

# Consistency of crisp and fuzzy preference relations

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## Abstract

In this paper we point out some difficulties in developing *rationality measures* of fuzzy preference relations, as defined by Cutello and Montero in a previous paper. In particular, we analyze some alternative approaches, taking into account that *consistency* can not be viewed as an univoque concept in a fuzzy framework, neither in the crisp context, where *consistency* should not be necessarily represented in terms of linear orders.

**Keywords:** Fuzzy Preferences, Rationality Measures, Decision Making.

## 1 Introduction

In the classical decision theory, a wide class of preference models have been introduced in order to capture coherence, rationality or consistency in the pairwise comparison: linear orders and complete preorders, where both the preference and indifference relations are transitive; semiorders, interval orders, semitransitivity and quasitransitivity, where the strict preference relation is transitive but the indifference relation is not necessarily transitive; and acyclicity, where no cycles are allowed (see for instance [26] and [12]).

In the fuzzy framework, transitivity plays again a crucial role in coherence modeling. However, there exists a great variety of fuzzy

transitivity properties, each one offering a different consistency assumption (see, e.g., [15]). The problem of studying consistency of fuzzy preferences, and choosing the most appropriate one in a given decision making problem, may require much more effort than in the ordinary crisp case.

In fact, a key argument in [3] was that most standard fuzzy transitivity conditions in literature were *crisp* in nature, i.e., they either hold or not hold. But it is obvious that some situations are *extremely intransitive* while sometimes we only find small transitivity violations that can be in some way bypassed in practice. We find these arguments both in the crisp case and the fuzzy case: a close enough transitive relation can be reached, just by introducing very few modifications in decision maker preferences (see, e.g., [20]). Consistency in most cases allows different degrees, and it should be measured.

The axiomatic approach of [3, 4] was a first proposal in this direction, proposing a particular family of conditions any *rationality measure* should verify. A rationality measure in [3] was defined as a mapping

$$\rho : \mathcal{P}(X) \rightarrow [0, 1]$$

where  $\mathcal{P}(X)$  represents the universe of all possible fuzzy preference relations on a finite set  $X$  of alternatives

$$\mu : X \times X \rightarrow [0, 1]$$

being  $\mu(x, y)$  the degree to which alternative  $x$  is weakly preferred to alternative  $y$ . Then,  $\rho(\mu)$  represents the degree of *consistency* of

$\mu$ , provided that such a mapping  $\rho$  verifies certain conditions (see [3]):

1. *Foundation* (linear orders have the maximum degree of rationality).
2. *Invariance* (with respect to permutations of alternatives).
3. *Symmetry* (with respect to *dual* opinions).
4. *Principle of persistent degree of rationality* (behavior with respect to *new* alternatives).
5. *Regularity* (with respect to preference modifications).

Of course, many different rationality measures can be defined, so a key problem is to find out a way of building up a particular one, being appropriated to the particular problem we are facing to. However, a positive test of mappings satisfying the proposed definition was missing in [3]. In this paper we propose to translate the particular view of consistency decision makers have in the crisp context into every  $\alpha$ -cut of their estimated fuzzy preference, so we can offer an alternative analysis of consistency fully based upon any given crisp consistency, which needs not to be the family of linear orders, as assumed in [3]. Hence, consistency degrees will depend on the previous choice of a particular family of crisp preference structures (not necessarily the family of all linear orders), to be taken as a basis for a rationality analysis of fuzzy preference relations.

## 2 Crisp consistency

A first objective should be to fix the concept of *consistency* within the crisp context.

We of course notice that the standard assumption in the crisp framework is to assimilate *consistency* to linear ordering, in such a way that for every pair of alternatives  $x$  and  $y$  either  $xPy$  or  $yPx$  hold, but not both, and then we talk about *strict preference*. But we should

also point out that this is not the only available proposal. For example, *quasitransitivity* imposes transitivity to strict preference, but not to indifference (the famous *sugar paradox* [16] is a nice argument, see also [19]).

