

# On the use of coherence measures for fuzzy preference relations

Camilo A. Franco de los Ríos, Javier Montero

Faculty of Mathematics

Complutense University

Madrid, 28040 (Spain)

[francodelosrios@gmail.com](mailto:francodelosrios@gmail.com), [javier\\_montero@mat.ucm.es](mailto:javier_montero@mat.ucm.es)

## Abstract

Although *consistency* in information and actions is a major argument in any decision making problem, most available tools are crisp, while observation strongly suggests to consider different degrees of *consistency*, i.e., *consistency* is fuzzy in nature. In this paper we propose to put together some of the works in the field, focusing on information *consistency* rather than on action *consistency*, showing in this way key classification features.

**Keywords:** Fuzzy Preferences, Consistency Measures, Decision Making.

## 1. Introduction

*Consistency* analysis represents a key approach to any problem subject to error observation. Within the field of decision making, for example, classical approaches refer to transitivity of crisp preferences. Surprisingly, most approaches to consistency within this field have been conceived under a crisp analytic framework, although most decision makers will agree on the existence of different degrees of consistency (see, for example, [4,5,8,10,16]). Anyway, we must stress with [12] how difficult it is to impose any model of consistency on human acts when Medicine shows the deep emotional basis of any decision. Hence, it looks more appropriate to pursue consistency models from a fuzzy classification approach [18], close perhaps to the framework proposed in [1] or [11].

In particular, the concept of coherence measure, as introduced in [16], examines the similarity and fuzziness of any pair of subsets. The fuzziness of any subset is understood as the distance from the extreme (crisp) cases  $\emptyset = \{0, 0, \dots, 0\}$  or  $\Omega = \{1, 1, \dots, 1\}$ .

Within a fuzzy preference representation, we should pursue methodologies with its whole preferential structure explicitly specified (as it is developed in [4,10] and stressed in [5], where misbehaviour of some standard strict preferences is proven because they do not allow to build up weak preferences that are consistent with those same strict preferences). Hence, if we are able to determine how different and how similar are any two fuzzy subsets, then we may try to order them according to their proximity to  $\emptyset$  or  $\Omega$ . If such a proximity to any of these two extreme cases is significant, a greater value for the coherence measure is then expected.

## 2. Coherence Measures

Following [16], and assuming a finite set of alternatives  $A$ ,  $R^f(A)$  will represent here the set of all fuzzy sets in  $A$ . And given a strong negation  $n$  defined on  $R^f(A)$ , see [17], then the function

$$\wp: R^f(A) \times R^f(A) \rightarrow [0,1]$$

is called *coherence measure* if and only if it satisfies the following three axioms:

- i) Symmetric measure:  $\wp(X, Y) = \wp(Y, X)$ , i.e., the mapping of the coherence measure  $\wp$  is symmetric for any pair of fuzzy subsets in  $A$ .
- ii) Coherence:  $\wp(X, n(Y)) = n(\wp(X, Y))$ , i.e., the coexistence of the two evaluations over  $X$  and  $Y$  has to be the opposite of the one between  $X$  and  $n(Y)$ .
- iii) Minimum coherence:  $\wp(\emptyset, A) = 0$ , i.e., minimum coherence is to be obtained when comparing the null set  $\emptyset$  with the referential set  $A$ .

The main aim of this investigation is to examine if we can use coherence measures over the fuzzy preference relations, and if so, to analyze its results.

### 3. Fuzzy preference relations and coherence measures

The preference model that we consider, following [7] (but see also [13]), expresses one unique relation  $R(a,b)$  as a composition of other four different relations

$$p, p^{-1}, i, j : [0,1]^2 \rightarrow [0,1]$$

such that

$$R(a,b) = \{p(a,b), p(b,a), i(a,b), j(a,b)\}$$

where  $p(a,b)$  stands for the strict preference intensity of  $a$  over  $b$ ;  $p(b,a)$  stands for strict preference of  $b$  over  $a$ ;  $i(a,b)$  for indifference between  $a$  and  $b$ ; and  $j(a,b)$  for incomparability between the two alternatives  $a$  and  $b$ . For simplicity, we will consider here only complete preference relations, such that  $R(a,b) = 1$  (i.e., the aggregated value of the previous decomposition into four intensities cover the total possible intensity).

