,/\, q \le x << c^j\}$$

It must be pointed out that though mathematical results are just the same, there will be differences in computation time (the effort in finding out those global paths and fellings can simplify the always tedious search for supremes -though in practice, they will be approached through maximums over arbitrary finite families of minimal paths or minimal cuts/fellings-):

Lemma 1.- Let us assume the notation given in the above theorems, with $s_j > \inf L_0$ $\forall j$. Then

$$q_i = \inf \{p_i / p \text{ minimal path of } \phi \text{ at level } s\}$$

holds for all i, and

$$\sup Pr\{x / x >> p\} = \sup Pr\{x \,/\, x_i > \max_{j=1,k} p_i^j \ \forall i\}$$

Lemma 2.- Let us assume the notation given in the above theorems, with $s_j > \inf L_0$ $\forall j$. Then

$$r_i = \sup \{f_i / f \text{ minimal felling of } \phi \text{ at any level } \alpha \ll s\}$$

holds for all i. Moreover,

$$\sup \Pr(x/x \ll f) = \sup \Pr(x/x \ll c^j)$$

Moreover, the above reliability bounds, being them so general, can trivially be improved in two ways:

(1) Making use of any particular analytic property of our structure function (e.g., if components and criteria space of states are discrete, or if we are dealing with a continuous real mapping), or upon the probability measure P (e.g., if it is a discrete or a continuous distribution).

(2) By repeating the underlying process in theorem 3: each type I reliability bound (theorem 1) has been improved with the aid of one extra-point (searching for supreme), but of course such combined bound can be improved if additional points are considered. In practice it could be easier to obtain such type I reliability bounds and then get improvements just taking into account a finite number of paths or cuts/fellings.

MANUFACTURING PROCESSES

An interesting case in which the above results can be applied are those modeling some manufacturing processes, where the items are elaborated successively through k stages and each criteria j for the overall operating level can be represented by a product structure function

$$\phi_j(x_1, x_2, \dots, x_n) = \prod_{i=1}^{k(j)} x_i$$

with n independent Beta-distributed components. Results for bicriteria cases (k=2) can be then compared -making use of *ad hoc* approximations in order to avoid the always tedious search for supremes- with the exact reliabilities (estimated by standard simulation) and other reliability bounds based on theoretical properties of the product of independent Beta random variables.

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