AIP Journal of Mathematical Physics

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Citation: Journal of Mathematical Physics **40**, 2782 (1999); doi: 10.1063/1.532728
View online: http://dx.doi.org/10.1063/1.532728
View Table of Contents: http://scitation.aip.org/content/aip/journal/jmp/40/6?ver=pdfcov
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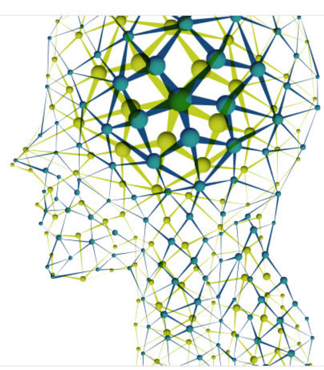

JOURNAL OF MATHEMATICAL PHYSICS VOLUME 40, NUMBER 6 JUNE 1999

Symmetries of discrete dynamical systems involving two species

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(Received 24 December 1998; accepted for publication 4 March 1999)

The Lie point symmetries of a coupled system of two nonlinear differential-difference equations are investigated. It is shown that in special cases the symmetry group can be infinite dimensional, in other cases up to ten dimensional. The equations can describe the interaction of two long molecular chains, each involving one type of atoms. © *1999 American Institute of Physics.* [S0022-2488(99)03206-5]

I. INTRODUCTION

Our purpose in this article is to perform a symmetry analysis of a system of two coupled differential-difference equations of the form

$$\begin{aligned} E_1 &= \ddot{u}_n - F_n(t, u_{n-1}, u_n, u_{n+1}, v_{n-1}, v_n, v_{n+1}) = 0, \\ E_2 &= \ddot{v}_n - G_n(t, u_{n-1}, u_n, u_{n+1}, v_{n-1}, v_n, v_{n+1}) = 0. \end{aligned} \tag{1.1}$$

The overdots denote time derivatives. The discrete variable n plays the role of a space variable; it labels positions along a one-dimensional lattice. The functions F_n and G_n represent interactions, e.g., between different atoms along a double chain of molecules (see Fig. 1). The functions F_n and G_n are *a priori* unspecified; our aim is to classify equations of the type (1.1) according to the Lie point symmetries that they allow. The interactions in such a model depend on up to six neighboring particles. For instance, we can interpret u_n and v_n as deviations from equilibrium positions of two different types of atoms, say type U and type V. The accelerations $\ddot{u}_n$ and $\ddot{v}_n$ depend on the deviations u and v of both types of atoms at the neighboring sites $n-1$, n, and $n+1$. We do not restrict to two-body forces, nor do we impose translational invariance for the chain. We do, however, assume there is no dissipation, i.e., system (1.1) does not involve first derivatives with respect to time.

Such differential-difference equations typically arise when modeling phenomena in molecular physics, biophysics, or simply coupled oscillations in classical mechanics.[1–3]

A recent article[4] was devoted to a similar problem, but was concerned with a single species, i.e., one dependent variable $u_n(t)$. The approach adopted here is similar to that of Ref. 4. Thus, we shall consider only symmetries acting on the continuous variables t, u_n, and v_n. Transformations of the discrete variable n must then be studied separately.

Several different treatments of Lie symmetries of difference and differential-difference equations exist in the literature.[4–13] The one adopted in this article is that of Refs. 4–6. It has been

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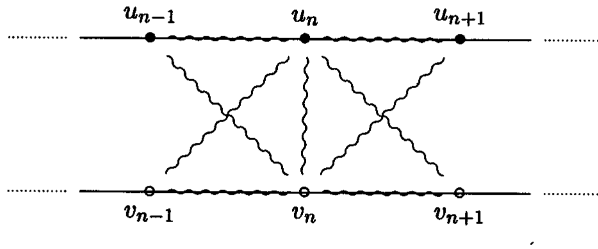

FIG. 1. Double molecular chain with two types of atoms.

called the ''intrinsic method,'' makes use of a Lie algebraic approach, and is entirely algorithmic. The Lie algebra of the symmetry group, the ''symmetry algebra'' for short, is realized by vector fields of the form

$$\hat{X}=\tau(t,u_n,v_n)\partial_t+\phi_n(t,u_n,v_n)\partial_{u_n}+\psi_n(t,u_n,v_n)\partial_{v_n}. \tag{1.2}$$

The algorithm for finding the functions τ, ϕ_n, and ψ_n in (1.2) is to construct the appropriate prolongation $\mathrm{pr}\,\hat{X}$ of $\hat{X}$ (see Refs. 4–6 and Sec. II) and to impose that it should annihilate the studied system of equations on their solution set,

$$\mathrm{pr}\,\hat{X}E_1|_{E_1=E_2=0}=0, \quad \mathrm{pr}\,\hat{X}E_2|_{E_1=E_2=0}=0. \tag{1.3}$$

Our first step is to find and classify all interactions (F_n,G_n) for which the system (1.1) allows at least a one-dimensional symmetry algebra. The next step is to specify the interactions further and to find all those that allow a higher-dimensional, possibly infinite-dimensional, symmetry algebra.

As in previous articles,[4,14] our classification will be up to conjugacy under a group of ''allowed transformations.'' These are fiber preserving locally invertible point transformations,

$$u_n=\Omega_n(\tilde{u}_n,\tilde{v}_n,\tilde{t}), \quad v_n=\Gamma_n(\tilde{u}_n,\tilde{v}_n,\tilde{t}), \quad t=t(\tilde{t}), \tag{1.4}$$

which preserve the form of Eqs. (1.1), but not necessarily the functions F_n and G_n (they go into new functions $\tilde{F}_n$ and $\tilde{G}_n$ of the new arguments).

Throughout the article we assume that both F_n and G_n depend on at least one of the quantities $u_{n-1},u_{n+1},v_{n-1},v_{n+1}$, so that nearest neighbors are genuinely involved. In the bulk of the article the interaction is assumed to be nonlinear.

In Sec. II we formulate the problem, establish the general form of the elements of the symmetry algebra, and present the determining equations for the symmetries. We also derive the ''allowed transformations'' under which we classify the interactions and their symmetries. Section III is devoted to a classification of interactions F_n, G_n, allowing at least a one-dimensional symmetry algebra. Ten classes of such interactions exist, each involving two arbitrary functions of six variables. In Sec. IV we study higher-dimensional symmetry algebras and introduce an important restriction. We first prove that four equivalence classes of symmetry algebras isomorphic to $\mathrm{sl}(2,\mathbb{R})$ exist. Then we restrict to just one of them, $\mathrm{sl}(2,\mathbb{R})_1$ generating a gauge group acting only on the fields u_n and v_n (in a global, coordinate-independent manner). We describe all symmetry algebras, containing the chosen $\mathrm{sl}(2,\mathbb{R})$ as a subalgebra. In Sec. V we obtain the invariant interactions for all algebras containing $\mathrm{sl}(2,\mathbb{R})_1$. The results are summed up and discussed in Sec. VI, where we also outline future work to be done.

