ON THE DIMENSION OF FUZZY PREFERENCE RELATIONS

Javier Montero, Javier Yáñez
Department of Statistics and O.R.
Complutense University
Madrid, Spain

Vincenzo CUTELLO
Department of Mathematics
University of Catania
Catania, Italy

Abstract— **The concept of Dimension plays a key role in the context of crisp (classical) preference modelling. Dimension analysis allows a better insight of structure of individual preferences, giving a hint about the complexity of the preference modelling problem. Several attempts can be found in the literature in order to translate such a concept and its associated procedures into fuzzy preference models. In this paper, we shall first of all discuss some of these approaches, showing that no more extra knowledge about the problem is sometimes attained, mainly due to some a priori introduced restrictions. Then we shall propose an alternative approach based upon the analysis of all associated crisp α-cuts. Some consequences in related fuzzy preference modelling problems are also discussed.**

Key words: Fuzzy Preference Relation, Fuzzy Binary Relation.

I. Introduction.

Searching for useful graphical representations of each problem indeed is a main research topic in any applied field. A nice graphical representation of a complex problem should allow a better comprehension of main characteristics to be taken into account, at least for a first approach to the problem. Such a representation is most of the cases a simplified model of reality, but it is built according to the degree of knowledge of each user. Once this user has exploided that representation, a more sophisticated model will be try, by considering either a more global view of the present status or a more specific look to some particular side of the problem. This is usually the way we most of the time learn about reality, by means of an underlying picture we build in our mind, which allows us to acquire some supported intuition.

When faced to real decision making problems, most of the time we realize that our individual preferences are difficult not only to be established but also to be explained. As usual, when the problem we are facing is too complex, we trie in some way to decompose it into pieces, in such a way that each one of these pieces defines a problem of an appropriate size for our capabilities, and in such a way that such decomposing procedure is also appropriate, so complexity is not excesive in any step and we can actually deal with such a decomposed problem.

From a multicriteria point of view, Dimension Theory should be very useful in order to find out the number of underlying criteria explaining our confusing view of a complex problem. The dimension of a binary preference relation is telling us the minimum number of linear criteria explaining undecided comparisions. This is obviously an interesting information in order to get a representation in a real space. We should look for at least as many relevant criteria as the dimension value of the binary preference relation we are defining. Dimension value is also giving a hint about how complex to explain is the set of preferences we are dealing with.

Of course, several simplified hypothesis are being introduced. On one hand, we are accepting the existence of those simple criteria, being each one linearly ordered and representable on the real line. On the other hand, it is also assumed that the reason for each undecided comparison between two alternatives is the relevant opposition in order with respect to two of those basic underlying simple criteria. For example, when comparing two cars A and B, it may be the case that car A is much cheaper than car B, but car B is much faster than car A. None of both cars is majorized by the other, and since there are two (in this case explicit) criteria, such a situation is naturally represented in $\Re^2$.

Crisp Dimension Theory, as proposed by Dushnik-Miller [4], has been widely developed from a theoretical point of view for crisp binary relations (see Trotter [8]), although serious algorithmic difficulties may be implied (see Yannakakis [9]). But not much has been done for fuzzy preference relations, being them a more close representation of individual preferences. The approach proposed in Adnadjevic [1], for example, is based on the number of *multichains* allowing a representation by means of their superposition. But it does not exploid all the information supplied by fuzzy preferences, since such a representation is based only on the fact that either $\mu(x,y) > 0$ or $\mu(x,y) = 0$. Such an approach is therefore basically crisp, searching for a classical like representation that in this fuzzy framework does not seem to be robust ($\mu(x,y)$ can take very small values, but still not equal to 0). Moreover, it is surprising the fact that dimension of a fuzzy preference is not the same as the Dushnik-Miller dimension of the natural associated crisp relation (it is in general lower). Adnadjevic's approach is no generalization of classical crisp approach.

II. Some basic definitions.

Let us remind here some classic definitions, in order to settle up the basis of our approach.

Let us consider X a finite set of n alternatives. A fuzzy preference relation is just a fuzzy binary relation, that is, a mapping

$$\mu : X \times X \rightarrow [0,1]$$

It is common to assume that each value $\mu(x,y)$ represents the degree to which alternative x is better than alternative y, i.e., the degree to which the assertion "$x \geq y$" is true, meaning here "$\geq$" the crisp binary relation "to be better or equal than" (*weak* preference). If this is the case, from such a weak preference values a complete structure of preferences can be built up, assigning values to strict preferences, indifferences and incomparabilities. For example (see, e.g., [7]), one possibility is the *additive* preference structure which assigns to each pair of alternatives a degree of *indifference*

$$\mu_i(x,y) = max\{\mu(x,y) + \mu(y,x) - 1, 0\}$$

a degree of *strict* preference

$$\mu_s(x,y) = \mu(x,y) - \mu_i(x,y)$$

and a degree of *incomparability*

$$\mu_n(x,y) = 1 - \{\mu_s(x,y) + \mu_i(x,y) + \mu_s(y,x)\}$$

in such a way that

$$\mu(x,y) = \mu_s(x,y) + \mu_i(x,y) + \mu_s(y,x) + \mu_n(x,y)$$

(see Fodor-Roubens [5] for a complete more general approach).