In fact, as pointed out in [13], there are many alternative definitions of *consistency*, each one still allowing a rich enough decision making model, fitting main restrictions of decision makers. But the family of consistent crisp preference relations may not be the family of all linear orders (see [13] where it is argued that decision makers may identify as consistent only *short* chains of alternatives: if decision makers can not deal with more than seven alternatives at once, it is not clear at all why the mathematical model should force them to assume long linear orders that decision makers will never be able to check).

Following [13], we shall simply assume that our decision maker has been able to define what consistency is, by listing a family  $\mathcal{C}(X)$  of *consistent* crisp binary preference relations ( $\mu(x, y) \in \{0, 1\}, \forall x, y \in X, \forall \mu \in \mathcal{C}(X)$ ), in such a way that  $\rho(\mu) = 1$  if and only if  $\mu \in \mathcal{C}(X)$ .

For example, invariance with respect permutations of the set of alternatives seems a natural condition that can be assumed for our family  $\mathcal{C}(X)$  of *consistent* crisp preference relations. But we also need to assure that such a family is being built according to a *rule*, in such a way that elements in  $\mathcal{C}(X)$  are connected by means of some unifying common criteria (perhaps a recursive construction can be tried, telling us how a new alternative can be *consistently* added to any given *consistent* crisp preference relation, as proposed in [5] in a different context).

## 3 Strong and weak preferences

As already pointed out, consistency of strict preference and consistency of indifference may require each one a different model, even in the crisp context. Main arguments can be translated into the fuzzy context, always taking into account the whole fuzzy preference structure that contains information about strict

preference and indifference but also about incomparability (neither  $xRy$  nor  $yRx$  hold, see [9], but also [21]). Consistency should be developed allowing both indifference and incomparability, although it is quite often assumed in practice that they can not simultaneously appear.

For example, suppose individuals who have to show their crisp preferences among the alternatives of the set  $X = \{x_1, \dots, x_n\}$ , where  $n \geq 3$ . A way of introducing crisp preference concepts is taking the strong (or strict) preference as primitive notion, through a binary relation  $P$ , where  $xPy$  means “ $x$  is preferred to  $y$ ” or “ $x$  is better than  $y$ ”. A basic assumption for  $P$  is asymmetry:  $P \cap P^{-1} = \emptyset$ . In this case the indifference relation  $I$  can be defined by absence of preference:  $x$  is indifferent to  $y$  when neither  $x$  is preferred to  $y$  nor  $y$  is preferred to  $x$  in such a way that  $I = (P \cup P^{-1})^c = P^c \cap (P^{-1})^c$ . Then,  $I$  is reflexive and symmetric, and  $R = P \cup I$  (the weak preference relation) is complete. But once asymmetry is being assumed in order to assure the strict preference role, indifference is fixed but incomparability can not be represented.

According to [18] and [7], we can use an index  $d_{ij}$  for distinguish among the three possible cases of preference and indifference between  $x_i$  and  $x_j$ :

$$d_{ij} = \begin{cases} 1, & \text{if } x_i P x_j, \\ 0, & \text{if } x_i I x_j, \\ -1, & \text{if } x_j P x_i. \end{cases}$$

Taking  $r_{ij} = \frac{d_{ij}+1}{2}$ , we have

$$r_{ij} = \begin{cases} 1, & \text{if } x_i P x_j, \\ 0.5, & \text{if } x_i I x_j, \\ 0, & \text{if } x_j P x_i. \end{cases}$$

We can therefore consider fuzzy binary relations as a generalization of the crisp ones by considering the above indices  $r_{ij}$ , but now belonging into the unit interval  $[0, 1]$  instead of the set  $\{0, 0.5, 1\}$ . If  $R$  is a fuzzy binary relation on  $X$  with membership function

$$\mu_R : X \times X \longrightarrow [0, 1]$$

we denote  $r_{ij} = \mu_R(x_i, x_j)$ . This value  $r_{ij}$  has been interpreted in the literature in mainly two ways (see [10]). For example, some authors (e.g., [1, 24, 25]) understand  $r_{ij}$  as the degree of certainty or confidence in the (strict or weak) preference of  $x_i$  over  $x_j$ .

