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Equivalence of Fuzzy Rationality Measures

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Abstract

The notion of fuzzy rationality can naturally be extended to fuzzy preference relations which formalize intensity of individual preferences over fixed sets of alternatives. Once fuzzy rationality measures have been axiomatically introduced, it is natural to investigate on the practical equivalence of two or more fuzzy rationality measures and to provide some characterization theorems for such an equivalence relation.

Keywords: aggregation rules, fuzzy preferences, decision making.

1 Introduction and preliminaries

In some recent scientific investigations (see [3, 4]) a set of axioms that characterize fuzzy rationality measures has been proposed. The underlying idea is that rationality of individuals is a fuzzy concept. Therefore, it must be possible to assign to each individual a value of rationality between 0 (absolute irrationality) and 1 (absolute rationality). This value (degree) of rationality can be assigned in many different ways, and each of these different assignments corresponds to a different criterion for measuring the rationality of an individual. Such criteria are called *fuzzy rationality measures*.

In particular, in [3] it is analyzed the problem of defining rationality of individuals based only on their opinions expressed over a fixed set of alternatives and in [4] such a problem is given an intuitively sound solution by introducing a collection of conditions to be satisfied by any fuzzy rationality measure.

Individuals expressing their opinions over pairs of elements of a finite set of alternatives are characterized by fuzzy preference relations. The latter were introduced by Zadeh [9] in order to capture degrees of preferences. Given a finite set of alternatives X a fuzzy binary preference relation μ is defined as a fuzzy set over all pairs of the cartesian product $X \times X$, so that for each pair (x, y) $\mu(x, y)$ represents the strength or intensity of preference between x and y, measured in the unit interval.

Throughout this paper we will also assume that fuzzy preference relations are complete, i.e.

$$\mu(x, y) + \mu(y, x) \geq 1 \quad \forall x, y \in X.$$

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Following [2], completeness is required in order to assure that all individuals consider the set of alternatives on which they are expressing their opinions as feasible and comprehensive. The values

$$\begin{aligned} \mu_I(x, y) &= \mu(x, y) + \mu(y, x) - 1 \\ \mu_B(x, y) &= \mu(x, y) - \mu_I(x, y) \\ \mu_W(x, y) &= \mu(y, x) - \mu_I(x, y) \end{aligned} \quad (1.1)$$

are understood, respectively, as the degree to which the two alternatives are indifferent (xIy), the degree of strict preference of x over y, (xBy, x is better than y) and the degree of strict preference of y over x (xWy, x is worse than y). Thus, preference between two alternatives x and y are explained by means of these three intensity values (xBy, xIy, xWy), each one associated to one possible crisp relation between both alternatives in such a way that

$$\mu_B(x, y) + \mu_I(x, y) + \mu_W(x, y) = 1 \quad \forall x, y \in X.$$

The collection of complete fuzzy binary preference relations defined over the finite set of alternatives X will be denoted by $\mathcal{P}(X)$.

In the next sections, we will first review the definition of fuzzy rationality measures given in [4] and subsequently we will introduce the notion of equivalence among fuzzy rationality measures and give theorems that characterize such an equivalence relation.

2 Fuzzy rationality measures

Many measures for individual fuzzy rationality have been proposed in the past.

In the case in which the relations are crisp, i.e. $\mu(x, y) \in \{0, 1\}$ for all $x, y \in X$, rationality has been characterized with the absence of cycles (see [8]).

In the context of fuzzy binary preference relations, a standard assumption to characterise rationality is max-min transitivity (see [8] and [5, 10]). Under this hypothesis if x is better than y with intensity $\mu(x, y)$ and y is better than z with intensity $\mu(y, z)$ then the degree to which x is better than z, i.e. $\mu(x, z)$, can never be lower than both values $\mu(x, y)$ and $\mu(y, z)$. However, the property of being max-min transitive is crisp, that is each relation either is or is not max-min transitive. Therefore, assuming max-min transitivity as key property for rationality does not allow a fuzzy classification.

This fuzzy classification of individuals is obtained if one uses the rationality measure given by Montero in [6, 7]. Such a measure was initially introduced in order to formalise the problem of rational aggregation rules in group decision making (see [4] for a characterization of such rational aggregation rules).