II. FORMULATION OF THE PROBLEM

To find the Lie point symmetries of the system (1.1), we write the second prolongation of the vector field (1.2) in the form[4–6]

$$\mathrm{pr}^{(2)}\hat{X}=\tau(t,u_n,v_n)\partial_t+\sum_{k=n-1}^{n+1}\phi_k(t,u_n,v_n)\partial_{u_k}+\sum_{k=n-1}^{n+1}\psi_k(t,u_n,v_n)\partial_{v_k}+\phi_n^{tt}\partial_{\ddot{u}_n}+\psi_n^{tt}\partial_{\ddot{v}_n}, \tag{2.1}$$

with

$$\begin{aligned}\phi_n^{tt}&=D_t^2\phi_n-(D_t^2\tau)\dot{u}_n-2(D_t\tau)\ddot{u}_n,\\ \psi_n^{tt}&=D_t^2\psi_n-(D_t^2\tau)\dot{v}_n-2(D_t\tau)\ddot{v}_n,\end{aligned} \tag{2.2}$$

where D_t is the total time derivative. The determining equations for the symmetries are obtained by requiring that Eq. (1.3) be satisfied. The obtained equations will involve terms like $\dot{u}^k$, $\dot{v}^k$, and $\dot{u}^k\dot{v}^l$. The coefficients of each linearly independent term must vanish and this provides 16 linear differential equations that are easy to solve and do not involve the interaction functions F_n, G_n. The result is that an element $\hat{X}$ of the symmetry algebra must have the form

$$\hat{X}=\tau(t)\partial_t+\left[\left(\frac{\dot{\tau}}{2}+a_n\right)u_n+b_nv_n+\lambda_n(t)\right]\partial_{u_n}+\left[c_nu_n+\left(\frac{\dot{\tau}}{2}+d_n\right)v_n+\mu_n(t)\right]\partial_{v_n}, \tag{2.3}$$

where the overdots denote time derivatives. The functions $\tau(t)$, $\lambda_n(t)$, $\mu_n(t)$, a_n, b_n, c_n, and d_n satisfy the two remaining determining equations, namely,

$$\begin{aligned}&\frac{\dddot{\tau}}{2}u_n+\ddot{\lambda}_n+\left(a_n-\frac{3}{2}\dot{\tau}\right)F_n+b_nG_n-\tau F_{n,t}-\sum_{k=n-1}^{n+1}F_{n,u_k}\left[\left(\frac{\dot{\tau}}{2}+a_k\right)u_k+b_kv_k+\lambda_k(t)\right]\\ &\quad-\sum_{k=n-1}^{n+1}F_{n,v_k}\left[\left(\frac{\dot{\tau}}{2}+d_k\right)v_k+c_ku_k+\mu_k(t)\right]=0,\end{aligned} \tag{2.4}$$

$$\begin{aligned}&\frac{\dddot{\tau}}{2}v_n+\ddot{\mu}_n+\left(d_n-\frac{3}{2}\dot{\tau}\right)G_n+c_nF_n-\tau G_{n,t}-\sum_{k=n-1}^{n+1}G_{n,u_k}\left[\left(\frac{\dot{\tau}}{2}+a_k\right)u_k+b_kv_k+\lambda_k(t)\right]\\ &\quad-\sum_{k=n-1}^{n+1}G_{n,v_k}\left[\left(\frac{\dot{\tau}}{2}+d_k\right)v_k+c_ku_k+\mu_k(t)\right]=0.\end{aligned} \tag{2.5}$$

In Eqs. (2.3), (2.4), and (2.5) the quantities a_n, b_n, c_n, and d_n are independent of t. To proceed further, one could specify the interaction functions F_n and G_n. Instead, we shall assume that at least one symmetry generator (2.3) exists and make use of allowed transformations to simplify this vector. The second step is to find interactions F_n and G_n compatible with such a symmetry.

Substituting (1.4) into Eq. (1.1) and requiring that the form of these two equations be preserved, we find that the allowed transformations are quite restricted, namely,

$$\begin{pmatrix}u_n(t)\\ v_n(t)\end{pmatrix}=\begin{pmatrix}Q_n & R_n\\ S_n & T_n\end{pmatrix}\dot{\tilde{t}}^{-1/2}\begin{pmatrix}\tilde{u}_n(\tilde{t})\\ \tilde{v}_n(\tilde{t})\end{pmatrix}+\begin{pmatrix}\alpha_n(t)\\ \beta_n(t)\end{pmatrix},\quad \tilde{t}=\tilde{t}(t),\quad \frac{d\tilde{t}}{dt}\neq 0. \tag{2.6}$$

The entries Q_n, R_n, S_n, and T_n are independent of t; $\tilde{t}(t)$ is an arbitrary locally invertible function of t; α_n, β_n are arbitrary functions of n and t, and the matrix

$$M_n=\begin{pmatrix}Q_n & R_n\\ S_n & T_n\end{pmatrix},\quad \det M_n\neq 0, \tag{2.7}$$

is nonsingular.

It will be convenient to use a shorthand notation for the vector field X_n of Eq. (2.3), namely,

$$\left\{\tau(t),A_n,\begin{pmatrix}\lambda_n(t)\\ \mu_n(t)\end{pmatrix}\right\},\quad A_n=\begin{pmatrix}a_n & b_n\\ c_n & d_n\end{pmatrix}. \tag{2.8}$$

If we perform an allowed transformation (2.6), then Eq. (1.1) goes into an equation of the same form, with F_n and G_n replaced by

$$\begin{pmatrix}\tilde{F}_n\\ \tilde{G}_n\end{pmatrix}=\dot{\tilde{t}}^{-3/2}M_n^{-1}\left[\begin{pmatrix}F_n\\ G_n\end{pmatrix}-\begin{pmatrix}\ddot{\alpha}_n\\ \ddot{\beta}_n\end{pmatrix}\right]+\left(\frac{1}{2}\dddot{\tilde{t}}\dot{\tilde{t}}^{-3}-\frac{3}{4}\ddot{\tilde{t}}^2\dot{\tilde{t}}^{-4}\right)\begin{pmatrix}\tilde{u}_n\\ \tilde{v}_n\end{pmatrix}, \tag{2.9}$$

where $\tilde{F}_n$ and $\tilde{G}_n$ are functions of the new variables.

The vector field characterized by the triplet (2.3) goes into a new one of the same form,

$$\left\{\tilde{\tau}(\tilde{t}),\tilde{A}_n,\begin{pmatrix}\tilde{\lambda}_n(\tilde{t})\\ \tilde{\mu}_n(\tilde{t})\end{pmatrix}\right\}, \tag{2.10}$$

with

$$\tilde{\tau}(\tilde{t})=\tau(t(\tilde{t}))\dot{\tilde{t}},$$

$$\tilde{A}_n=M_n^{-1}A_nM_n,$$

$$\begin{pmatrix}\tilde{\lambda}_n(\tilde{t})\\ \tilde{\mu}_n(\tilde{t})\end{pmatrix}=M_n^{-1}\dot{\tilde{t}}^{1/2}\left[\left(A_n+\frac{\dot{\tau}}{2}\right)\begin{pmatrix}\alpha_n\\ \beta_n\end{pmatrix}-\tau\begin{pmatrix}\dot{\alpha}_n\\ \dot{\beta}_n\end{pmatrix}+\begin{pmatrix}\lambda_n\\ \mu_n\end{pmatrix}\right].$$

We shall use the allowed transformations to simplify the vector field, rather than the equation itself.

III. SYSTEMS WITH ONE-DIMENSIONAL SYMMETRY GROUPS

Let us now assume that the system (1.1) has at least a one-dimensional symmetry group, generated by a vector field of the type (2.3). Using allowed transformations (2.6), we take $\hat{X}$ into one of ten inequivalent classes.

Indeed, for $\tau\neq 0$ we can choose the function $\tilde{t}(t)$ so as to transform $\tau(t)$ into $\tau=1$, the functions $\alpha_n(t)$ and $\beta_n(t)$ so as to annul $\lambda_n(t)$, and $\mu_n(t)$ and the matrix M_n so as to take A_n into its canonical Jordan form.

For $\tau=0$ the standardized form of $\hat{X}$ depends on the rank of the matrix A_n. For rank $A_n=2$, we can again transform λ_n and μ_n into $\lambda_n=\mu_n=0$ and take A_n into one of three canonical forms. For rank $A_n=1$, only one of the functions λ_n or μ_n can be annulled. We choose it to be $\lambda_n(t)=0$. Then A_n can be taken into one of the two standard matrices of rank 1 in $\mathbb{R}^{2\times 2}$. For rank $A_n=0$ both $\lambda_n(t)$ and $\mu_n(t)$ survive.