But for other authors  $r_{ij}$  denotes the intensity in which  $x_i$  is preferred to  $x_j$  (e.g., [2, 10, 22, 23, 28]). Reciprocity is in this framework a common hypothesis:  $r_{ij} + r_{ji} = 1$  for all pair of alternatives  $x_i, x_j \in X$ . The set of reciprocal fuzzy binary relations on  $X$  will be denoted by  $\mathcal{R}(X)$ . But notice that some authors (see, e.g., [2, 23]) propose reciprocity with an exception:  $r_{ii} = 0$  (these authors assume that  $r_{ij} = 0.5$  indicates indifference between  $x_i$  and  $x_j$ , and since the alternative  $x_i$  must be indifferent to itself, it should be  $r_{ii} = 0.5$ , just as happens under reciprocity. Thus, as the mentioned authors assert,  $r_{ii} = 0$  is a convention).

Anyway, given  $\alpha \in [0, 1]$ , we can define the  $\alpha$ -cut of  $R \in \mathcal{R}(X)$ :

$$P_\alpha = \{(x_i, x_j) \in X \times X \mid r_{ij} \geq \alpha\}$$

Analogously, for any  $\alpha \in [0, 1]$  we can also define another ordinary binary relations also associated with  $R$ :

$$P_{\bar{\alpha}} = \{(x_i, x_j) \in X \times X \mid r_{ij} > \alpha\}$$

Hence,  $x_i P_\alpha x_j \Leftrightarrow r_{ij} \geq \alpha$  and, analogously,  $x_i P_{\bar{\alpha}} x_j \Leftrightarrow r_{ij} > \alpha$ .

We note that every reciprocal fuzzy binary relation defines, in a natural way, a set of preference ordinary relations. In this case,  $P_\alpha$  and  $P_{\bar{\alpha}}$  are ordinary preference relations, for each  $\alpha \in (0.5, 1]$  and  $\alpha \in [0.5, 1)$ , respectively. The indifference relations associated to  $P_\alpha$  (i.e., neither  $x_i P_\alpha x_j$  nor  $x_j P_\alpha x_i$  hold), and  $P_{\bar{\alpha}}$  (i.e., neither  $x_i P_{\bar{\alpha}} x_j$  nor  $x_j P_{\bar{\alpha}} x_i$  hold), can be respectively defined by

$$x_i I_\alpha x_j \Leftrightarrow 1 - \alpha < r_{ij} < \alpha$$

and

$$x_i I_{\bar{\alpha}} x_j \Leftrightarrow 1 - \alpha \leq r_{ij} \leq \alpha.$$

Fuzzy binary relations generalize ordinary binary relations. However, no reciprocal fuzzy

binary relation is ordinary: from  $r_{ii} = 0.5$  for all  $x_i \in X$ , we have  $0.5 \in \mu_R(X \times X)$ ; hence,  $\mu_R(X \times X) \subseteq \{0, 1\}$  is not being verified. Then, we say that  $R \in \mathcal{R}(X)$  is *crisp* if  $r_{ij} \in \{0, 0.5, 1\}$  for all  $x_i, x_j \in X$  ( $r_{ij} = 0.5$  implies  $r_{ji} = 0.5$  and it is then being understood that  $x_i$  and  $x_j$  are indifferent).

#### 4 Max-\* transitivity

Many alternative transitivity conditions can be found in the fuzzy literature, in order to assure some kind of *consistency*. Among those transitivity definitions, the most frequent is *max-min* transitivity:

$$r_{ik} \geq \min\{r_{ij}, r_{jk}\} \quad \forall x_i, x_j, x_k \in X.$$

Such a definition can be easily generalized by considering another t-norm instead of the *minimum* operator (see, e.g., [14]): a t-norm is a mapping

$$* : [0, 1] \times [0, 1] \rightarrow [0, 1]$$

being monotonous, commutative, associative, and verifying a particular boundary condition ( $a * 1 = 1 * a = a$ ,  $\forall a \in [0, 1]$ ).

Being a fuzzy preference relation *max-\** transitive means that

$$r_{ik} \geq r_{ij} * r_{jk} \quad \forall x_i, x_j, x_k \in X.$$

(see, e.g., [15] but also [20]).