In order to exploid already well developed crisp concepts and methods, a standard approach in the fuzzy field is to analize the familiy of all α-cuts. Applied this principle to the dimension problem leads us to study the family of Dushnik-Miller dimension values for each crisp α-cut.

Let $\alpha \in [0,1]$ be fixed. Given a fuzzy preference relation $\mu : X \times X \rightarrow [0,1]$, its associated crisp α-cut R^α is defined as the binary relation such that $(x,y) \in R^\alpha y$ if and only if $\mu(x,y) \geq \alpha$ (it can also be denoted as $x \geq_\alpha y$). That is,

$$R^\alpha = \{(x,y) \in X \times X / \mu(x,y) \geq \alpha\}$$

Obviously, R^0 is non informative, and every fuzzy binary relation is characterized by means of the family of its α-cuts, $R^\alpha, \alpha \in (0,1]$.

Some properties are commonly assumed for fuzzy preference relations, meaning at least a partially *normative* approach. Of course, everybody will agree that one should be indifferent when faced to choose between two alternatives being actually the same alternative, that is, $\mu(x,x) = 1$ (reflexivity). But this is only true if we realize those alternatives are actually the same, and this may be not so obvious in practice. From a pure *descriptive* point of view, we can not directly accept neither such an easy reflexivity. Again, a condition of transitivity can be also assumed from a *normative* point of view. It is quite frequent, for example, to imposse fuzzy max-min transitivity, in such a way that

$$\mu(x,y) \geq \min\{\mu(x,z), \mu(z,y)\} \quad \forall z \in X$$

whenever $x \neq y$. It is explained that the strengh of a chain should not be lower than the strengh of any of its links. Anyway, such an assumption is also subject to similar criticism crisp transitivity is subject to (we may be willing to accept as true values those implying a non transitive preference relation).

Max-min transitivity is a particular case of max-$*$ transitivity, being $* : [0,1] \times [0,1] \to [0,1]$ an appropriate connective:

$$\mu(x,y) \geq \mu(x,z) * \mu(z,y) \quad \forall z \in X$$

(see, e.g., [3] and [6]). In our context, the interesting property is the following.

THEOREM 1 *Let $* : [0,1] \times [0,1] \to [0,1]$ non-decreasing in each coordenate and idempotent (i.e., $a * a = a$ for all a). A fuzzy preference μ is max-$*$ transitive if and only if for all α we have that R^α is transitive.*

III. Dimension of α-cuts.

The key question is wether or not we can evaluate the dimension of each α-cut, so it can be represented in $\Re^k$, being k its dimension in the sense of Dushnik-Miller. Such a representation can be obtained when such α-cut is transitive and verifies antisymmetry. Obviously, if our fuzzy preference is assumed to verify any max-* transitivity, such a representation of an α-cut depends only on wether or not antisymmetry holds.

Antisymmetry of a fuzzy relation is usually defined in the following way:

$$\mu(x,y) > 0 \Rightarrow \mu(y,x) = 0$$

Such a condition obviously simplifies our representation problem, again by impossing that the associated crisp binary relation (obtained by taking into account only wether or not the intensity value is 0) verifies antisymmetry. Moreover, it is clear that such a condition can be applied only to preference relations meaning strict intensity values, instead of weak intensities ($\mu(x,x) = 0$ must always hold, otherwise we get into a contradiction). Anyway, dealing with fuzzy relations, such a condition of antisymmetry does not seem to be fully justified neither from a *normative* or a *descriptive* point of view. We just need α-cuts to verify crisp antisymmetry in order to allow a dimension in the sense of Dushnik-Miller (i.e., $\mu(x,y) = 1 \Rightarrow \mu(y,x) \neq 1$). Crisp antisymmetry and crisp transitivity are imposed for crisp preference relations in order to assure that there is no cycle $x > y > x$ and no cycle $x > y > z > x$, respectively (as a consequence, non existence of longer cycles is also assured).