In particular, in [16] it is shown an interesting coherence measure based upon the Euclidian metric  $d$ , defined for any pair of fuzzy subsets  $A, B$ , such that

$$d(A,B) = \frac{1}{m} \sum_{i=1}^m |a_i - b_i|$$

where  $m$  is the number of elements in the fuzzy subset. Then, the coherence measure  $\beta$  is defined as (see [16])

$$\beta(A,B) = \frac{1 + d(A, B^c) - d(A,B)}{2}$$

For example, let us consider the following fuzzy sets, given by a complete binary fuzzy preference relation such that the intensity of membership of each one of the elements is as follows,

$$A : \{0.1, 0.8, 0.0, 0.1\}$$

$$B : \{0.7, 0.0, 0.1, 0.2\}$$

$$C : \{0.9, 0.0, 0.0, 0.1\}$$

$$E : \{0.0, 0.9, 0.1, 0.0\}$$

$$F : \{0.0, 0.9, 0.0, 0.1\}$$

$$I : \{0.0, 0.0, 0.9, 0.1\}$$

Then, applying the particular coherence measure stated above over these fuzzy preference relations, we have that

$$\beta(A,B) = 0.55$$

$$\beta(B,C) = 0.85$$

$$\beta(A,C) = 0.55$$

in such a way that coherence of  $A$  over  $B$  or  $C$  can be considered *low*, while coherence between  $B$  and  $C$  can be considered *high*. These results are as expected, because  $B$  and  $C$  are two relations where the highest intensity of membership corresponds to  $p(b,a)$ , while in  $A$  the highest one corresponds to  $p(a,b)$ , with very similar intensity values for  $i(a,b)$  and  $j(a,b)$  in the three sets.

Now, if we compare two very distinct sets such as  $E$  and  $C$ , where the intensity values of membership of  $p(a,b)$  and  $p(b,a)$  are so distant, the coherence measure should be very close to 0.5, as it certainly occurs for  $(E,C)$ ,  $(F,C)$  and  $(I,C)$ , according to the following values:

$$\beta(E,C) = 0.500$$

$$\beta(F,C) = 0.525$$

$$\beta(I,C) = 0.525$$

In this way, the coherence measure is equally low when we compare  $C$ , a preference relation with a high membership intensity for  $p(a,b)$ , with  $F$ , a relation with the same intensity value but for  $p(b,a)$ , or with  $I$ , a preferente relation with the same intensity value but for  $i(a,b)$ .

Let us now introduce the following relation  $K$ ,

$$K : \{0.5, 0.0, 0.5, 0.0\}$$

and let us compare  $K$  with  $I$  and  $C$ . The corresponding coherence measures are,

$$\beta(I,K) = 0.635$$

$$\beta(C,K) = 0.635$$

The information given by the coherence measure tells us that there exists the same distance between the fuzzy preference relations

with a strong intensity for  $p(a,b)$  or for  $i(a,b)$  in relation to  $K$ , a relation with the same value of intensity for  $p(a,b)$  and  $i(a,b)$ .

Hence, if we consider the next fuzzy relation,

$$J : \{0.0, 0.9, 0.0, 0.1\},$$

we have that its coherence with  $K$ , according to the above measure, is,

$$\beta(J, K) = 0.5,$$

which means that they are completely different.

#### 4. Reflection on the use of coherence measures over fuzzy preference relations.

The similarity between two fuzzy sets can be stated by measuring the distance between the respective membership intensities of their corresponding elements. The coherence measures are practical and efficient tools for accomplishing the task proposed by [16].

Evidently, if we see the comparisons made between  $I$ ,  $C$  and  $K$ , different fuzzy relations, in our case  $I$  and  $C$ , if compared with another one, such as  $K$ , may have exactly the same value of coherence. Now, if we examine the coexistence evaluation between  $I$  and  $C$ , we notice that they are not similar, because their measure is 0.525. These two fuzzy sets are different, but when compared through a third one, they share the same coherence measure. Taking into account this information solely, we can not figure out just how or in which way the fuzzy sets are really different or what is the meaning of the fact that the distance between them is so wide.