We characterize below max-min transitivity in the framework of the reciprocal fuzzy binary relations: the first item says us that this property is a natural extension of transitivity to the fuzzy case; the second item justifies why this property is called “*max-min*”. The fourth item will shows us an undesirable side effect of this property.

**Proposition 1.** For every  $R \in \mathcal{R}(X)$  the following statements are equivalent:

1.  $P_\alpha$  is transitive for all  $\alpha \in [0, 1]$ .
2.  $r_{ik} \geq \max\{\min\{r_{ij}, r_{jk}\} \mid x_j \in X\}$  for all  $x_i, x_k \in X$ .
3.  $r_{ik} \geq \min\{r_{ij}, r_{jk}\}$  for all  $x_i, x_j, x_k \in X$ .

4.  $\min\{r_{ij}, r_{jk}\} \leq r_{ik} \leq \max\{r_{ij}, r_{jk}\}$  for all  $x_i, x_j, x_k \in X$ .

PROOF: Obviously 2 and 3 are equivalent.

1  $\Rightarrow$  3: If  $\alpha = \min\{r_{ij}, r_{jk}\}$ , we have  $r_{ij}, r_{jk} \geq \alpha$ . Then,  $x_i P_\alpha x_j$  and  $x_j P_\alpha x_k$ ; consequently,  $x_i P_\alpha x_k$ , i.e.,

$$r_{ik} \geq \alpha = \min\{r_{ij}, r_{jk}\}.$$

3  $\Rightarrow$  4: We only need to justify the second inequality. By hypothesis we have

$$r_{ki} \geq \min\{r_{kj}, r_{ji}\}$$

for all  $x_i, x_j, x_k \in X$ . Then,

$$-r_{ki} \leq -\min\{r_{kj}, r_{ji}\} = \max\{-r_{kj}, -r_{ji}\}$$

and

$$\begin{aligned} r_{ik} &= 1 - r_{ki} \leq 1 + \max\{-r_{kj}, -r_{ji}\} = \\ &= \max\{1 - r_{kj}, 1 - r_{ji}\} = \max\{r_{jk}, r_{ij}\} \end{aligned}$$

i.e.,  $r_{ik} \leq \max\{r_{ij}, r_{jk}\}$  for all  $x_i, x_j, x_k \in X$ .

4  $\Rightarrow$  1: Suppose  $x_i P_\alpha x_j$  and  $x_j P_\alpha x_k$ ; then,  $r_{ij}, r_{jk} \geq \alpha$ . By hypothesis we have

$$r_{ik} \geq \min\{r_{ij}, r_{jk}\} \geq \alpha$$

i.e.,  $x_i P_\alpha x_k$ .  $\blacksquare$

Therefore, if  $R \in \mathcal{R}(X)$  is max-min transitive, we have that

1. For each  $\alpha \in (0.5, 1]$  the ordinary preference and indifference relations  $P_\alpha$ ,  $I_\alpha$  and  $P_\alpha \cup I_\alpha$  are transitive.
2. For each  $\alpha \in [0.5, 1)$  the ordinary preference and indifference relations  $P_\alpha$ ,  $I_\alpha$  and  $P_\alpha \cup I_\alpha$  are transitive.
3. Paradoxically, if  $r_{ij} = r_{jk} = 0.9$ , then necessarily  $r_{ik} = 0.9$ . This aspect could be considered as a drawback of the max-min property. In this way some restrictions have been considered in the literature. “Weak” (or “restricted”) conditions are considered by [28] and [6], among others, when certain additional hypotheses are required. In this paper we consider preference intensities greater than 0.5 (or greater than or equal to 0.5) in order to avoid the mentioned drawback.

The next property, appearing in three equivalent ways, is a restricted version of the max-min transitivity. It has been considered under the reciprocity assumption by [17] and [8], among others, in the framework of the probabilistic choice theory, with the name of *moderate stochastic transitivity*, and by [28], within the fuzzy decision theory, under the name of *fuzzy preference order*.