Notice that transitivity, if written as

$$\mu(x,y) \geq \min\{\mu(x,z), \mu(z,y)\} \quad \forall x,y,z \in X$$

implies that

$$\mu(x,x) = 0 \Rightarrow \min\{\mu(x,y), \mu(y,x)\} = 0$$

in such a way that it should be $\mu(y,x) = 0$ or $\mu(x,y) = 0$, whenever $\mu(x,x) = 0$ has been assumed. But from our point of view, $\mu(x,y) = \mu(y,x) = 1$ should be allowed under a structure of preferences approach (see [7]), meaning the existence of different underlying criteria with opposite evaluation with respect to those two alternatives. This can be properly modeled as *incomparability* between alternatives (see also [5]).

From now on we shall assume just that we are dealing with *strict* fuzzy preference relations

$$\mu_s : X \times X \to [0,1]$$

meaning a fuzzy binary relation such that $\mu_s(x,x) = 0$ for all $x \in X$. Then, being such a strict fuzzy preference μ_s given, we shall search within the family of all $\alpha - cuts$,

$$R^\alpha = \{(x,y) \in X \times X / \mu_s(x,y) \geq \alpha\}$$

if they allow a representation in terms of its Dushnik-Miller dimension.

Example 1 *Let X a set of three alternatives, $X = \{\mathbf{1},\mathbf{2},\mathbf{3}\}$, and let us consider the following strict fuzzy preference on X, given in its matrix form:*

$$\mu_s = \begin{pmatrix} 0 & 0.7 & 0.4 \\ 0.3 & 0 & 0.6 \\ 0.2 & 0.4 & 0 \end{pmatrix}$$

Then, for $\alpha = 0.35$ we have that

$$R^{0.35} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

does not verify antisymmetry, and therefore it is not representable in the above sense (notice that for any $x \neq y$ we have that $\mu_s(x,y) \geq \min\{\mu_s(x,z), \mu_s(z,y)\}$ holds for all $z \in X$). Analogously,

$$R^{0.55} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

is not transitive (though antisymmetry now holds).

We then propose the following main definition, relative to the *maximum* level of non-representability understood as the level above it all α-cuts are representable (in other words, the *minimal* representation level).

DEFINITION 1 *A fuzzy strict preference relation is α_0 non-representable if each α-cut, $R^\alpha, \alpha > \alpha_0$, verifies crisp antisymmetry and crisp transitivity.*

Existence of such a maximum level α_0 of *non-representability level* can be assured by impossing representability of R^1, that is, when

1. $\mu_s(x,y) = 1 \Rightarrow \mu_s(y,x) < 1$.
2. $\mu_s(x,z) = \mu_s(z,y) = 1 \Rightarrow \mu_s(x,y) = 1$.

If these two conditions hold, we talk about *representable* fuzzy crisp preference relations. Of course, the lower the minimal representation level α_0 is, the better, since we can get more information about preference by representing each R^α for every $\alpha \geq \alpha_0$.

Notice that from the previous theorem 1 we can assure transitivity of all α-cuts whenever we are dealing with max-min transitive fuzzy preference relations.

Notice also that in case decision maker gives weak preference values $\mu(x,y) \in [0,1]$, such that $\mu(x,y) = \mu(y,x) = 1$, then the *additive* preference structure given above would lead to $\mu_n(y,x) = \mu_i(x,y) = 0$ and $\mu_s(x,y) = \mu_s(y,x) = 0.5 \neq 1$. Therefore, under such a particular model, $\mu_s(x,y) = 1$ and $\mu_s(y,x) = 1$ will never happen simmultaneously.

Moreover, it is immediate that R^α verifying antisymmetry implies that R^β also verifies antisymmetry, for all $\beta > \alpha$.

THEOREM 2 *Let μ_s be a fuzzy strict preference relation being max-min transitive. Then the maximum level of non-representability is*

$$\alpha_0 = \max_{x \neq y} \min\{\mu_s(x,y), \mu_s(y,x)\}$$

Therefore, μ_s being max-min transitive, we have

- All α-cuts are representable if and only if $\alpha_0 = 0$, that is, when
$$\min\{\mu_s(x,y), \mu_s(y,x)\} = 0 \quad \forall x, y$$
- No α-cut is representable if and only if $\alpha_0 = 1$, that is,
$$\exists x, y \quad / \quad \mu_s(x,y) = \mu_s(y,x) = 1$$
- In case $\alpha_0 \in (0,1)$, we get representability for each $R^\alpha, \alpha > \alpha_0$, but R^{α_0} is still non-representable.