We thus obtain ten mutually inequivalent one-dimensional symmetry algebras, listed below. The statement now is that any single vector field $\hat{X}$ of the form (2.3) can be transformed by an allowed transformation into precisely one of these vector fields.

The next step is to determine the interactions for which a one-dimensional symmetry group exists. To do this, we run through the canonical vector fields just obtained, substitute the corresponding τ (=1 or 0), A_n, $\lambda_n(t)$, and $\mu_n(t)$ into Eqs. (2.4) and (2.5), and solve these equations for F_n and G_n.

Following this procedure, we arrive at the following list of interactions and their one-dimensional symmetry algebras:

$A_{1,1}$
$$\hat{X}=\partial_t+a_n u_n\partial_{u_n}+d_n v_n\partial_{v_n},$$
$$F_n=e^{a_n t}f_n(\xi_k,\eta_k),$$
$$G_n=e^{d_n t}g_n(\xi_k,\eta_k),$$
$$\xi_k=u_k e^{-a_k t},\quad \eta_k=v_k e^{-d_k t},$$
$$k=n-1,n,n+1;$$

$A_{1,2}$
$$\hat{X}=\partial_t+(a_n u_n+v_n)\partial_{u_n}+a_n v_n\partial_{v_n},$$
$$F_n=e^{a_n t}[f_n(\xi_k,\eta_k)+t g_n(\chi_k,\eta_k)],$$
$$G_n=e^{a_n t}g_n(\xi_k,\eta_k),$$
$$\xi_k=(u_k-t v_k)e^{-a_k t},\quad \eta_k=v_k e^{-a_k t},$$
$$k=n-1,n,n+1;$$

$A_{1,3}$
$$\hat{X}=\partial_t+(a_n u_n+b_n v_n)\partial_{u_n}+(-b_n u_n+a_n v_n)\partial_{v_n},\quad b_n>0,$$
$$\begin{pmatrix}F_n\\ G_n\end{pmatrix}=e^{a_n t}\begin{pmatrix}\cos b_n t & \sin b_n t\\ -\sin b_n t & \cos b_n t\end{pmatrix}\begin{pmatrix}f_n(\xi_k,\eta_k)\\ g_n(\xi_k,\eta_k)\end{pmatrix},$$
$$\xi_k=r_k e^{-a_k t},\quad \eta_k=\theta_k+b_k t,$$
$$u_k=r_k\cos\theta_k,\quad v_k=r_k\sin\theta_k,$$
$$k=n-1,n,n+1;$$

$A_{1,4}$
$$\hat{X}=a_n u_n\partial_{u_n}+d_n v_n\partial_{v_n},\quad |a_n|\geqslant|d_n|,$$
$$F_n=u_n f_n(\xi_\alpha,\eta_k,t),$$
$$G_n=v_n g_n(\xi_\alpha,\eta_k,t),$$
$$\xi_a=u_\alpha^{a_n}u_n^{-a_\alpha},\quad \eta_k=v_k^{a_n}u_n^{-d_k},$$
$$k=n-1,n,n+1,\quad \alpha=n-1,n+1;$$

$A_{1,5}$
$$\hat{X}=(a_n u_n+v_n)\partial_{u_n}+a_n v_n\partial_{v_n},\quad a_n\neq 0,$$
$$F_n=v_n f_n(\eta_\alpha,\xi_k,t)+v_n\ln(v_n)g_n(\eta_\alpha,\xi_k,t),$$
$$G_n=a_n v_n g_n(\eta_\alpha,\xi_k,t),$$
$$\xi_k=a_k\frac{u_k}{v_k}-\ln(v_k),\quad \eta_\alpha=v_\alpha^{a_n}v_n^{-a_\alpha},$$
$$k=n-1,n,n+1,\quad \alpha=n-1,n+1;$$

$A_{1,6}$
$$\hat{X}=v_n\partial_{u_n},$$
$$F_n=f_n(v_k,\xi_\alpha,t)+u_n g_n(v_k,\xi_\alpha,t),$$
$$G_n=v_n g_n(v_k,\xi_\alpha,t),$$
$$\xi_\alpha=-v_\alpha u_n+v_n u_\alpha,$$
$$k=n-1,n,n+1,\quad \alpha=n-1,n+1;$$

$A_{1,7}$
$$\hat{X}=(a_n u_n+b_n v_n)\partial_{u_n}+(-b_n u_n+a_n v_n)\partial_{v_n},\quad b_n>0,$$
$$\begin{pmatrix}F_n\\ G_n\end{pmatrix}=e^{-(a_n/b_n)\theta_n}\begin{pmatrix}\cos\theta_n & -\sin\theta_n\\ \sin\theta_n & \cos\theta_n\end{pmatrix}\begin{pmatrix}f_n(\xi_k,\eta_\alpha,t)\\ g_n(\xi_k,\eta_\alpha,t)\end{pmatrix},$$
$$\xi_k=r_k^{b_n}e^{a_k\theta_n},\quad \eta_\alpha=b_n\theta_\alpha-b_\alpha\theta_n,$$
$$u_k=r_k\cos\theta_k,\quad v_k=r_k\sin\theta_k,$$
$$k=n-1,n,n+1,\quad \alpha=n-1,n+1;$$

$A_{1,8}$
$$\hat{X}=a_n u_n\partial_{u_n}+\mu_n(t)\partial_{v_n},\quad \mu_n\neq 0,$$
$$F_n=u_n f_n(\eta_\alpha,\xi_k,t),$$
$$G_n=\frac{\ddot{\mu}_n}{\mu_n}v_n+g_n(\eta_\alpha,\xi_k,t),$$
$$\eta_\alpha=\mu_n v_\alpha-\mu_\alpha v_n,\quad \xi_k=u_k e^{-\alpha_k v_n/\mu_n},$$
$$k=n-1,n,n+1,\quad \alpha=n-1,n+1;$$

$$
\begin{aligned}
A_{1,9} \qquad & \hat{X}=v_n\partial_{u_n}+\mu_n(t)\partial_{v_n}, \quad \mu_n\neq 0,\\
& F_n=\frac{1}{2}\frac{\ddot{\mu}_n}{\mu_n^2}v_n^2+v_n g_n(\eta_\alpha,\eta_n,\xi_\alpha,t)+f_n(\eta_\alpha,\eta_n,\xi_\alpha,t),\\
& G_n=\frac{\ddot{\mu}_n}{\mu_n}v_n+\mu_n g_n(\eta_\alpha,\eta_n,\xi_\alpha,t),\\
& \eta_\alpha=\mu_n^2u_\alpha+\tfrac{1}{2}\mu_\alpha v_n^2-\mu_n v_n v_\alpha, \quad \xi_\alpha=\mu_\alpha v_n-\mu_n v_\alpha,\\
& \eta_n=\mu_n u_n-\tfrac{1}{2}v_n^2, \quad \alpha=n-1,n+1;\\
A_{1,10} \qquad & \hat{X}=\lambda_n(t)\partial_{u_n}+\mu_n(t)\partial_{v_n}, \quad \lambda_n, \ \mu_n\neq 0,\\
& F_n=\frac{\ddot{\lambda}_n}{\lambda_n}u_n+f_n(\eta_k,\xi_\alpha,t),\\
& G_n=\frac{\ddot{\mu}_n}{\mu_n}u_n+g_n(\eta_k,\xi_\alpha,t),\\
& \xi_\alpha=\lambda_n u_\alpha-\lambda_\alpha u_n, \quad \eta_k=\mu_k u_n-\lambda_n v_k,\\
& k=n-1,n,n+1, \quad \alpha=n-1,n+1.
\end{aligned}
$$

We mention that the variables ξ_k and η_k are to be taken exactly as above. For instance, ξ_{n+1} is not an upshift of ξ_n.