**Proposition 2.** For every  $R \in \mathcal{R}(X)$ , the following statements are equivalent:

1.  $P_\alpha$  is transitive for all  $\alpha \in [0.5, 1]$ .
2.  $r_{ik} \geq \max\{\min\{r_{ij}, r_{jk}\} \mid x_j \in X, r_{ij} \geq 0.5, r_{jk} \geq 0.5\}$  for all  $x_i, x_k \in X$ .
3.  $(r_{ij} \geq 0.5 \text{ and } r_{jk} \geq 0.5) \Rightarrow r_{ik} \geq \min\{r_{ij}, r_{jk}\}$  for all  $x_i, x_j, x_k \in X$ .

In a similar way, 0.5 intensities can be excluded from the prerequisites, leading to the next generalization of quasitransitivity within the fuzzy framework.

**Proposition 3.** For every  $R \in \mathcal{R}(X)$  the following statements are equivalent:

1.  $P_\alpha$  is transitive for all  $\alpha \in (0.5, 1]$ .
2.  $P_{\overline{\alpha}}$  is transitive for all  $\alpha \in [0.5, 1)$ .
3.  $r_{ik} \geq \max\{\min\{r_{ij}, r_{jk}\} \mid x_j \in X, r_{ij} > 0.5, r_{jk} > 0.5\}$  for all  $x_i, x_k \in X$ .
4.  $(r_{ij} > 0.5 \text{ and } r_{jk} > 0.5) \Rightarrow r_{ik} \geq \min\{r_{ij}, r_{jk}\}$  for all  $x_i, x_j, x_k \in X$ .

Now we introduce two classes of fuzzy transitivity properties, depending on binary operations which allow enforcement of preference intensities among preference-connected triplets of alternatives.

**Definition 1.** Let  $R \in \mathcal{R}(X)$  and  $*$  a binary operation on  $[0.5, 1]$  (i.e.,  $a * b \in [0.5, 1]$  for all  $a, b \in [0.5, 1]$ ) satisfying commutativity ( $a * b = b * a$ , for all  $a, b \in [0.5, 1]$ ), monotonicity ( $a' * b' \geq a * b$  whenever  $a' \geq a$  and  $b' \geq b$ , for all  $a, b \in [0.5, 1]$ ) and continuity. Then

1.  $R$  is called *moderate max-\* transitive* if  $(r_{ij} \geq 0.5 \text{ and } r_{jk} \geq 0.5) \Rightarrow r_{ik} \geq r_{ij} * r_{jk}$  for all  $x_i, x_j, x_k \in X$ .

2.  $R$  is called *moderate max-\* quasitransitive* if  $(r_{ij} > 0.5 \text{ and } r_{jk} > 0.5) \Rightarrow r_{ik} \geq r_{ij} * r_{jk}$  for all  $x_i, x_j, x_k \in X$ .

Examples of binary operations verifying the required conditions are:  $a *_1 b = 0.5$ ,  $a *_2 b = \max\{a + b - 1, 0.5\}$ ,  $a *_3 b = \max\{ab, 0.5\}$ ,  $a *_4 b = \min\{a, b\}$ ,  $a *_5 b = \frac{a+b}{2}$  and  $a *_6 b = \max\{a, b\}$ . An empirical study of the fulfillment of moderate max-\* transitivity with respect to the above binary operations can be found in [11] (see also [27]).

We note that if  $R$  is moderate max-max (max-\*<sub>6</sub>) transitive, then  $r_{ij} = r_{jk} = 0.5 \Rightarrow r_{ik} = 0.5$  for all  $x_i, x_j, x_k \in X$ ; in other words,  $I_{\overline{0.5}}$  is transitive.

## 5 Coherence measures

In the classical decision theory there are many available models for rationality. Among the great variety of preference structures we can find in the literature (linear orders, complete preorders, semiorders, interval orders, semi-transitivity, quasitransitivity and acyclicity, among others), at a first stage we propose to concentrate our attention in three of them (see, for instance, [26] and [12]). Given  $P$  be an asymmetric ordinary binary relation on  $X$  representing strong preference:

1. If  $P$  is transitive and  $I$  is antisymmetric we call it *linear order*.
2. If  $P$  and  $I$  are transitive, we call it *complete preorder*. It is worth to emphasize that complete preorders satisfy the following properties (see, for instance, [12]):
  - (a)  $(x_i P x_j \text{ and } x_j I x_k) \Rightarrow x_i P x_k$  for all  $x_i, x_j, x_k \in X$ .
  - (b)  $(x_i I x_j \text{ and } x_j P x_k) \Rightarrow x_i P x_k$  for all  $x_i, x_j, x_k \in X$ .
3. If transitivity is imposed just to  $P$ , we call it *quasitransitive*.