In [4] rationality measures are formally characterised as maps of type

$$\rho : \mathcal{P} \to [0, 1]$$

where

$$\mathcal{P} = \bigcup_{X \text{ finite}} \mathcal{P}(X),$$

and a set of conditions that any fuzzy rationality measure must satisfy is introduced. Such conditions are the following ones:

(R1) For any finite set of alternatives X, $\rho(\mu) = 1$ for any crisp binary preference μ which is a strict chain on X;

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(R2) Given $\mu \in \mathcal{P}$ and a permutation $\pi : X \to X$ then

$$\rho(\mu^\pi) = \rho(\mu)$$

where $\mu^\pi(x,y) = \mu(\pi(x),\pi(y))$ for all $x, y \in X$.

(R3) For all $\mu \in \mathcal{P}$, $\rho(\neg\mu) = \rho(\mu)$
where $\neg\mu(x,y) = \mu(y,x)$.

(R4) Let Y be a non-empty finite set of alternatives and let z be an extra alternative not belonging to Y. Let us consider a fuzzy preference $\mu : Y \times Y \to [0,1]$ such that $\mu(y,z) = 1, \mu(z,y) = 0, \forall y \in Y_1, \forall z \in Y_2$ for some Y_1, Y_2 partition of Y, and an extension μ' such that

$$\mu'(y,z) = \mu(y,z), \forall y, z \in Y$$

$$\mu'(y,z) = 1, \mu'(z,y) = 0, \forall y \in Y_1$$

$$\mu'(z,x) = 1, \mu'(x,z) = 0 \ \forall x \in Y_2$$

$$\mu'(z,z) = 1$$

Then it must be

$$\rho(\mu') \geq \rho(\mu).$$

(R5) Let $\mu \in \mathcal{P}(X)$ be fixed. Given an arbitrary ordered pair of alternatives $(\bar{x},\bar{y})$, an arbitrary point $(\bar{a},\bar{b}) \in [0,1] \times [0,1]$, and real numbers γ and λ such that $0 \leq \bar{a} + \lambda\cos\gamma \leq 1$, $0 \leq \bar{b} + \lambda\sin\gamma \leq 1$ and $\bar{a} + \bar{b} + \lambda(\sin\gamma + \cos\gamma) \geq 1$, we denote by $\Gamma_\mu((\bar{x},\bar{y}),(\bar{a},\bar{b}),\gamma,\lambda)$ the fuzzy preference relation defined as

$$\Gamma_\mu((\bar{x},\bar{y}),(\bar{a},\bar{b}),\gamma,\lambda)(x,y) = \begin{cases} \bar{a} + \lambda\cos\gamma & \text{if } (x,y) = (\bar{x},\bar{y}) \\ \bar{b} + \lambda\sin\gamma & \text{if } (x,y) = (\bar{y},\bar{x}) \\ \mu(x,y) & \text{otherwise} \end{cases}$$

Let $(\bar{x},\bar{y})$, $(\bar{a},\bar{b})$, γ be fixed and let us consider the fuzzy preference relation $\Gamma^*(\lambda)$ defined as

$$\Gamma^*(\lambda)(x,y) = \Gamma_\mu((\bar{x},\bar{y}),(\bar{a},\bar{b}),\gamma,\lambda)(x,y)$$

Then, one of the following three properties must be verified by ρ:

(R5.1) $\rho(\Gamma^*(\lambda))$ is monotone, i.e. either

$$\rho(\Gamma^*(\lambda)) \leq \rho(\Gamma^*(\lambda'))$$

for any $\lambda \leq \lambda'$ or

$$\rho(\Gamma^*(\lambda)) \geq \rho(\Gamma^*(\lambda'))$$

for any $\lambda \leq \lambda'$.

(R5.2) there exists a value λ such that

$$\rho(\Gamma^*(\lambda)) \geq \rho(\Gamma^*(\lambda_1)) \geq \rho(\Gamma^*(\lambda_2))$$

for all λ_1, λ_2 such that either $\lambda < \lambda_1 < \lambda_2$ or $\lambda > \lambda_1 > \lambda_2$.

