

Special construction of Atanassov's intuitionistic fuzzy S-implications that maintain the Atanassov's intuitionistic index

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Abstract

In this paper we present a method for the construction of Atanassov's intuitionistic fuzzy S-implications that satisfy the following property: if in the intuitionistic fuzzy conditional the antecedent is equal to the consequent, then the Atanassov's intuitionistic fuzzy implication operator has the same Atanassov's intuitionistic fuzzy index as the antecedent and the consequent.

Keywords: Atanassov's intuitionistic fuzzy implication operators; Atanassov's intuitionistic fuzzy index; Atanassov's intuitionistic fuzzy S-implication operators.

1 Introduction

In 1983 K. Atanassov (see [3]) introduced A-IFSs (Atanassov's intuitionistic fuzzy sets) in the following way:

Let U be an ordinary finite non-empty set. An *A-IFS* in U is an expression A given by

$$A = \{(u, \mu_A(u), \nu_A(u)) | u \in U\} \text{ where}$$

$$\mu_A : U \longrightarrow [0, 1]$$

$$\nu_A : U \longrightarrow [0, 1]$$

with the condition $\mu_A(u) + \nu_A(u) \leq 1$ for all $u \in U$.

The numbers $\mu_A(u)$ and $\nu_A(u)$ represent respectively the degree of membership and the degree of non-membership of element u to set A .

In [3] *Atanassov's intuitionistic fuzzy index* is defined in the following way:

$$\pi_A(u) = 1 - \mu_A(u) - \nu_A(u) \text{ for all } u \in U.$$

In this paper we will study the conditions in which Atanassov's intuitionistic fuzzy S-implications satisfy the following property: if in the Atanassov's intuitionistic fuzzy conditional the antecedent is equal to the consequent, then the Atanassov's intuitionistic fuzzy implication operator has the same Atanassov's intuitionistic fuzzy index as the antecedent and the consequent. In [6] this special property has been applied in image processing to calculate the contrast in grayscale images.

This paper is organized as follows. In Section 2 we present the basic concepts and notations that we will use throughout the paper. In Section 3 we review the concept Atanassov's intuitionistic fuzzy implication operators and in Section 4 we provide a method for the construction of Atanassov's intuitionistic fuzzy S-implications that satisfy a special property. Finally we draw conclusions and indicate future lines of research.

2 Preliminaries

Let us take the following set:

$$L([0, 1]) = \{(x, y) | (x, y) \in [0, 1] \times [0, 1] \text{ and } x + y \leq 1\}.$$

For every $(x, y), (z, t) \in L([0, 1])$ the following expressions are known ([3]-[11]):

- $(x, y) \leq (z, t)$ if and only if $x \leq z$ and $y \geq t$.
- $(x, y) = (z, t)$ if and only if $(x, y) \leq (z, t)$ and $(z, t) \leq (x, y)$.
- $(x, y) \sqsubseteq (z, t)$ if and only if $1 - y \leq z$.
- For all $(x, y) \in L([0, 1])$ we define the *complementary* of (x, y) (see [4]) and express it as $c((x, y))$ in this manner: $c((x, y)) = (y, x)$.

Definition 1 Let n be any strong negation (see [15],[12],[18],[19]), that is, an involutory order reversing bijection of the closed unit interval. A strong negation on $L([0, 1])$ is defined as follows:

$$\begin{aligned} \mathbf{n} : L([0, 1]) &\rightarrow L([0, 1]), \text{ given by} \\ \mathbf{n}((x, y)) &= (n(1 - y), 1 - n(x)). \end{aligned} \quad (1)$$

It is important to note that if in expression (1), n is the standard negation, that is, if $n(x) = N(x) = 1 - x$, then $\mathbf{n}((x, y)) = (N(1 - y), 1 - N(x)) = (y, x)$. Throughout the paper we will only use the strong negation $n(x) = N(x) = 1 - x$.

A good study of the concept of negation and its characterization when working with A-IFSs can be found in [12].

In fuzzy set theory the intersection and union of fuzzy sets are modeled by means of t-norms and t-conorms defined in $[0, 1]$. In analogous way intuitionistic fuzzy t-norms and intuitionistic fuzzy t-conorms are also used to model the intersection and union of A-IFSs.