Since each strong ordinary preference relation  $P$  on  $X$  can be considered as a crisp reciprocal

fuzzy binary relation  $R \in \mathcal{R}(X)$  by means of

$$r_{ij} = \begin{cases} 1, & \text{if } x_i P x_j, \\ 0.5, & \text{if } x_i I x_j, \\ 0, & \text{if } x_j P x_i \end{cases}$$

we can define some key ordinary preference structures through properties on  $R$ :

1.  $P$  is a linear order if and only if  $R$  is moderate max-min quasitransitive and  $r_{ij} \in \{0, 1\}$  for all  $x_i, x_j \in X$ .
2.  $P$  is a complete preorder if and only if  $R$  is moderate max-max transitive.
3.  $P$  is quasitransitive if and only if  $R$  is moderate max-min quasitransitive.

Now we introduce a way of defining a class of coherence measures in the framework of reciprocal fuzzy binary relations, taking into account the above crisp properties: let

$$\mathcal{R} = \bigcup_{X \text{ finite}} \mathcal{R}(X).$$

Then a *fuzzy coherence measure* should be a mapping  $\rho : \mathcal{R} \rightarrow [0, 1]$  satisfying at least the following properties (compare them with those in [3]):

1.  $\rho(R) = 1$  for all  $R \in \mathcal{C}$ , being  $\mathcal{C}$  a fixed family of binary relations which decision maker identifies as *consistent*.
2. Neutrality: Let

$$\pi : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$$

be a bijection and  $R \in \mathcal{R}(X)$ . If  $R^\pi$  is defined by

$$r_{ij}^\pi = r_{\pi(i)\pi(j)}$$

then  $\rho(R^\pi) = \rho(R)$ .

3. Reciprocity: Let  $R \in \mathcal{R}(X)$ . If  $R^{-1}$  is defined by  $r_{ij}^{-1} = r_{ji}$ , then  $\rho(R^{-1}) = \rho(R)$ .
4. Monotonicity: Let  $R, R' \in \mathcal{R}(X)$  such that  $R'$  and  $R$  coincides in  $X \times X \setminus \{(x_i, x_k), (x_k, x_i)\}$ . If  $r'_{ik} \geq r_{ik}$  for each  $x_j$  such that  $r'_{ij} = r_{ij} > 0.5$  and  $r'_{jk} = r_{jk} \geq 0.5$ , then  $\rho(R') \geq \rho(R)$ .

Let us suppose now an ordered list

$$\mathbf{T} = \{T_1, \dots, T_s\}$$

of moderate max-\* transitivity (quasitransitivity) properties, where  $T_i$  is associated with the binary operation  $*_i$  on  $[0.5, 1]$  such that  $a *_i b \leq a *_j b$  for all  $a, b \in [0.5, 1]$ , whenever  $i < j$ . Consequently,  $T_j$  implies  $T_i$  if  $i < j$ .

Lets consider  $\mathbf{w} = (w_1, \dots, w_s) \in [0, 1]^s$  a vector of weights, such that  $w_1 + \dots + w_s = 1$ . Given  $R \in \mathcal{R}(X)$ , with  $p_i(R)$  we denote the rate of triplets of  $X$  satisfying the property  $T_i$ ; consequently,  $p_i(R)$  is a relative measure of the accomplishment of  $T_i$ . The map  $\mathbf{p} : \mathcal{R}(X) \rightarrow [0, 1]^s$  assigns the vector of fulfillment rates of properties  $T_i$  to each reciprocal fuzzy binary relation,  $\mathbf{p}(R) = (p_1(R), \dots, p_s(R))$ . Since the binary operations  $*_i$  provide greater results when  $i$  increases, the components of the vector  $\mathbf{p}(R)$  are ordered in a non-decreasing manner.