The above results are summed up quite simply. Namely, the existence of a one-dimensional symmetry algebra restricts the interaction terms F_n and G_n to two arbitrary functions of six variables, rather than the original seven variables. The algebras $A_{1,1}$, $A_{1,2}$ and $A_{1,3}$ involve time translations. Hence, the time dependence in these cases is restricted: F_n and G_n depend on time explicitly and via invariant variables ξ_k and η_k that, in turn, depend explicitly on t. The algebras $A_{1,4},\ldots,A_{1,10}$ correspond to gauge transformations: the group transformations act on dependent variables only. The time variable figures in the arbitrary functions, as does the discrete independent variable n.

IV. HIGHER-DIMENSIONAL SYMMETRY ALGEBRAS

A. General strategy

The commutator of two symmetry operators (2.3) is an operator $X_3=[X_1,X_2]$ of the same form, satisfying

$$
\begin{gathered}
\tau_3=\tau_1\dot{\tau}_2-\tau_2\dot{\tau}_1, \quad A_{n,3}=-[A_{n,1},A_{n,2}],\\
\begin{pmatrix}\lambda_{n,3}\\ \mu_{n,3}\end{pmatrix}=\tau_1\begin{pmatrix}\dot{\lambda}_{n,2}\\ \dot{\mu}_{n,2}\end{pmatrix}-\tau_2\begin{pmatrix}\dot{\lambda}_{n,1}\\ \dot{\mu}_{n,1}\end{pmatrix}-\left(A_{n,1}+\frac{\dot{\tau}_1}{2}\right)\begin{pmatrix}\lambda_{n,2}\\ \mu_{n,2}\end{pmatrix}+\left(A_{n,2}+\frac{\dot{\tau}_2}{2}\right)\begin{pmatrix}\lambda_{n,1}\\ \mu_{n,1}\end{pmatrix}.
\end{gathered}
\tag{4.1}
$$

To obtain a finite-dimensional Lie algebra of symmetry operators, we see that the ''differential components'' $\tau_i(t)\partial_t$ must form a Lie algebra L_d, the ''matrix components'' $A_{n,i}$ must also form a Lie algebra L_m, homomorphic to L_d. Moreover, Eq. (4.1) shows that the ''functional components'' $(\lambda_{n,i}(t),\mu_{n,i}(t))$ must satisfy certain cohomology conditions.

The algebra of diffeomorphisms of $\mathbb{R}^1$, $\{\tau(t)\partial_t\}$ has only three mutually nondiffeomorphic finite-dimensional subalgebras, namely $\mathrm{sl}(2,\mathbb{R})$ and its subalgebras, realized, e.g., as

$$
\{\partial_t,t\partial_t,t^2\partial_t\}, \quad \{\partial_t,t\partial_t\}, \quad \text{and } \{\partial_t\}, \tag{4.2}
$$

respectively.

For n fixed, the matrices A_n generate the Lie algebra of $\mathrm{gl}(2,\mathbb{R})$. However, since the dependence on n is arbitrary, an unlimited number of copies of $\mathrm{gl}(2,\mathbb{R})$ and its subalgebras is available.

We shall not perform a complete classification of possible symmetry algebras here. Instead, we shall first concentrate on sl(2,$\mathbb{R}$) symmetry algebras and show that, up to allowed transformations, four different sl(2,$\mathbb{R}$) symmetry algebras can be constructed. We then consider just one of these four and study its extensions to higher-dimensional Lie algebras.

B. Equivalence classes of sl(2,$\mathbb{R}$) symmetry algebras

Since sl(2,$\mathbb{R}$) is a simple Lie algebra, it has no ideals. Hence, a homomorphism between sl(2,$\mathbb{R}$) algebras is either an isomorphism, or one of the algebras is mapped onto zero. Correspondingly, we have three possibilities to explore: we shall call them $\mathrm{sl}(2,\mathbb{R})_d$, $\mathrm{sl}(2,\mathbb{R})_m$, and $\mathrm{sl}(2,\mathbb{R})_c$ (where d stands for ''differential,'' m for ''matrix,'' and c for ''combined'').

1. The algebra $\mathrm{sl}(2,\mathbb{R})_d$

We have *a priori*

$$X_1=\partial_t+\lambda_n(t)\partial_{u_n}+\mu_n(t)\partial_{v_n},$$

$$X_2=t\partial_t+(\tfrac{1}{2}u_n+\rho_n(t))\partial_{u_n}+(\tfrac{1}{2}v_n+\sigma_n(t))\partial_{v_n}, \tag{4.3}$$

$$X_3=t^2\partial_t+(tu_n+\omega_n(t))\partial_{u_n}+(tv_n+\kappa_n(t))\partial_{v_n}.$$

Using allowed transformations we transform $\lambda_n\to 0$, $\mu_n\to 0$. The commutation relation $[X_1,X_2]=X_1$ then requires $\dot{\rho}_n=\dot{\sigma}_n=0$. A further allowed transformation (2.6) with $\tilde{t}(t)=t$, $M_n=I$, and (α_n,β_n) constant will not change X_1, but take $\rho_n\to 0$, $\sigma_n\to 0$ (while leaving $\lambda_n=\mu_n=0$). The commutation relations $[X_2,X_3]=X_3$ and $[X_1,X_3]=2X_2$ then imply $\omega_n=\kappa_n=0$.

2. The algebra $\mathrm{sl}(2,\mathbb{R})_m$

A priori we have

$$X_1=b_nv_n\partial_{u_n}+\lambda_n(t)\partial_{u_n}+\mu_n(t)\partial_{v_n},$$

$$X_2=a_n(u_n\partial_{u_n}-v_n\partial_{v_n})+\rho_n(t)\partial_{u_n}+\sigma_n(t)\partial_{v_n}, \tag{4.4}$$

$$X_3=c_nu_n\partial_{v_n}+\omega_n(t)\partial_{u_n}+\kappa_n(t)\partial_{v_n}.$$

The structure constants cannot depend on n, so the commutation relations imply

$$a_n=a, \quad b_nc_n=bc. \tag{4.5}$$

Given that the product b_nc_n does not depend on n, we can use an allowed transformation to take $b_n\to b$, $c_n\to c$. A further allowed transformation will take $\rho_n\to 0$, $\sigma_n\to 0$. The commutation relations then imply $\lambda_n=\mu_n=0$ and $\omega_n=\kappa_n=0$.

3. The combined algebra $\mathrm{sl}(2,\mathbb{R})_c$

In view of the above results, we can write a ''combined'' algebra as

$$X_1=\partial_t+\alpha v_n\partial_{u_n}+\xi_n\partial_{u_n}+\eta_n\partial_{v_n}, \quad \alpha\neq 0,$$

$$X_2=t\partial_t+[(\tfrac{1}{2}+\beta)u_n+\lambda_n]\partial_{u_n}+[(\tfrac{1}{2}-\beta)v_n+\mu_n]\partial_{v_n}, \tag{4.6}$$

$$X_3=t^2\partial_t+(tu_n+\rho_n)\partial_{u_n}+(\gamma u_n+tv_n+\sigma_n)\partial_{v_n}.$$

We use allowed transformations to set $\alpha=1$, $\xi_n=\eta_n=0$. The commutation relations then determine $\beta=\frac{1}{2}$, $\gamma=-1$. The functions $\lambda_n(t)$, $\mu_n(t)$, $\rho_n(t)$, and $\sigma_n(t)$ are greatly restricted by the commutation relations. As a matter of fact, we either have $\lambda_n=\mu_n=\rho_n=\sigma_n=0$, or we can use allowed transformations to obtain $\lambda_n=t$, $\mu_n=1$, $\rho_n=2t^2$, $\sigma_n=2t$.

We arrive at the following result.