Any pair formed by a classical t-norm T in $[0, 1]$ and a classical t-conorm S in $[0, 1]$ (see [15],[17]), such that $S \leq S^*$, S^* being the dual t-conorm of T with respect to the negation $N(x) = 1 - x$, enables us to construct a t-norm \mathbf{T} and t-conorm \mathbf{S} (as functions that act on $L([0, 1])^2$ in $L([0, 1])$) in the following way:

$$\mathbf{T}((x, y), (z, t)) = (T(x, z), S(y, t)) \quad (2)$$

$$\mathbf{S}((x, y), (z, t)) = (S(x, z), T(y, t)). \quad (3)$$

In [11] Cornelis, Deschrijver and Kerre defined the expressions (2) and (3) as t-representable intuitionistic fuzzy t-norm and s-representable intuitionistic fuzzy t-conorm respectively (a thorough study on the non-representable ones can be found in [12]). Throughout the paper we will only use these expressions.

3 Atanassov's intuitionistic fuzzy implication operator

Atanassov and Gargov [5] and later Cornelis and Deschrijver [10] gave the definition of Atanassov's intuitionistic fuzzy implication operator. In the same way, as many authors do in fuzzy theory, Bustince in [7] gives a definition from which he demands more conditions than those imposed by Gargov, Cornelis and Deschrijver so that this definition satisfies the

conditions given in [5] and [10] and recovers Fodor's (see [15]) definition of fuzzy implication operator when the sets considered are fuzzy.

Definition 2 ([7]) *An Atanassov's intuitionistic fuzzy implication operator is a function:*

$$I_I : L([0, 1])^2 \rightarrow L([0, 1]),$$

having the following properties:

I_{I0} . If $(x, y), (z, t) \in L([0, 1])$ are such that $x + y = 1$ and $z + t = 1$, then

$$\pi_{I_I((x,y),(z,t))} = 0;$$

I_{I1} . If $(x, y) \leq (x', y')$ then $I_I((x, y), (z, t)) \geq I_I((x', y'), (z, t))$ for all $(z, t) \in$

$$L([0, 1]);$$

I_{I2} . If $(z, t) \leq (z', t')$ then $I_I((x, y), (z, t)) \leq I_I((x, y), (z', t'))$ for all $(x, y) \in$

$$L([0, 1]);$$

I_{I3} . $I_I((0, 1), (x, y)) = (1, 0)$ for all $(x, y) \in L([0, 1])$;

I_{I4} . $I_I((x, y), (1, 0)) = (1, 0)$ for all $(x, y) \in L([0, 1])$;

I_{I5} . $I_I((1, 0), (0, 1)) = (0, 1)$.

Atanassov's intuitionistic fuzzy implication operators can be demanded to satisfy other properties (see [5],[7]) in addition to $I_{I0} - I_{I5}$. We will divide these properties into two groups: the truly Atanassov's intuitionistic and the properties inherited when the fuzzy implications are generalized to the Atanassov's intuitionistic case. Below we present the properties used in this work (the numeration is the one used in [7]-[9]).

From the first group:

I_{I7} . If $(x, y) = (z, t)$, then $\pi_{I_I((x,y),(z,t))} = \pi_{(x,y)}$.

The properties in the second group are:

I_{I9} . $I_I((1, 0), (x, y)) = (x, y)$.

I_{I10} . $I_I((x, y), I_I((z, t), (r, s))) = I_I((z, t), I_I((x, y), (r, s)))$.

I_{I12} . $I_I((x, y), (0, 1)) = (y, x)$.

I_{I15} . $I_I((t, z), (y, x)) = I_I((x, y), (z, t))$.

I_{I18} . $I_I((x, y), (y, x)) = (y, x)$.

In [8] we can find an interpretation of every property and a construction method of I_I . Also, the authors study the conditions in which operators I_I satisfy each one of the presented properties.

4 Construction of Atanassov's intuitionistic fuzzy S-implications with special properties

In this section we set out to study a method for the construction of Atanassov's intuitionistic fuzzy implication operators that satisfy the exclusively intuitionistic property I_{I7} .

This objective leads us to introduce the concept of Atanassov's intuitionistic fuzzy S-implication associated with a De Morgan triple.

Assume that S is a t-conorm in $[0,1]$, T is a t-norm $[0,1]$ and N the standard negation. We say that (S, T, N) is a De Morgan triple if and only if

$$S(x, y) = 1 - T(1 - x, 1 - y)$$

is satisfied (this concept has been generalized for other negations, see [12],[13]-[15], [18]).

Cornelis and Deschrijver present the following definition in [10].

Definition 3 *An Atanassov's intuitionistic fuzzy S-implication associated with a De Morgan triple (S, T, N) is defined by*

$$I_I((x, y), (z, t)) = \mathbf{S}((y, x), (z, t)) = (S(y, z), T(x, t)).$$

It is easy to see that $I_I((x, y), (z, t)) = (S(y, z), T(x, t))$ satisfies the properties $I_{I0} - I_{I5}$, therefore, it is an Atanassov's intuitionistic fuzzy implication in the sense of Definition 2.