**Definition 2.** Given a list  $\mathbf{T}$  of fuzzy transitivity properties, a vector of weights  $\mathbf{w}$  and a non-decreasing function  $\varphi : [0, 1] \rightarrow [0, 1]$  such that  $\varphi(0) = 0$  and  $\varphi(1) = 1$ , we define the fuzzy coherence measure  $\rho : \mathcal{R} \rightarrow [0, 1]$  associated with  $\langle \mathbf{T}, \mathbf{w}, \varphi \rangle$  by

$$\rho(R) = w_1 \cdot \varphi(p_1(R)) + \dots + w_s \cdot \varphi(p_s(R)).$$

Now we show some simple examples of fuzzy coherence measures by considering concrete vectors of weights and functions  $\varphi$ .

### Examples.

1. Absolute fulfillment of  $T_i \in \mathbf{T}$ :

$$w_j = \begin{cases} 1, & \text{if } j = i, \\ 0, & \text{if } j \neq i, \end{cases}$$

$$\varphi(x) = \begin{cases} 1, & \text{if } x = 1, \\ 0, & \text{if } x < 1. \end{cases}$$

In this case,  $\rho(R) = 1$  if and only if  $R$  satisfies  $T_i$ .

2. Relative fulfillment of  $T_i \in \mathbf{T}$ :

$$w_j = \begin{cases} 1, & \text{if } j = i, \\ 0, & \text{if } j \neq i, \end{cases} \quad \varphi(x) = x.$$

Now  $\rho(R)$  is the rate of fulfillment of  $T_i$ .

3. Detection of maximum fulfillment of properties in  $\mathbf{T}$ :

$$\mathbf{w} = \left( \frac{1}{s}, \dots, \frac{1}{s} \right), \varphi(x) = \begin{cases} 1, & \text{if } x = 1, \\ 0, & \text{if } x < 1. \end{cases}$$

In this case

$$s \cdot \rho(R) = \max\{i \mid p_i(R) = 1\}$$

indicates the maximum index  $i$  such that  $R$  satisfies  $T_i$ .

4. Average of relative fulfillment of the properties in  $\mathbf{T}$ :

$$\mathbf{w} = \left( \frac{1}{s}, \dots, \frac{1}{s} \right), \quad \varphi(x) = x.$$

Now  $\rho(R)$  is the average of the rates of fulfillment of the properties in  $\mathbf{T}$ .

Of course, *consistency* can be addressed by defining a certain *distance* in some way telling us how close we are to a *consistent* binary relation. This is an underlying argument in [3], and indeed it allowed to get compositions and mixtures of rationality measures (see [4]). But apart from that rationality measure initially proposed in [19], non binary (crisp) rationality measures are difficult to be defined. Alternatively, we can consider all  $\alpha$ -cuts, and evaluate the distribution of *consistent* crisp relations, i.e., whether each one belongs to  $\mathcal{C}$ .

## 6 Final comments

This paper points out practical difficulties when the approach of [3] has to be developed. In particular, we point out that *consistency* should not be necessarily associated to linear ordering, but the decision maker has to declare what should be understood as consistency, by defining a family of crisp preference relations. Distance to consistency of an arbitrary fuzzy preference relation can be then analyzed by means of its sequence of  $\alpha$ -cuts if compared to elements in that family representing crisp consistency.