Theorem 1: *Precisely four classes of* $\mathrm{sl}(2,\mathbb{R})$ *algebras can be realized by vector fields of the form (2.3). Any such* $\mathrm{sl}(2,\mathbb{R})$ *algebra can be taken by an allowed transformation (2.6) into precisely one of the following algebras:*

$$\mathrm{sl}(2,\mathbb{R})_1: \quad \begin{aligned} X_1&=v_n\partial_{u_n},\\ X_2&=\tfrac{1}{2}(u_n\partial_{u_n}-v_n\partial_{v_n}),\\ X_3&=u_n\partial_{v_n}, \end{aligned} \tag{4.7}$$

$$\mathrm{sl}(2,\mathbb{R})_2: \quad \begin{aligned} X_1&=\partial_t,\\ X_2&=t\partial_t+\tfrac{1}{2}(u_n\partial_{u_n}+v_n\partial_{v_n}),\\ X_3&=t^2\partial_t+t(u_n\partial_{u_n}+v_n\partial_{v_n}), \end{aligned} \tag{4.8}$$

$$\mathrm{sl}(2,\mathbb{R})_3: \quad \begin{aligned} X_1&=\partial_t+v_n\partial_{u_n},\\ X_2&=t\partial_t+u_n\partial_{u_n},\\ X_3&=t^2\partial_t+tu_n\partial_{u_n}+(tv_n-u_n)\partial_{v_n}, \end{aligned} \tag{4.9}$$

$$\mathrm{sl}(2,\mathbb{R})_4: \quad \begin{aligned} X_1&=\partial_t+v_n\partial_{u_n}\\ X_2&=t\partial_t+(u_n+t)\partial_{u_n}+\partial_{v_n}\\ X_3&=t^2\partial_t+(tu_n+2t^2)\partial_{u_n}+(tv_n-u_n+2t)\partial_{v_n}. \end{aligned} \tag{4.10}$$

C. Indecomposable Lie algebras containing $\mathrm{sl}(2,\mathbb{R})_1$

A Lie algebra L is called indecomposable if it cannot be written as a direct sum, $L=L_1\oplus L_2$. A Lie algebra over $\mathbb{R}$ containing $\mathrm{sl}(2,\mathbb{R})$ is either simple or it allows a nontrivial Levi decomposition,[15]

$$L=S\rhd R, \tag{4.11}$$

where S is a semisimple Lie algebra and R is the radical, that is, the maximal solvable ideal of L.

It follows from the results of Sec. IV A that the only simple Lie algebras that can be constructed from operators of the form (2.3) are the four $\mathrm{sl}(2,\mathbb{R})$ algebras obtained in Sec. IV B. We can hence concentrate on Lie algebras of the form (4.11).

The algebra S is either $\mathrm{sl}(2,\mathbb{R})_1$ itself, or the direct sum of $\mathrm{sl}(2,\mathbb{R})_1$ with one or more other $\mathrm{sl}(2,\mathbb{R})$ algebras.

Requiring that a symmetry operator Y should commute with all elements of $\mathrm{sl}(2,\mathbb{R})_1$, we find that Y must have the form

$$Y_0=\tau\partial_t+(\tfrac{1}{2}\dot{\tau}+a_n)(u_n\partial_{u_n}+v_n\partial_{v_n}). \tag{4.12}$$

It is hence possible to construct precisely one semisimple Lie algebra properly containing $\mathrm{sl}(2,\mathbb{R})_1$, namely, the direct sum $\mathrm{sl}(2,\mathbb{R})_1\oplus\mathrm{sl}(2,\mathbb{R})_2$ with $\mathrm{sl}(2,\mathbb{R})_2$ defined in Eq. (4.8).

Let us introduce some notations for vector fields, to be used below. We put

$$V(a_n)=a_n(u_n\partial_{u_n}+v_n\partial_{v_n}), \tag{4.13}$$

$$T(a_n)=\partial_t+a_n(u_n\partial_{u_n}+v_n\partial_{v_n}), \tag{4.14}$$

$$D(a_n)=t\partial_t+(\tfrac{1}{2}+a_n)(u_n\partial_{u_n}+v_n\partial_{v_n}), \tag{4.15}$$

$$P(a_n)=t^2\partial_t+(t+a_n)(u_n\partial_{u_n}+v_n\partial_{v_n}), \tag{4.16}$$

$$R(a_n)=(t^2+1)\partial_t+(t+a_n)(u_n\partial_{u_n}+v_n\partial_{v_n}), \tag{4.17}$$

$$Y_u(\lambda_n)=\lambda_n(t)\partial_{u_n}, \quad Y_v(\lambda_n)=\lambda_n(t)\partial_{v_n}. \tag{4.18}$$

In all cases we have $\dot{a}_n=0$, but $\lambda_n(t)$ can be a function of t. Both a_n and $\lambda_n(t)$ can be functions of n.

Let us consider $S=\mathrm{sl}(2,\mathbb{R})_1$ and $S=\mathrm{sl}(2,\mathbb{R})_1\oplus\mathrm{sl}(2,\mathbb{R})_2$ in Eq. (4.11) separately.

1. $S=\mathrm{sl}(2,\mathbb{R})_1$

The considered Lie algebras will have a basis $\{X_1,X_2,X_3,Y_1,\dots,Y_n\}$ with X_i given in Eq. (4.7). The basis elements $\{Y_1,\dots,Y_n\}$ span the radical R. The algebra S acts on R according to some linear, not necessarily irreducible, finite-dimensional representation.

We start with the Cartan subalgebra $\{X_2\}$ of $\mathrm{sl}(2,\mathbb{R})$. It can be represented by a diagonal matrix in any finite-dimensional representation. Consider $Y\in R$. We have

$$[X_2,Y]=pY, \tag{4.19}$$

with Y as in Eq. (2.3). Equation (4.19) implies

$$p\tau=0,$$
$$p\left(\frac{\dot{\tau}}{2}+a_n\right)=0, \quad \left(p+\frac{1}{2}\right)\lambda_n=0, \quad (p+1)b_n=0, \tag{4.20}$$
$$p\left(\frac{\dot{\tau}}{2}+d_n\right)=0, \quad \left(p-\frac{1}{2}\right)\mu_n=0, \quad (p-1)c_n=0.$$

For $p=0$ we obtain an operator that commutes not only with X_2, but with all of $\mathrm{sl}(2,\mathbb{R})_1$, namely, Y_0 of Eq. (4.12). This is a singlet representation of $\mathrm{sl}(2,\mathbb{R})$.

For $p=1$, or $p=-1$, Eq. (4.19) forces Y to be an element of $\mathrm{sl}(2,\mathbb{R})_1$, in other words, no such $Y\in R$ exists.

For $p=\pm\frac{1}{2}$ we obtain $Y_1=\lambda_n(t)\partial_{u_n}$ and $Y_2=\mu_n(t)\partial_{v_n}$, respectively. Acting with X_1 and X_3 on these operators, we find that the only representation of $\mathrm{sl}(2,\mathbb{R})_1$ that can be realized is a doublet one, namely $\{Y_u(\lambda_n),Y_v(\lambda_n)\}$ of Eq. (4.18), with $\lambda_n(t)$ an arbitrary function of n and t. The indecomposable Lie algebra $\{X_1,X_2,X_3,Y_u(\lambda_n),Y_v(\lambda_n)\}$ is isomorphic to the special affine Lie algebra $\mathrm{saff}(2,\mathbb{R})$.

All further indecomposable symmetry algebras containing $\mathrm{sl}(2,\mathbb{R})_1$ must be extensions of $\mathrm{saff}(2,\mathbb{R})$. The objects that we can add to $\mathrm{saff}(2,\mathbb{R})$ are either $\mathrm{sl}(2,\mathbb{R})$ doublets or singlets. Let us run through all possibilities.

