

A DISCUSSION ON CONSISTENCY OF GLOBAL AGGREGATION OPERATORS

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Summary

Several authors have introduced *global aggregation operators* as families of aggregation operators $\{T_n\}_n$, being each one of these T_n the n -ary operator in charge of aggregation, if and only if the number of items being actually aggregated is n . But according to this general definition, it is clear that operators within the same family may be not related at all. In this paper we shall point out the absolute need of an additional condition allowing some kind of *consistency* for such a family of operators, in order to be considered an *aggregation rule*. In particular, we shall discuss some advantages and limits of the concept of *recursiveness*, which basically assumes that the aggregation can be worked out by means of a recursive process. A comparative analysis with some alternative consistency conditions is also presented.

1 Introduction.

Quite a number of papers have been published in the past stressing the role of t -norms and t -conorms [27] within Fuzzy Sets Theory (see, e.g., the recent books of Klement *et al.* [14] and Calvo *et al.* [6]). The classical structure of binary logic was directly translated into the fuzzy context, with two basic conjunction and disjunction operators generalizing each one the classical binary connectives, now allowing any value within the unit interval $[0, 1]$ as a possible degree for *truth* instead of just the two extreme values $\{0, 1\}$, respectively associated to *falsehood* and *truth*. Negation was generalized in a similar way, showing the potential existence of a rich family of solutions for each one of these operators within a $[0, 1]$ context (see Trillas [28]). But it was very soon pointed out that key aggregation operators, like weighted means, for example, were neither t -norms

or t -conorms. A key effort in this direction is the work of Yager [29] on OWA operators and Yager-Rybalov on other *uninorms* [30] (see also [11]). Nevertheless, as already pointed out in [8] (see also [9, 10]), we realize that each OWA operator can deal only with an exact number of items, and we usually do not know such a number in advance. We do need to consider *global aggregation operators* as a family of n -ary operators, so we can always evaluate aggregation, no matter the number of items to be aggregated: a procedure for each case has to be defined (see [15, 19, 21, 23] but also [12]).

The approach of Cutello and Montero [8], initially conceived only for OWA operators, was then brought into a more general framework, leading to the concept of *recursive rule* (see [9, 10]). The recursive model seems to offer nice properties: for example, taking into account classical results [1], the solution of a key generalized associativity equation (in the sense of Mak [17]) will lead to a quasi-additive recursive aggregation, as shown in Amo *et al.* [3].

In the next sections we shall analyze main consequences of the recursive approach, pointing out that this assumption implies a strong consistency on the family of n -ary operators. Hence, recursiveness appears as a natural way of assuring that our family of n -ary operators is a proper rule. As pointed out in [26], not every family of n -ary operators should be considered a proper rule. Needless to say, recursiveness is not the only way of obtaining proper aggregation rules and some alternative approaches can be tried.

2 Recursive rules.

The key idea of recursiveness is that an aggregation rule, in order to be operational, should be based upon an iterative application of binary operators, taking advantage of previous aggregations. Data are therefore being assumed to be aggregated one by one, and each particular arrangement of data will tell us the sequence

of items to be aggregated. Hence, see [10], we first rearrange data.

DEFINITION 1 *Let us denote*

$$\pi_n(a_1, a_2, \dots, a_n) = (a_{\pi_n(1)}, a_{\pi_n(2)}, \dots, a_{\pi_n(n)})$$

An ordering rule π is a consistent family of permutations $\{\pi_n\}_{n>1}$ such that for any possible finite collections of numbers, each extra item a_{n+1} is allocated keeping previous relative positions of items, i.e.,

$$\pi_{n+1}(a_1, a_2, \dots, a_n, a_{n+1}) = (a_{\pi_n(1)}, \dots, a_{\pi_n(j-1)}, a_{\pi_{n+1}(j)}, a_{\pi_n(j)}, \dots, a_{\pi_n(n)})$$

for some $j \in \{1, \dots, n+1\}$.

In other words, once relative position of two elements is being fixed by means of a permutation π_n , no permutation $\pi_m, m > n$, will change it.

The following definition was then proposed in [10].

DEFINITION 2 *A left-recursive connective rule is a family of connective operators*

$$\{\phi_n : [0, 1]^n \rightarrow [0, 1]\}_{n>1}$$

such that there exists a sequence of binary operators

$$\{L_n : [0, 1]^2 \rightarrow [0, 1]\}_{n>1}$$

verifying

$$\phi_2(a_1, a_2) = L_2(a_{\pi(1)}, a_{\pi(2)})$$

and

$$\phi_n(a_1, \dots, a_n) = L_n(\phi_{n-1}(a_{\pi(1)}, \dots, a_{\pi(n-1)}), a_{\pi(n)})$$

for all $n > 2$ and some ordering rule π .

Notice that in no way we are imposing a unique binary operator for the whole iterative process. This was in fact the main criticism argued in [25] against the restrictive result obtained by Fung-Fu [13].