IV. Poset-based representation.

Let us consider $\alpha_0 < 1$, and let $d(\alpha)$ be the dimension of the α-cut R^α, for all $\alpha > \alpha_0$. Then we know (see, e.g., [8]) that the set of alternatives X can be represented in $\Re^{d(\alpha)}$, in such a way that

$$x = (x_1^\alpha, \ldots, x_{d(\alpha)}^\alpha) \in \Re^{d(\alpha)} \quad \forall x \in X$$

and

$$x \geq_\alpha y \Longleftrightarrow x_i^\alpha \geq y_i^\alpha \quad \forall i \in \{1, \ldots d(\alpha)\} \quad \forall x, y \in X$$

We can then postulate the existence of $d(\alpha)$ implicit criteria in such a way that the coordinates of every element $x \in X$ represent the valoration of the elements with respect to each criterion.

We propose the analysis of the set of values

$$\{d(\alpha)\}_{\alpha > \alpha_0}$$

in order to get a better insigh of the set of preferences.

Example 2 *Let μ_s be a strict fuzzy preference relation defined on a set with three alternatives, $X = \{\mathbf{1}, \mathbf{2}, \mathbf{3}\}$, this time with the following matrix representation:*

$$\mu_s = \begin{pmatrix} 0 & 0.7 & 0.8 \\ 0.3 & 0 & 0.6 \\ 0.2 & 0.4 & 0 \end{pmatrix}$$

Then we get $\alpha_0 = 0.4$, and the following cases can be distinguished:

- $\alpha \in (0.4, 0.6]$:

$$R^\alpha = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

in such a way that $d(\alpha) = 1$. All three alternatives can be represented in the real line $\Re$.

- $\alpha \in (0.6, 0.7]$:

$$R^\alpha = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

in such a way that $d(\alpha) = 2$.

- $\alpha \in (0.7, 0.8]$:

$$R^\alpha = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and $d(\alpha) = 2$.

- $\alpha \in (0.8, 1]$:

$$R^\alpha = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with $d(\alpha) = 2$.

Notice that representation in $\Re^2$ is different in all previous cases, as shown in figure 1.

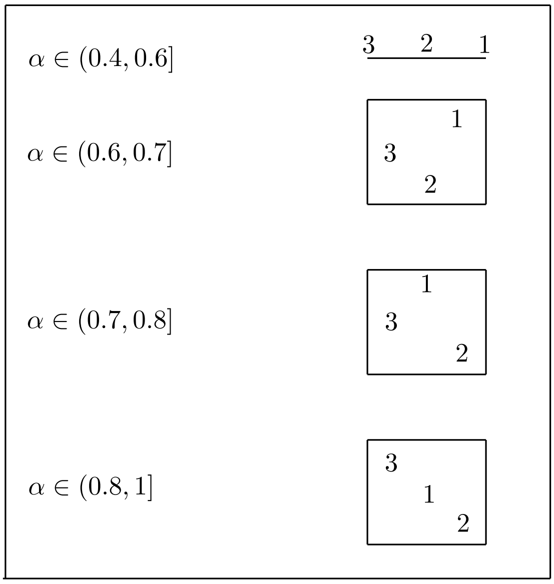

Figure 1

In more complex cases, a graphic representation of $d(\alpha)$ versus α allows a better insight on the number of possible underlying criteria, as shown in the following example.

Example 3 *Let us we consider the following strict preference relation being defined on a set of 6 alter-*

natives, $X = \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}\}$.

$$\mu_s = \begin{pmatrix} 0 & .4 & .4 & .5 & .7 & .8 \\ .1 & 0 & .4 & .8 & .5 & .8 \\ .2 & .3 & 0 & .8 & .8 & .5 \\ .1 & .1 & .2 & 0 & .4 & .4 \\ .1 & .1 & .2 & .1 & 0 & .4 \\ .2 & .1 & .3 & .1 & .2 & 0 \end{pmatrix}$$

It is easy to check that in this case $\alpha_0 = 0.3$, *leading to the dimensional mapping shown in figure 2.*

$d(\alpha)$
3
2
1
0.3 0.4 0.5 0.7 1.0
α

Figure 2

Alternatively to the above a graphic representation of the family of dimension α-cuts, $d(\alpha)$ versus α, we can consider relevant point information, like

$$d_{max} = \max_{\alpha > \alpha_0} \{d(\alpha)\}$$

or some kind of meaninful weigthed mean value.

V. Final comments.

Although the approach developed in last section assumes max-min transitivity and crisp antisymmetry, we must point out that those two restrictions can be avoided in two different ways. On one hand, crisp antisymmetry directly suggests the existence of an extra underlying criterion, still not taken into account. In this way, such a problem can potentially be solved as pure *incomparability*. On the other hand, max-min transitivity can be relaxed by allowing some bounded in size change of data. In fact, as pointed out in [2], max-min transitivity is properly a crisp concept (a preference relation does or it does not hold such a property, with no intermediate degree).

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