It may be necessary to have information about the type of sets that come into comparison. In our example, we need to know the kind of sets that make it possible for the measure to have those specific values. If we take a closer look, the fuzzy set  $K$  has the same value for  $p(a,b)$  and for  $i(a,b)$ . Our intuition tells us that if the intensity of  $i(a,b)$  has a similar value than the one of  $p(a,b)$ , then it should also be so with  $p(b,a)$ . Because this is not the case, it is necessary to take into consideration this type of information for a better understanding of the differences between any pair of fuzzy subsets.

In other words, we need a coherence measure over each fuzzy set so we can evaluate its real content. A measure of this type, one that examines the ratio between the membership

intensities of the elements of a fuzzy set, depends on the kind of fuzzy set that we are dealing with. Because preference relations are a particular class of fuzzy sets, we need to evaluate the way in which each fuzzy relation constitutes itself, examining its intrinsic content.

We therefore propose to use coherence measures over the binary relations of fuzzy preferences in such way that we are able to evaluate the distance between their components  $p(a,b)$ ,  $p(b,a)$ ,  $i(a,b)$ ,  $j(a,b)$ . By evaluating their distance we make reference to the way in which their values may be aggregated to obtain the unique value for any  $R(a,b)$ .

#### 5. Coherence as a single evaluation.

The preference model that we consider now, following the precedent analysis, expresses one unique relation  $R(a,b)$  as a composition of other three different relations

$$p, i, j : [0,1]^2 \rightarrow [0,1]$$

such that

$$R(a,b) = \{p(a,b), i(a,b), j(a,b)\}$$

where  $p(a,b)$  stands for the difference, in absolute value, between preference of  $a$  over  $b$  and preference of  $b$  over  $a$ ,  $i(a,b)$  for indifference between  $a$  and  $b$ , and  $j(a,b)$  for incomparability between the two alternatives  $a$  and  $b$ . We will continue to consider only complete preference relations, such that  $R(a,b) = 1$ , just as before.

The coherence measure we will use for evaluating the distance between the component relations of  $R(a,b)$  is the same as the one we used in the first place, but notice that now  $d(A,B) = 0$ . The distance that we will be now evaluating is the one between the fuzzy set and its complement or negation, given  $n$ , a strong negation defined over the set of fuzzy sets in  $A$ ,  $R^f(A)$ .

Examining our case given by the fuzzy preferences  $I$ ,  $C$  and  $K$ , we want to know in which way these fuzzy relations are similar and different at the same time, so we can calculate the degree in which they can be directly compared. In particular, we want to understand why two fuzzy preference relations that are so

different, when compared through a third one, share the same coherence measure.

Applying the coherence measure to each fuzzy preference relation, so that we can identify the degree of consistency of each subset in relation to itself, we have

$$\delta(I) = 0.93$$

$$\delta(C) = 0.93$$

$$\delta(K) = 0.66$$

Fuzzy preferences  $I$  and  $C$  have a similar composition in relation to themselves, because they share the same values of intensities but for different elements. Remember that  $I$  is a preference relation with a strong intensity value for  $i(a,b)$ , while  $C$  is a preference relation with a high membership intensity for  $p(a,b)$ . As we know from  $\beta(I,C) = 0.525$ ,  $I$  and  $C$  do not express similar preference relations, but completing this information with the coherence measure of the fuzzy relation to itself, they do represent equally consistent or coherent relations. This is an evaluation that focuses on the way they are constructed but not on their content or the message they represent.

The advantage of complementing the two coherence measures (one same measure but applied in different ways), is that we can take into account a measure of its content or semantics with respect to another fuzzy preference relation, along with a measure of its syntax or consistency of how it is constructed. If we see that fuzzy relation  $K$ , its value with respect to itself is 0.66, in such a way that we can notice its low level or degree of intrinsic coherence. That's why it's not meaningful the coherence measure obtained when we compared  $I$  and  $C$  with  $K$ , because  $K$  is a poorly constructed fuzzy preference relation.