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(R5.3) there exists a value λ such that

$$\rho(\Gamma^*(\lambda)) \leq \rho(\Gamma^*(\lambda_1)) \leq \rho(\Gamma^*(\lambda_2))$$

for all λ_1, λ_2 such that either $\lambda < \lambda_1 < \lambda_2$ or $\lambda > \lambda_1 > \lambda_2$.

(R1) is a foundational axiom and it expresses the fact that if ρ is a fuzzy rationality measure then it must be based upon some clear idea of rationality. In particular, whenever the individual can sort the alternatives in a precise way then the degree of rationality must be one.

(R2) is a consequence of the hypothesis that rationality is judged solely on the basis of the opinions (degree of preferences) expressed by individuals over a fixed set of alternatives. As a consequence, all the alternatives must be considered intrinsically equivalent, i.e. rationality must not depend upon the name of an alternative. Therefore, fuzzy rationality measures must be invariant with respect to any permutation of the set of alternatives.

(R3) reflects the common sense observation that two individuals that express opposite opinions must have the same degree of rationality.

(R4) derives from the belief that the rationality degree is persistent with respect to good crisp extensions of the individual opinions over an extended set of alternatives and in turn is a consequence of the hypothesis that rationality depends solely on the individuals preference intensities.

(R5) is a regularity conditions. It forces fuzzy rationality measures to change their values consistently with the changes in the individuals opinions (see [4] for more comments on the regularity condition).

We then have the following definition.

DEFINITION 2.1 *Any mapping $\rho : \mathcal{P} \to [0,1]$ is a*

1. normal fuzzy rationality measure *if it verifies conditions (R1)-(R5.1);*
2. pessimistic fuzzy rationality measure *if it verifies conditions (R1)-(R5.2);*
3. optimistic fuzzy rationality measure *if it verifies conditions (R1)-(R5.3).*

□

We will now give some examples of fuzzy rationality measures.

2.1 Normal fuzzy rationality measures

(Ex1) Let μ be a binary fuzzy preference relation. We consider all cycles of type

$$C = x_1 B x_2 B \cdots x_k B x_1$$

that we will call B-cycles. To a given B-cycle $C = x_1 B x_2 B \cdots x_k B x_1$ we associate the weight

$$W_\mu(C) = \prod_{i=1}^{k} \mu_B(x_i, x_{i+1})$$

where for convenience $x_{k+1} = x_1$. We then put

$$\rho_N(\mu) = 1 - \max\{W_\mu(C) | C \ B\text{-cycle}\}.$$

2.2 Pessimistic fuzzy rationality measures

The first example of pessimistic fuzzy rationality measure is given by max-min transitivity.

(Ex2)

$$\rho_{max-min}(\mu) = \begin{cases} 1 & \text{if } \mu \text{ is max-min transitive} \\ 0 & \text{otherwise} \end{cases}$$

An example of pessimistic fuzzy rationality measure which allows a fuzzy classification is given by the measure introduced by Montero in [6, 7].

Such a measure is computed by looking at all possible chains $G = (x_1 - x_2 - \cdots - x_k - x_1)$ of distinct alternatives and considering all cycles of type

$$x_1 P_1 x_2 P_2 \cdots x_k P_k x_1$$

where $P_h \in \{W, I, B\}$ for all $h = 1, 2, \ldots, k$. A cycle $x_1 P_1 x_2 P_2 \cdots x_k P_k x_1$ is irrational if either

- $P_h \in \{B, I\}$ for all $h = 1, 2, \ldots, k$ and $B \in \{P_h : h = 1, 2, \ldots, k\}$; or
- $P_h \in \{W, I\}$ for all $h = 1, 2, \ldots, k$ and $W \in \{P_h : h = 1, 2, \ldots, k\}$,

whereas a cycle is rational if it is not irrational.

Given a cycle $C \equiv x_1 P_1 \cdots x_k P_k x_1$ where $P_h \in \{B, I, W\}$ for all $h = 1, 2, \ldots, k$, the weight associated to C and denoted by $\Delta(C)$ is

$$\Delta(C) = \Pi_{h=1}^{k} \mu_{P_h}(x_h, x_{h+1})$$

where $x_{k+1} = x_1$ for convenience.