(1) We can add an arbitrary number k of doublets of the form (4.18), where the k functions $\{\lambda_n^1(t),\lambda_n^2(t),\dots,\lambda_n^k(t)\}$ must be linearly independent. However, we shall see in Sec. V that the presence of three such pairs forces the functions F_n and G_n in Eq. (1.1) to be linear. Moreover, even two such pairs are compatible with a nonlinear interaction only if they are of the form (or transformable into)

$$\lambda_n^1(t)=1, \quad \lambda_n^2(t)=t. \tag{4.21}$$

(2) We can add a singlet of the form (4.12). If we have $\tau=0$, then the commutation relations $[Y_0,Y_u]$ and $[Y_0,Y_v]$ imply $a_n=a_{n+1}$ and we can set $a_n=1$. We obtain an affine Lie algebra $\mathrm{gaff}(2,\mathbb{R})_1$ consisting of $\mathrm{saff}(2,\mathbb{R})$ and $V(1)$ of Eq. (4.13).
If we have $\tau\neq0$ in Eq. (4.12) and only one operator of this type, then we can use allowed transformations to take $\tau(t)$ into $\tau(t)=1$. The commutation relations $[Y_0,Y_u]$ and $[Y_0,Y_v]$ then imply

$$\lambda_n(t)=R_n e^{(a_n+k)t}, \quad \dot{R}_n=0.$$

For $k=0$, the algebra is decomposable. For $k\neq0$ we can use allowed transformations to put $k=-1$ and $R_n=1$. We obtain a second algebra isomorphic to $\mathrm{gaff}(2,\mathbb{R})$, but not conjugate to the previous one. We have

$$\mathrm{gaff}(2,\mathbb{R})_2\sim\{X_1,X_2,X_3,Y_u(e^{(a_n-1)t}),Y_v(e^{(a_n-1)t}),T(a_n)\}. \tag{4.22}$$

In the special case of $a_n=a_{n+1}$ in Eq. (4.22), a further extension is possible. We transform $\lambda=e^{(a-1)t}$ into $\lambda=1$; then $T(a_n)$ goes into $D(b_n)$ with $b_n=b_{n+1}\equiv b\neq-\frac{1}{2}$, since for $b=-\frac{1}{2}$ the algebra is decomposable.

(3) We can add two singlets of the form (4.12). If they commute, they must be $\{V(1),T(0)\}$. The obtained algebra is decomposable. If they do not commute, they must form a two-dimensional Lie algebra, namely, $\{T(0),D(a),a_n=a_{n+1}\equiv a\}$. This implies $\lambda_n(t)\sim1$, i.e., the entire radical is $\{Y_u(1),Y_v(1),T(0),D(a)\}$ with $a\neq\frac{1}{2}$ (the case $a=\frac{1}{2}$ corresponds to a decomposable algebra).

(4) If we add three singlets, the only case corresponds to the radical $\{Y_u(1),Y_v(1),V(1),T(0),D(0)\}$. There will then be no invariant interaction (see below).

(5) Let us consider the special case of two doublets of the form (4.18), namely,

$$Y_u(1)=\partial_{u_n}, \quad Y_v(1)=\partial_{v_n}, \quad Y_u(t)=t\partial_{u_n}, \quad Y_v(t)=t\partial_{v_n}. \tag{4.23}$$

This algebra can be extended by a further element, namely,

$$Z=(\tau_0+\tau_1 t+\rho_2 t^2)\partial_t+(\tfrac{1}{2}\tau_1+\tau_2 t+a)(u_n\partial_{u_n}+v_n\partial_{v_n}), \qquad a_n=a_{n+1}\equiv a, \tag{4.24}$$

where τ_0, τ_1, and τ_2 are constants. By allowed transformations we can take Z into one of the four operators $V(1)$, $T(a)$, $D(a)$, or $R(a)$ of (4.13), (4.14), (4.15), and (4.17), respectively.

(6) We can add a two-dimensional algebra to (4.23), namely,

$$\{T(0),D(a)\}, \quad \{T(0),V(1)\}, \quad \{V(1),D(0)\}, \text{ or } \{V(1),R(0)\}.$$

(7) We can add only one three-dimensional algebra to (4.23), namely,

$$\{T(0), \quad D(0), \quad V(1)\}.$$

This completes the list of indecomposable symmetry algebras of the form (4.11) with $S=\mathrm{sl}(2,\mathbb{R})_1$.

2. $S=\mathrm{sl}(2,\mathbb{R})_1\oplus\mathrm{sl}(2,\mathbb{R})_2$

The algebra S is itself decomposable. It gives rise to precisely two indecomposable symmetry algebras. First, we have the one obtained by adding the Abelian ideal (4.23) to $\mathrm{sl}(2,\mathbb{R})_1\oplus\mathrm{sl}(2,\mathbb{R})_2$. Second, we get an 11-dimensional algebra by adding $V(1)$ to the first case.

D. Decomposable Lie algebras containing $\mathrm{sl}(2,\mathbb{R})_1$

All decomposable Lie algebras L_D can be obtained from the indecomposable L_I ones, by adding their centralizers,

$$L_D = L_I \oplus C, \quad [C, L_I] = 0. \tag{4.25}$$

The centralizer C must commute with all elements of $\mathrm{sl}(2,\mathbb{R})_1$ and hence all of its elements will have the form of Y_0 of Eq. (4.12).

Let us consider the individual indecomposable algebras L_I.

1. $L_I = \mathrm{sl}(2,\mathbb{R})_1$

The centralizer C can be Abelian. Then we have the following possibilities: $C = \{V(a_{i,n}), i = 1,...,k\}$ or $C = \{V(a_{i,n}), T(b_n), i = 1,...,k\}$. The quantities $a_{1,n}, \ldots, a_{k,n}$ must form a set of k linearly independent functions of n. If the centralizer is non-Abelian, then we have either $C \sim \mathrm{sl}(2,\mathbb{R})_2$ or $C = \{T(0), D(a)\}$. Both of these centralizers can be further extended by adding $V(a_{i,n})$, $i = 1,...,k$, (with $a_{1,n}, \ldots, a_{k,n}$ linearly independent).

2. $L_I = \mathrm{saff}(2,\mathbb{R})$

We must require Y_0 of Eq. (4.12) to commute with $Y_u(\lambda_n)$ and $Y_v(\lambda_n)$ of Eq. (4.18). We obtain

$$\lambda_n(\tfrac{1}{2}\dot{\tau} + a_n) - \tau\dot{\lambda}_n = 0. \tag{4.26}$$

For $\tau = 0$, Eq. (4.26) implies $\lambda_n a_n = 0$, and this is not allowed. For $\tau \neq 0$ we take $\tau \to 1$ by an allowed transformation, and Eq. (4.26) then implies $\lambda_n(t) = \gamma_n e^{a_n t}$. A further allowed transformation will take $\gamma_n \to 1$. We obtain the decomposable Lie algebra $\mathrm{saff}(2,\mathbb{R}) \oplus T(a_n)$. In the special case $a_n = a_{n+1}$ we transform $\lambda_n(t) \to 1$ and obtain a larger centralizer, namely, $\{T(0), D(-\tfrac{1}{2})\}$.

3. $L_I = \mathrm{gaff}(2,\mathbb{R})_1$

A nontrivial centralizer exists only if we have $\lambda_n(t) = e^{a_n t}$ in $\mathrm{saff}(2,\mathbb{R})$. In the case $a_n \neq 0$, the centralizer is $C = \{T(a_n)\}$. If $a_n = 0$ the centralizer is $C = \{T(0), D(-\tfrac{1}{2})\}$.

4. $L_I = \mathrm{gaff}(2,\mathbb{R})_2$

The centralizer is $C = \{T(a_n) - V(1)\}$. This algebra corresponds to the first one obtained in the case $L_I = \mathrm{gaff}(2,\mathbb{R})_1$ above.

