

## An axiomatic approach to fuzzy rationality

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### Abstract

Fuzzy preference relations formalize intensity of individual preferences over fixed sets of alternatives. It is therefore natural to extend to fuzzy preferences the notion of rationality or consistency. With this goal in mind, we address in this paper the problem of giving an axiomatic basis for defining the concept of fuzzy rationality.

### 1 Introduction

Fuzzy preference relations formalize intensity of individual preferences over fixed sets of alternatives. It is therefore natural to extend to fuzzy preferences the notion of rationality or consistency. In doing so, we would intuitively expect that practical difficulties increase proportionally with the fuzziness of the preference relations. Thus, we believe to be of particular relevance to address the problem of giving an axiomatic basis to define the concept of fuzzy rationality.

Fuzzy preference relations were introduced by Zadeh [9] in order to capture intensity degrees of preferences. For instance, two different alternatives may be considered clearly better than a third alternative, but one preference may not be so intense as the other one. Given an arbitrary finite set of alternatives  $X$ , a fuzzy preference relation is defined as a mapping  $\mu : X \times X \rightarrow [0, 1]$  where  $\mu(x, y)$  represents the degree to which alternative  $x$  is considered not worse than alternative  $y$ . Obviously, this definition contains classical crisp preference relations, that is relations verifying  $\mu(x, y) \in \{0, 1\}$ .

An immediate question arises: when the values of  $\mu$  should be considered *consistent*, i.e. when  $\mu$  should be considered to be *rational*?

The idea of being *consistent* or *rational* is related to some practical problems that preference relations may create. In the crisp context, being consistent means that there are no cycles of preferences (see for example [7]), which is equivalent to classical

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max-min transitivity when dealing with crisp preference relations that are complete, i.e. satisfying  $\mu(x, y) = 1$  or  $\mu(y, x) = 1$  for all  $x, y \in X$ . Hence, classical crisp binary order preference relations, i.e. those relations verifying reflexivity, antisymmetry and transitivity, appeared in principle as the consistent crisp preference relations (an alternative idea of rationality underlies in the definition of quasi transitivity introduced in [8]). The assumption of crisp order as a basis for crisp rationality is therefore based upon practical considerations, identifying consistency with rationality.

## 2 Problem setting

A standard assumption in the context of fuzzy preference relations is *max-min transitivity*. A fuzzy preference relation  $\mu$  is max-min transitive if  $\mu(x, z) \geq \min\{\mu(x, y), \mu(y, z)\}$  for all  $x, y, z \in X$  (see [6]). In this way, if  $x$  is better than  $y$  with an intensity  $\mu(x, y)$  and  $y$  is better than  $z$  with an intensity  $\mu(y, z)$ , then the degree to which  $x$  is considered to be better than  $z$  can never be lower than both values  $\mu(x, y)$  and  $\mu(y, z)$ . The intuitive meaning of this property is clear and, in fact, it generalizes classical crisp transitivity. However, the property of being max-min transitive is crisp, in the sense that each relation will either be max-min transitive or it will not be max-min transitive. There is no gradation in the degree of membership to the set of max-min transitive fuzzy preference relations.

Such a gradation in rationality appears in the

concept of fuzzy rationality as defined in Montero [2, 3], initially introduced in order to formalize the problem of *rational aggregation rules* in group decision making for complete fuzzy preference relations, i.e. relations verifying  $\mu(x, y) + \mu(y, x) \geq 1$  for all  $x, y \in X$ , (see [1] for a characterization of such *rational aggregation rules*).

Our goal here is to axiomatically characterize fuzzy rationality measures, and we will obtain that Montero's fuzzy rationality and other rationality measures will be particular cases. In general, five seem to be the basic properties to be imposed to any fuzzy rationality measure:

1. the notion of being absolutely rational and absolutely irrational must both be well defined.
2. A rationality measure must be invariant with respect to any permutation on the set of alternatives.
3. *Opposite* fuzzy preference relations must have the same degree of rationality, where the opposite  $\neg\mu$  of a given fuzzy preference relation  $\mu$  is defined as  $\neg\mu(x, y) = \mu(y, x)$ , for all  $x, y$ .
4. Individuals do not lose rationality when their opinions are evaluated on a smaller subset of alternatives.
5. Consistent changes in the values of the fuzzy relations must produce consistent changes in their associated rationality values. This also implies that any rationality measure must be in some way coherent with the basic idea of

crisp rationality previously defined, assigning lower degrees of rationality to preferences farther away from crisp absolutely rational preferences.

The above five conditions are formalized and the importance of the last one is properly underlined. Some specific properties and examples are also analyzed, by taking into account some comments given in [4, 5].

Finally, it is pointed out that a strong inconvenience will appear in real applications when small changes in intensity values may lead to big changes in rationality evaluation measures. Therefore, an additional sixth condition related to this idea of stability should be imposed in practice. All crisp fuzzy rationality measures (those assigning values 0 or 1 to every fuzzy preference measures) will not be stable in this sense, showing how difficult decision making problems can be if a crisp rationality measure has been assumed.

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