Right recursiveness can be analogously defined, and then we can talk about a *recursive rule* when both left and right representations hold for the same ordering rule (*standard* recursive rules when they are based upon the identity ordering rule). Then it follows (see [2]) that a connective rule $\{\phi_n\}_{n>1}$ is recursive if and only if a set of general associativity equations (in the sense of Mak [17]) hold for each n , once the ordering rule π has been already applied:

$$\begin{aligned} \phi_n(a_1, \dots, a_n) &= \\ R_n(a_{\pi(1)}, \phi_{n-1}(a_{\pi(2)}, \dots, a_{\pi(n)})) &= \\ L_n(\phi_{n-1}(a_{\pi(1)}, \dots, a_{\pi(n-1)}), a_{\pi(n)}) & \end{aligned}$$

must hold for all n . Assuming certain regularity conditions in a recursive rule (mainly strict monotonicity), it was then shown by Amo *et al.* [3] that there exist

1. a continuous and strictly increasing function

$$p : [0, 1] \rightarrow [0, \infty)$$

2. a family of continuous and strictly increasing functions

$$\{\delta_n : [0, 1] \rightarrow [0, \infty)\}_{n>1}$$

3. and a sequence of positive real numbers

$$\{c_n\}_{n \geq 1}$$

in such a way that

$$\phi_n(a_1, \dots, a_n) = \delta_n^{-1} \left(\prod_{j=2}^{n-2} c_j \sum_{k=1}^n c_1^{k-1} p(a_k) \right) \quad (1)$$

for all $(a_1, \dots, a_n) \in [0, 1]^n$, and $n \geq 2$, being $\prod_{j=2}^{\ell} c_j = 1$ whenever $\ell \leq 2$.

Additional results can be tried proposing alternative regularity conditions, or imposing extra conditions [3]. But it is important to notice in the above result that as soon as we decide the very first aggregation for the first couple of items, we are restricting our possibilities for the whole family of n -ary aggregations. Every decision will take away some degree of freedom, and eventually the aggregation rule may be fully characterized. This is a main consequence, consistent with intuition.

3 Operationality and consistency.

A key argument supporting recursiveness is indeed the iterative calculus. Items are aggregated one by one (implying an accepted linear order on the sequence data to be aggregated), and therefore any finite family of items can be obtained by means of binary operators. But imposing that each one of these sequential aggregations will be based upon the previous aggregation links all n -ary aggregation operators in a very particular way. Consistency is therefore a consequence of the way operationality has been understood (the existence of a recursive calculus). And it is the above generalized associativity equation the core of this consistency.

4 Some alternative approaches.

Some quite similar approaches can be found in the literature, and they become an alternative to recursiveness as far as some *consistency* restriction is being

implied. An aggregation rule should never be understood just as a family of n -ary operators: all those aggregation operators must be deeply related, following some building procedure which should be in principle always the same. There must be some idea behind and not only a mathematical expression with no particular meaning for users.

For example, in Mas *et al.* [19] a general associativity equation plays also a key role, but their *modularity condition* appears as a particular case:

$$F(x, G(y, z)) = G(F(x, y), z)$$

where F and G are assumed uninorms and/or t -operators (see also [20]). From our point of view, neither commutativity or associativity should be assumed as granted. Commutativity is kind of contradictory with the fact that data have been previously ordered. And associativity has no support when the binary operator is in principle not be applied into a sequential aggregation $F(x, F(y, z))$ or $F(F(x, y), z)$.

Deeply related to recursiveness seems to be the property of being *decomposable* (see [7, 12, 18]), which assumes that each item of any given subfamily of items can be substituted by the aggregated value of such a subfamily of items. Indeed, recursiveness assumes that calculus is sequentially decomposable, but recursiveness is not assuming such a particular extra behavior. Moreover, it is not clear the need of an arbitrary decomposition, if we have assumed that operationality is in some way related to a potential sequential decomposable calculus.

Finally, the compensatory condition [22] (see also [15]) seems also deeply related to recursiveness (see the seminal paper of Zimmermann-Zysno [31]). In fact, a link between each n -ary operator and the next $(n + 1)$ -ary operator is being introduced. The existence of an iterative calculus is therefore being assumed, but the basic definition does not properly link consecutive operators (Kolesárová-Komorníková [16] obtain important results but restricted to *triangular norm-based* iterative compensatory operators). A similar iterative approach is suggested in Mayor-Calvo [21] with their *self identity condition*, but again this condition should not be considered a proper link between consecutive operators (moreover, they assume that every operator is idempotent, by definition).

5 Final comments

In order to be considered a proper rule, all its n -ary operators should be deeply related, and one should not expect too much freedom once first aggregations have been fixed. Recursiveness introduces such a link from a particular computational argument. Of course

we realize some quite standard rules are not recursive rules (the *median* rule is an important one, see [12]), so we should search some more general or alternative approaches, still keeping some underlying operational principle:

- allowing a more arbitrary structure of data, as in some classification problems; for example, in [4, 5] data are organized in a surface (see [21]);
- not imposing recursiveness but the ability of taking advantage of some previous calculus that can be kept in memory (see, for example, [24]);
- introducing some computational complexity restrictions, based upon the particular calculus capacity of user (notice for example that our previous re-arrangement of data has $O(n^2)$ computational complexity).

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