E. Summary of possible symmetry algebras containing $\mathrm{sl}(2,\mathbb{R})_1$

The classification of possible symmetry algebras can now be summed up rather simply. In addition to $\mathrm{sl}(2,\mathbb{R})_1$ of Eq. (4.7), we have a further algebra L_C (the ''complementary'' algebra). The structure of each symmetry algebra is

$$L = \mathrm{sl}(2,\mathbb{R})_1 \dotplus L_C, \quad [\mathrm{sl}(2,\mathbb{R})_1, L_C] \subseteq L_C, \quad [L_C, L_C] \subseteq L_C. \tag{4.27}$$

The symbol $\dotplus$ denotes a direct sum of vector spaces. Moreover, Eq. (4.27) shows that L is either a direct sum or a semidirect one. The algebra L_C is also a representation space for $\mathrm{sl}(2,\mathbb{R})_1$. Irreducible representations in this case can be of dimension 1 or 2. All higher-dimensional representations are completely reducible into sums of one- and two-dimensional representations.

For further use it is convenient to split the symmetry algebras into four series, according to the structure of the Lie algebra L_C. In all cases L contains $\mathrm{sl}(2,\mathbb{R})_1$. We shall just specify L_C.

1. Series A

L_C is solvable and each element is a $\mathrm{sl}(2,\mathbb{R})_1$ singlet. There exist three different infinite-dimensional Lie algebras of this type:

$$A_1.\ \{V(a_{k,n})\}, \tag{4.28}$$

$$A_2.\ \{T(b_n), V(a_{k,n})\}, \tag{4.29}$$

$$A_3.\ \{T(0),D(b_n),V(a_{k,n})\}. \tag{4.30}$$

In each case we have $k=1,2,\ldots,$ and the expressions a_k must be linearly independent functions of n. Taking $1\leqslant k\leqslant N$ for some finite N, we obtain finite-dimensional subalgebras.

2. Series B

L_C is solvable and contains precisely one $\mathrm{sl}(2,\mathbb{R})_1$ doublet,

$$B_1=\{Y_u(\lambda_n),Y_v(\lambda_n)\}. \tag{4.31}$$

This is the indecomposable algebra $\mathrm{saff}(2,\mathbb{R})$ [B_1 together with $\mathrm{sl}(2,\mathbb{R})_1$]. We have $\dim L=5$,

$$B_2=\{Y_u(\lambda_n),Y_v(\lambda_n),V(1)\}. \tag{4.32}$$

B_2 corresponds to the indecomposable algebra $\mathrm{gaff}(2,\mathbb{R})_1$ with $\dim L=6$,

$$B_3=\{Y_u(e^{(a_n-1)t}),Y_v(e^{(a_n-1)t}),T(a_n)\}. \tag{4.33}$$

B_3 corresponds to the Lie algebra $\mathrm{gaff}(2,\mathbb{R})_2$, isomorphic but not conjugate to B_2,

$$B_4=\{Y_u(e^{a_nt}),Y_v(e^{a_nt}),T(a_n)\}. \tag{4.34}$$

This algebra is $\mathrm{saff}(2,\mathbb{R})\oplus T(a_n)$,

$$B_5=\{Y_u(1),Y_v(1),T(0),D(a)\}. \tag{4.35}$$

The algebra B_5 is indecomposable (except if $a=-\frac{1}{2}$),

$$B_6=\{Y_u(e^{(a_n-1)t}),Y_v(e^{(a_n-1)t}),T(a_n),V(1)\}. \tag{4.36}$$

The algebra B_6 is decomposable,

$$B_7=\{Y_u(1),Y_v(1),T(0),D(0),V(1)\}. \tag{4.37}$$

The algebra B_7 is indecomposable.

3. Series C

L_C contains two $\mathrm{sl}(2,\mathbb{R})$ doublets. The doublets could be characterized by any two functions $\lambda_{1,n}(t)$ and $\lambda_{2,n}(t)$. However, we shall only be interested in the case $\lambda_1=1$, $\lambda_2=t$. The others do not lead to invariant interactions. Similarly, we do not need algebras containing three or more doublets. In all cases the algebra L_C contains the elements (4.23). For $\dim L_C\geqslant 5$ it contains further elements. We have

$$C_1=\{Y_u(1),Y_v(1),Y_u(t),Y_v(t)\}. \tag{4.38}$$

Further, we just list the additional elements,

$$C_2.\ \{T(a)\},\quad a=0\ \text{ or }\ 1, \tag{4.39}$$

$$C_3.\ \{D(a)\}, \tag{4.40}$$

$$C_4.\ \{R(a)\}, \tag{4.41}$$

$$C_5.\ \{V(1)\}, \tag{4.42}$$

$$C_6.\ \{T(0),D(a)\}. \tag{4.43}$$

In all cases above, a does not depend on $n(a_{n+1}=a_n)$,

$$C_7.\ \{V(1),T(0)\}, \tag{4.44}$$

$$C_8.\ \{V(1),D(0)\}, \tag{4.45}$$

$$C_9.\ \{V(1),R(0)\}, \tag{4.46}$$

$$C_{10}.\ \{T(0),D(0),P(0)\}\sim \mathrm{sl}(2,\mathbb{R})_2, \tag{4.47}$$

$$C_{11}.\ \{T(0),D(0),V(1)\}, \tag{4.48}$$

$$C_{12}.\ \{T(0),D(0),P(0),V(1)\}. \tag{4.49}$$

4. Series D

L_C contains $\mathrm{sl}(2,\mathbb{R})_2$ and (possibly) further elements, namely,

$$D_1.\ \text{None}, \tag{4.50}$$

$$D_2.\ \{V(a_n)\}, \tag{4.51}$$

$$D_3.\ \{V(a_{1,n}),V(a_{2,n})\}, \tag{4.52}$$

$$D_4.\ \{Y_u(1),Y_v(1),Y_u(t),Y_v(t)\}, \tag{4.53}$$

$$D_5.\ \{Y_u(1),Y_v(1),Y_u(t),Y_v(t),V(1)\} \tag{4.54}$$

(D_4 coincides with C_{10} and D_5 with C_{12}).

V. THE INVARIANT INTERACTIONS

A. General procedure and interactions invariant under SL$(2,\mathbb{R})_1$

In this section we shall find all interaction functions, invariant under symmetry groups, containing $\mathrm{SL}(2,\mathbb{R})_1$. We make use of the subalgebra classification provided in Sec. IV.

We first establish the form of the interaction, invariant under $\mathrm{SL}(2,\mathbb{R})_1$ itself. To do this we set $\tau(t)=\lambda_n(t)=\mu_n(t)=0$ in the determining equations (2.4) and (2.5) and consider the equations obtained for $a_n=-d_n=1$, $b_n=c_n=0$, then $b_n=1$, $a_n=-d_n=c_n=0$, and, finally, $c_n=1$, $a_n=-d_n=b_n=0$. The general solution of the obtained system of six equations can be written in the following form:

$$F_n=u_{n+1}f_n+u_ng_n,\quad G_n=v_{n+1}f_n+v_ng_n, \tag{5.1}$$

where f_n and g_n are functions of four variables each, namely,

$$t,\ \ \xi_n=u_{n+1}v_{n-1}-u_{n-1}v_{n+1},\quad \xi_\alpha=u_\alpha v_n-u_nv_\alpha,\quad \alpha=n\pm 1. \tag{5.2}$$

Note that ξ_n, ξ_{n+1}, and ξ_{n-1} are as given in Eq. (5.2). They are not upshifts or downshifts of each other.

We shall proceed further by dimension of the symmetry algebra and by its structure. Thus, we can successively add $\mathrm{sl}(2,\mathbb{R})$ singlets of the form (4.12) or doublets of the form (4.18). We continue adding symmetry elements until the interaction is completely specified, i.e., it involves no further arbitrary functions. We then solve the ''inverse problem.'' That is, we substitute the functions F_n and G_n back into the determining equations and solve for the symmetries. This provides a verification of previous calculations. More important, this procedure will find the largest symmetry algebra allowed by any given interaction.

Obviously, all invariant interactions will have the form (5.1). It is the functions f_n and g_n that will be further refined, and their dependence on the variables ξ_k and t will be restricted.