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## References

- [1] Barrett, C.R.; Pattanaik, P.K.; and Salles, M. (1986): "On the structure of fuzzy social welfare functions". *Fuzzy Sets and Systems* 19: 1–10.
- [2] Bezdek, J.C.; Spillman, B.; and Spillman, R. (1978): "A fuzzy relation space for group decision theory: An application". *Fuzzy Sets and Systems* 1: 255–268.
- [3] Cutello, V.; and Montero, J. (1994): "Fuzzy rationality measures". *Fuzzy Sets and Systems* 62: 39–54.
- [4] Cutello, V.; and Montero, J. (1997): "Equivalence and compositions of fuzzy rationality measures". *Fuzzy Sets and Systems* 85: 31–43.
- [5] Cutello, V.; and Montero, J. (1999): "Recursive connective rules". *International Journal of Intelligent Systems* 14: 3–20.
- [6] Dasgupta, M.; and Deb, R. (1996): "Transitivity and fuzzy preferences". *Social Choice and Welfare* 13: 305–318.
- [7] Fishburn, P.C. (1973): *The Theory of Social Choice*. Princeton University Press, Princeton.
- [8] Fishburn, P.C. (1973): "Binary choice probabilities: on the varieties of stochastic transitivity". *Journal of Mathematical Psychology* 10: 327–352.
- [9] Fodor, J.; and Roubens, M. (1994): "Valued preference structures". *European Journal of Operational Research* 79: 277–286.
- [10] García-Lapresta, J.L.; and Llamazares, B. (2000): "Aggregation of fuzzy preferences: Some rules of the mean". *Social Choice and Welfare* 17: 673–690.

- [11] García-Lapresta, J.L.; and Meneses, L.C. (forthcoming): “An empirical analysis of transitivity with four scaled preferential judgment modalities”. *Review of Economic Design*.
- [12] García-Lapresta, J.L.; and Rodríguez-Palmero, C. (forthcoming): “Some algebraic characterizations of preference structures”. *Journal of Interdisciplinary Mathematics*.
- [13] Gómez, D; Montero, J.; Yáñez, J.; González-Pachón, J.; and Cutello, V. (forthcoming): “Crisp dimension theory and valued preference relations”. *International Journal of General Systems*.
- [14] Klement, E.P.; Mesiar, R.; and Pap, E. (2000): *Triangular Norms*. Kluwer, Dordrecht.
- [15] Klir, G.J.; and Folger, T.A. (1988): *Fuzzy Sets, Uncertainty and Information*. Prentice Hall, Englewood Cliffs, NJ.
- [16] Luce, R.D. (1956): “Semiorders and a theory of utility discrimination”. *Econometrica* 24: 178–191.
- [17] Luce, R.D.; and Suppes, P. (1965): “Preferences, utility and subjective probability”. In *Handbook of Mathematical Psychology III*, eds. R.D. Luce *et al*, Wiley, New York, chap. 19.
- [18] May, K.O. (1952): “A set of independent necessary and sufficient conditions for simple majority decision”. *Econometrica* 20: 680–684.
- [19] Montero, J. (1987): “Arrow’s theorem under fuzzy rationality”. *Behavioural Science* 32: 267–273.
- [20] Montero, J.; and Tejada, J. (1986): “Some problems on the definition of fuzzy preference relations”. *Fuzzy Sets and Systems* 20: 45–53.
- [21] Montero, J.; Tejada, J.; and Cutello, V. (1997): “A general model for deriving preference structures from data”. *European Journal of Operational Research* 98: 98–110.
- [22] Nakamura, K. (1992): “On the nature of intransitivity in human preferential judgements”. In: Novák, V.; and Ramík, J. *et al* (eds.) (1992) *Fuzzy Approach to Reasoning and Decision-Making*, pp. 147–162. Kluwer Academic Publishers, Dordrecht.
- [23] Nurmi, H. (1981): “Approaches to collective decision making with fuzzy preference relations”. *Fuzzy Sets and Systems* 6: 249–259.
- [24] Orlovsky, S.A. (1978): “Decision making with a fuzzy preference relation”. *Fuzzy Sets and Systems* 1: 155–167.
- [25] Ovchinnikov, S.V. (1981): “Structure of fuzzy binary relations”. *Fuzzy Sets and Systems* 6: 169–195.
- [26] Roubens, M.; and Vincke, P. (1985): *Preference Modelling*. Lecture Notes in Economics and Mathematical Systems, vol. 250. Springer-Verlag, Berlin.
- [27] Switalski, Z. (2003): “General transitivity conditions for fuzzy reciprocal preference matrices”. *Fuzzy Sets and Systems* 137: 85–100.
- [28] Tanino, T. (1984): “Fuzzy preference orderings in group decision making”. *Fuzzy Sets and Systems* 12: 117–131.