Given a chain $G = (x_1 - x_2 - \cdots - x_k - x_1)$ the degree of rationality associated to G and denoted by $A_\mu(G)$ is defined as

$$A_\mu(G) = \sum_{C \in \text{rat.cycles}} \Delta(C).$$

In particular (see [1]) $A_\mu(G)$ verifies

$$A_\mu(G) = 1 - (\Pi_{h=1}^{k} \mu(x_h, x_{h+1}) + \Pi_{h=1}^{k} \mu(x_{h+1}, x_h) - 2\Pi_{h=1}^{k} \mu_I(x_h, x_{h+1})). \quad (2.2)$$

(Ex3) In view of 2.2, Montero's rationality is defined as the fuzzy property $\rho_M : \mathcal{P} \to [0,1]$ with

$$\rho_M(\mu) = \min_G A_\mu(G) \quad (2.3)$$

2.3 Optimistic fuzzy rationality measure

We now give an example of an optimistic fuzzy rationality measure ρ_O using the fuzzy rationality measure ρ_P above introduced

(Ex4) We define ρ_O as

$$\rho_O(\mu) = \max_{Y \subseteq X : |Y| \geq 2} \left\{ \frac{|Y|}{|X|} \rho_P(\mu_Y) \right\}$$

where μ_Y denotes for simplicity the restriction of the fuzzy preference $\mu : X \times X \to [0,1]$ to $Y \times Y$.

3 The notion of equivalence

As we have seen, there are many ways in which we can measure rationality of individuals. From a practical point of view, however, two fuzzy rationality measures can be considered equivalent if they always agree when comparing individual rationalities. We can therefore introduce the following definition

DEFINITION 3.1 *Given two fuzzy rationality measures ρ_1 and ρ_2 we will say that ρ_1 and ρ_2 are equivalent if and only if for any pair of individuals μ_1 and μ_2, $\rho_1(\mu_1) \geq \rho_1(\mu_2)$ if and only if $\rho_2(\mu_1) \geq \rho_2(\mu_2)$.* □

It is easy to see that the above definition is indeed an equivalence relation. Moreover, any two equivalent fuzzy rationality measures are based on the same idea of rationality as expressed by conditions (R.2) and (R5). Indeed, the following theorem holds.

THEOREM 3.1 *If ρ_1 and ρ_2 are equivalent then*

1. *$\rho_1(\mu) = 0$ [resp. $\rho_1(\mu) = 1$] if and only if $\rho_2(\mu) = 0$ [resp. $\rho_2(\mu) = 1$].*
2. *ρ_1 is normal [resp. pessimistic, optimistic] if and only if ρ_2 is normal [resp. pessimistic, optimistic].*

■

An analytical characterisation of the equivalence relations among fuzzy rationality measures is given by the following theorem.

THEOREM 3.2 *Let ρ_1 and ρ_2 be two fuzzy rationality measures. Then ρ_1 and ρ_2 are equivalent if and only if there exists a strictly increasing function $f : [0,1] \to [0,1]$ such that*

- *$f(0) = 0$ and $f(1) = 1$;*
- *$\rho_2 \equiv f \circ \rho_1$, i.e. for all μ, $\rho_2(\mu) = f(\rho_1(\mu))$.*

■

Theorem 3.2 can be generalised as follows

THEOREM 3.3 *Let ρ be a fuzzy rationality measure. If $f : [0,1] \times [0,1] \to [0,1]$ is a strictly increasing function such that $f(0) = 0$ and $f(1) = 1$, then $f \circ \rho$ is a fuzzy rationality measure.* ■

The above theorem shows how we can obtain infinitely many equivalent fuzzy rationality measures starting from a given one.

4 Concluding remarks

In this paper we have introduced the notion of equivalence among fuzzy rationality measures as axiomatically characterised in [3, 4]. Such an equivalence condition is analytically characterised by means of theorems that on one hand show that equivalent fuzzy rationality measures must be based on the same underlying idea of rationality, and on the other hand make possible to create hierarchies of infinitely many rationality measures pairwise equivalent.

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