It is important to mention that in [11] Cornelis, Deschrijver and Kerre present a characterization of Atanassov's intuitionistic fuzzy S-implications, however, we want to obtain in a general way Atanassov's intuitionistic fuzzy S-implications in the sense of Definition 3 that satisfy property I_{I7} . To achieve our objective we will now present a proposition and a theorem.

Proposition 1 *If I_I is an Atanassov's intuitionistic fuzzy S-implication associated with a De Morgan triple (S, T, N) , then I_I satisfies I_{I9} , I_{I10} and I_{I15} .*

Proof 1 *Direct.*

The study of the conditions under which the reciprocal of Proposition 1 holds can be found in [9].

In order to use I_{I7} we will base ourselves on the most important properties of the Frank family of t-norms and t-conorms (see [16]) and also on the continuity of I_I .

To study the continuity of the Atanassov's intuitionistic fuzzy S-implication, a metric is needed (we will use the definition and notation of Jenei in [17]).

Let $(x, y), (z, t) \in L([0, 1])$. Then let \mathcal{D} denote the well-known Hausdorff metric for the elements of \mathcal{D} , that is: $\mathcal{D}((x, y), (z, t)) = \vee(|x - z|, |y - t|)$.

Lemma 1 *An Atanassov's intuitionistic fuzzy S-implication associated to a De Morgan triple (S, T, N) is continuous if and only if S and T are continuous.*

Proof. *Direct bearing in mind the Hausdorff metric.*

Let $m > 0, m \neq 1$ be a real number. Define a parametric family of continuous Archimedean t-norms in the following way:

$$T^m(x, y) = \log_m(1 + \frac{(m^x - 1)(m^y - 1)}{m - 1}).$$

We can extend this definition for $m = 0, m = 1$ and $m = \infty$ by taking limits. Thus, $T^0(x, y) = \lim_{m \rightarrow 0} T^m(x, y) = \wedge(x, y)$, $T^1(x, y) = \lim_{m \rightarrow 1} T^m(x, y) = xy$ and $T^\infty(x, y) = \lim_{m \rightarrow \infty} T^m(x, y) = \vee(x + y - 1, 0)$.

The family $(T^m)_{m \in [0, \infty]}$ is called the *Frank family* of t-norms. The De Morgan law enables us to define the Frank family of t-conorms $(S^m)_{m \in [0, \infty]}$ by

$$S^m(x, y) = 1 - T^m(1 - x, 1 - y) \text{ for any } m \in [0, \infty].$$

In [16] one can find the following interesting characterization of these parametrized families.

Theorem 1 *A continuous t-norm T and a continuous t-conorm S satisfy the functional equation $T(x, y) + S(x, y) = x + y$*

if and only if

- (a) *there is a number $m \in [0, \infty]$ such that $T = T^m$ and $S = S^m$, or*
- (b) *T is representable as an ordinal sum of t-norms, each of which is a member of the family (T^m) , $0 < m \leq \infty$ and S is obtained from T via*

$$S^m(x, y) = 1 - T^m(1 - x, 1 - y).$$

The following theorem enables us construct functions I_I that satisfy $I_{I0} - I_{I5}$ and the property I_{I7} .

Theorem 2 *If I_I is an Atanassov's intuitionistic fuzzy S-implication associated with a De Morgan triple (S, T, N) with continuous S and T such that*

- (a) *there is a number $m \in [0, \infty]$ such that $T = T^m$ and $S = S^m$, or*

(b) T is representable as an ordinal sum of t -norms, each of which is a member of the family (T^m) , $0 < m \leq \infty$, and S is obtained from T via:

$$S^m(x, y) = 1 - T^m(1 - x, 1 - y),$$

then I_I is a continuous function such that I_{I10} , I_{I12} and I_{I7} are satisfied.

Proof 2 To carry out the proof it is necessary to bear in mind the existing interdependences between the properties of I_I presented in [9], Lemma 1, Proposition 1 and Theorem 1.

Applying Theorem 2 to Atanassov's intuitionistic fuzzy S-implication (according to Definition 3), we can obtain different expressions all of which satisfy the property I_{I7} . For example:

1. If $S = \vee$ and $T = \wedge$, the expression obtained is the Atanassov's intuitionistic fuzzy implication operator of K. Atanassov and G. Gargov

$$I_I((x, y), (z, t)) = (\vee(y, z), \wedge(x, t))$$

that satisfies I_{I7} . It also satisfies: $I_{I0} - I_{I10}$, I_{I12} , I_{I13} , $I_{I15} - I_{I18}$.