## 6. Final comments

Coherence measures, as presented in [16], are necessary and efficient tools for developing comparisons between different fuzzy sets. In the case of fuzzy preference relations, they can be clearly applied and pertinent information can come out of them. Although its necessary character, it does not seem sufficient to solely consider coherence measures as a double evaluation between any pair of fuzzy sets.

We extend here the concept of coherence measures between pairs of fuzzy sets, and propose to also consider coherence measures as an intrinsic property for any fuzzy set. In this way, the information of the first can be complemented by the second one, examining the way in which the fuzzy set has been constructed. The first one is an evaluation of its existence with respect to other subsets of its kind, while the second one makes reference to a foundational property for any fuzzy set.

It remains as a pending task to consider coherence measures that take into account a kind of negation that doesn't consist in its strong version (an extremely relevant issue in most classification structures, see [11,17]). If we consider only the complement as the negative version for any proposition, we are ignoring the classical intuitionistic principle that says, every assertion has to first, be constructed, and then, because of the same construction, it can be demonstrated (see [2,3]). That is why the principle of the excluded middle or its strong negation may not be solidly grounded on verifiable scientific grounds, and the arguments in [14] may become a key stone of future research, together with standard arguments from logic (see, e.g., [15]).

## Acknowledgements.

This research has been partially supported by grant TIN2006-06190 from the Government of Spain.

## References

- [1] A. Amo, J. Montero, G. Biging, V. Cutello (2004): Fuzzy classification systems. *European Journal of Operational Research* 156:459-507.
- [2] K. Atanassov (1999): *Intuitionistic Fuzzy Sets* (Physica-Verlag, Heidelberg).
- [3] L.E.J. Brouwer (1975): *Collected Works* (North-Holland, Amsterdam).
- [4] V. Cutello, J. Montero (1993): A characterization of rational amalgamation operations. *International Journal of Approximate Reasoning* 8:325-344.
- [5] V. Cutello, J. Montero (1994): Fuzzy rationality measures. *Fuzzy Sets and Systems* 62:39-44.

- [6] D. Dubois, S. Gottwald, P. Hajek, J. Kacprzyk, H. Prade (2005): Terminological difficulties in fuzzy set theory-The case of "Intuitionistic Fuzzy Sets". *Fuzzy Sets and Systems* 156:485-491.
- [7] J. Fodor, M. Roubens (1994): *Fuzzy Preference Modelling and Multicriteria Decision Support* (Kluwer Academic Publishers).
- [8] J.L. García-Lapresta, J. Montero (2004): Consistency of crisp and fuzzy preference relations. *Proceedings of the IPMU'04 Conference* (Universita La Sapienza, Roma), pp. 527-534.
- [9] A. Heyting (1966): *Intuitionism, an Introduction* (North-Holland, Amsterdam).
- [10] J. Montero (1987): Arrow's theorem under fuzzy rationality. *Behavioral Science* 32:267-273.
- [11] J. Montero, D. Gómez, H. Bustince (2007): On the relevance of some families of fuzzy sets. *Fuzzy Sets and Systems* 158:2429-2442
- [12] J. Montero, V. López, D. Gómez (2007): The role of fuzziness in decision making. *Studies in Fuzziness and Soft Computing* 215:337-349.
- [13] J. Montero, J. Tejada, V. Cutello (1997): A general model for deriving preference structures from data. *European Journal of Operational research* 98:98-110.
- [14] M. Öztürk, A. Tsoukiàs (2008): A. Bipolar preference modelling and aggregation in decision support. *International Journal of Intelligent Systems* 23:970-984.
- [15] N. Rescher (1969): *Many Valued Logics* (New York, McGraw-Hill).
- [16] A. Sancho-Royo, J.L. Verdegay (2000): Coherence measures on finite fuzzy sets, *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems* 8:541-663.
- [17] E. Trillas (2006): On the use of words and fuzzy sets. *Information Sciences* 176 :1463-1487.
- [18] L.A. Zadeh (1965): Fuzzy sets. *Information and Control* 8:338-353.