2. If $S(x, y) = x + y - x \cdot y$ and $T(x, y) = x \cdot y$, then

$$I_I((x, y), (z, t)) = (y + z - y \cdot z, x \cdot t)$$

that satisfies the properties: $I_{I0} - I_{I5}$, I_{I7} , I_{I9} , I_{I10} , I_{I12} , I_{I15} and it also satisfies: If $I_I((x, y), (z, t)) = (1, 0)$ then $(x, y) \sqsubseteq (z, t)$.

Example 1 In figure 1 we show the Atanassov's intuitionistic fuzzy sets A and B defined on the finite non-empty referentials U and V respectively. For each sets we draw the membership function, the non-membership function and the Atanassov's intuitionistic fuzzy index, which is constant and equal to 0.2 in both cases. In figure 2 are depicted the values of

$\pi_{I_I((\mu_A(u), \nu_A(u)), (\mu_B(v), \nu_B(v)))} = \pi_{(\nu_A(u) + \mu_B(v) - \nu_A(u) \cdot \mu_B(v), \mu_A(u) \cdot \nu_B(v))}$ when it is applied to the Atanassov's intuitionistic fuzzy sets A and B (figure 1). In figure 2 (c) it is verified that I_I satisfies I_{I7} . We can observe that $\pi_{I_I((\mu_A(u), \nu_A(u)), (\mu_B(v), \nu_B(v)))} = 0.2$ when the elements $(\mu_A(u), \nu_A(u))$, $(\mu_B(v), \nu_B(v))$ are equal.

5 Conclusions and future lines of research

We have presented a method for the construction of the Atanassov's intuitionistic fuzzy S-implications that satisfy the conservation of the

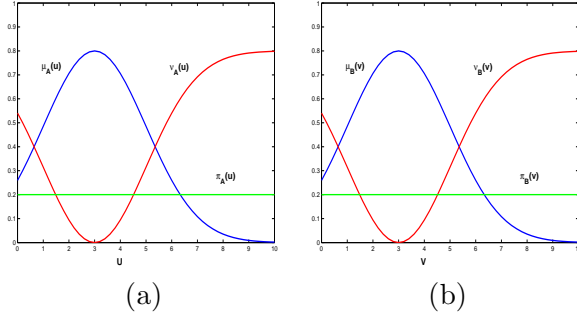


Figure 1: (a) Atanassov's intuitionistic fuzzy set A (b) Atanassov's intuitionistic fuzzy set B.

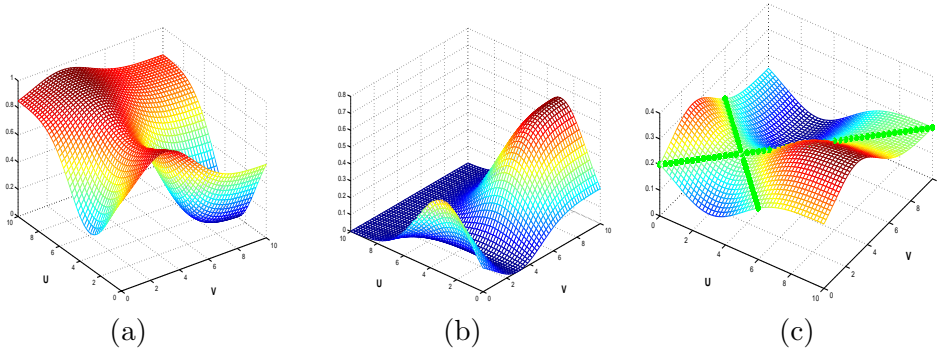


Figure 2: (a) Membership function $\mu_{I_I}((\mu_A(u), \nu_A(u)), (\mu_B(v), \nu_B(v)))$ (b) Non-membership function $\nu_{I_I}((\mu_A(u), \nu_A(u)), (\mu_B(v), \nu_B(v)))$ (c) $\pi_{I_I}((\mu_A(u), \nu_A(u)), (\mu_B(v), \nu_B(v)))$. The thick lines show the elements which $\pi_{I_I}((\mu_A(u), \nu_A(u)), (\mu_B(v), \nu_B(v)))=0.2$

Atanassov's intuitionistic fuzzy index (I_{I7}). This property is closely related to the Atanassov's intuitionistic fuzzy index. Therefore, it is very important to take it into account in approximate reasoning systems that represent knowledge by means of A-IFSs. In the future we want to carry out a complete study of the conditions under which the reciprocal of Theorem 2 holds.

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