For future convenience we write down two further forms of the $\mathrm{SL}(2,\mathbb{R})_1$ invariant interaction functions, equivalent to (5.1). The first is

$$F_n = u_{n+1}\frac{\xi_{n-1}}{\xi_n}h_n + u_n k_n, \quad G_n = v_{n+1}\frac{\xi_{n-1}}{\xi_n}h_n + v_n k_n, \tag{5.3}$$

where h_n and k_n are arbitrary functions of the variables (5.2). The second convenient form is

$$\begin{aligned} F_n &= (\lambda_{n-1}u_{n+1} - \lambda_{n+1}u_{n-1})\phi_n + (\lambda_{n+1}u_n - \lambda_n u_{n+1})\psi_n + \frac{\ddot{\lambda}_n}{\lambda_{n+1}}u_{n+1}, \\ G_n &= (\lambda_{n-1}v_{n+1} - \lambda_{n+1}v_{n-1})\phi_n + (\lambda_{n+1}v_n - \lambda_n v_{n+1})\psi_n + \frac{\ddot{\lambda}_n}{\lambda_{n+1}}v_{n+1}, \end{aligned} \tag{5.4}$$

where $\lambda_n(t)$ is some arbitrary function of n and t and ϕ_n and ψ_n depend in an unspecified manner on the variables (5.2).

B. Interactions invariant under four-dimensional symmetry groups

As was shown in Sec. IV, two types of four-dimensional symmetry algebras containing $\mathrm{sl}(2,\mathbb{R})_1$ can exist. Both are decomposable according to the pattern $4=3+1$. Here and below we shall always list the operators that we can add to $\mathrm{sl}(2,\mathbb{R})_1$.

1. $V(a_n) = a_n(u_n\partial_{u_n} + v_n\partial_{v_n})$

The invariant interactions will have the form (5.3), but h_n and k_n will depend on three variables only.

(i) $a_{n-1} + a_{n+1} \neq 0$. The variables are

$$t, \quad \eta_\alpha = (\xi_\alpha)^{a_{n-1}+a_{n+1}}(\xi_n)^{-a_n - a_\alpha}, \quad \alpha = n \pm 1. \tag{5.5}$$

(ii) $a_{n-1} + a_{n+1} = 0$. The variables are

$$t, \quad \xi_n, \quad \eta = (\xi_{n+1})^{a_{n+1}-a_n}(\xi_{n-1})^{a_{n+1}+a_n}. \tag{5.6}$$

2. $T(b_n) = \partial_t + b_n(u_n\partial_{u_n} + v_n\partial_{v_n})$

The invariant interaction will again have the form (5.3), however, in this case h_n and k_n are arbitrary functions of the three variables,

$$\zeta_n = \xi_n e^{-(b_{n-1}+b_{n+1})t}, \quad \zeta_\alpha = \xi_\alpha e^{-(b_n + b_\alpha)t}, \quad \alpha = n \pm 1. \tag{5.7}$$

We see that adding further singlets of the type $V(a_n)$ will restrict the variables in the functions h_n and k_n, not, however, the general form of Eq. (5.3).

C. Five-dimensional symmetry groups

From the results of Sec. IV, we know that three decomposable and one indecomposable symmetry algebras of dimension 5 can exist. Let us run through all four possibilities.

1. Decomposition 5=3+1+1

a. $V(a_{i,n}) = a_{i,n}(u_n\partial_{u_n} + v_n\partial_{v_n})$ $i=1,2$, $a_{2,n} \neq \lambda a_{1,n}$. The interaction is of the form (5.3). The functions h_n and k_n depend on two variables each, namely, time t and

$$\eta = (\xi_{n-1})^A(\xi_{n+1})^B(\xi_n)^C, \tag{5.8}$$

$$A=a_{1,n}(a_{2,n+1}+a_{2,n-1})+a_{1,n+1}(a_{2,n-1}-a_{2,n})-a_{1,n-1}(a_{2,n+1}+a_{2,n}),$$
$$B=-a_{1,n}(a_{2,n+1}+a_{2,n-1})+a_{1,n+1}(a_{2,n-1}+a_{2,n})-a_{1,n-1}(a_{2,n+1}-a_{2,n}), \quad (5.9)$$
$$C=a_{1,n}(a_{2,n+1}-a_{2,n-1})-a_{1,n+1}(a_{2,n-1}+a_{2,n})+a_{1,n-1}(a_{2,n+1}+a_{2,n}).$$

Note that the variable η always exists since the condition $A=B=C=0$ (and hence η=const) only occurs for $a_{1,n-1}a_{2,n}-a_{1,n}a_{2,n-1}=0$, which implies $a_{2,n}=\lambda a_{1,n}$, λ=const, and this is not allowed.

b. $V(a_n)=a_n(u_n\partial_{u_n}+v_n\partial_{v_n})$, $T(b_n)=\partial_t+b_n(u_n\partial_{u_n}+v_n\partial_{v_n})$. The invariant interaction is as in Eq. (5.3) with h_n and k_n functions of two variables each. Namely, the following.

(i) $a_{n+1}+a_{n-1}\neq 0$:

$$\rho_\alpha=(\zeta_\alpha)^{a_{n+1}+a_{n-1}}(\zeta_n)^{-a_\alpha-a_n}, \quad \alpha=n\pm 1, \quad (5.10)$$

with ζ_α, ζ_n as in Eq. (5.7).

(ii) $a_{n+1}+a_{n-1}=0$:

$$\rho_n=\zeta_n, \quad \sigma_n=(\zeta_{n-1})^{a_{n+1}+a_n}(\zeta_{n+1})^{a_{n+1}-a_n}. \quad (5.11)$$

2. Decomposition 5=3+2

a. $T(0)=\partial_t$, $D(b_n)=t\partial_t+(\frac{1}{2}+b_n)(u_n\partial_{u_n}+v_n\partial_{v_n})$. We impose $b_n\neq-\frac{1}{2}$; otherwise we have no invariant interaction. We must distinguish two subcases here.

(1) $b_{n+1}+b_{n-1}+1\neq 0$. The interaction as in Eq. (5.3), with

$$h_n=(\xi_n)^{-2/(b_{n+1}+b_{n-1}+1)}p_n, \quad k_n=(\xi_n)^{-2/(b_{n+1}+b_{n-1}+1)}q_n, \quad (5.12)$$

where p_n and q_n depend on two variables, namely,

$$\chi_\alpha=(\xi_\alpha)^{b_{n+1}+b_{n-1}+1}(\xi_n)^{-b_n-b_\alpha-1}, \quad \alpha=n\pm 1. \quad (5.13)$$

(2) $b_{n+1}+b_{n-1}+1=0$, $b_{n+1}+b_n+1\neq 0$:

$$h_n=(\xi_{n+1})^{-2/(b_{n+1}+b_n+1)}p_n, \quad k_n=(\xi_{n+1})^{-2/(b_{n+1}+b_n+1)}q_n, \quad (5.14)$$

where p_n and q_n depend on

$$\chi_n=(\xi_{n-1})^{b_{n+1}+b_n+1}(\xi_{n+1})^{-b_{n-1}-b_n-1}, \quad \xi_n. \quad (5.15)$$

Note that for $b_{n+1}+b_{n-1}+1=0$, $b_{n+1}+b_n+1=0$, we have $b_n=-\frac{1}{2}$, and there is no invariant interaction.

3. Indecomposable Lie algebra

$$Y_u(\lambda_n)=\lambda_n(t)\partial_{u_n}, \quad Y_v(\lambda_n)=\lambda_n(t)\partial_{v_n}. \quad (5.16)$$

The invariant interaction is as in Eq. (5.4), but the functions ϕ_n and ψ_n depend on only two variables, namely,

$$t, \quad \omega=\lambda_{n-1}\xi_{n+1}-\lambda_n\xi_n-\lambda_{n+1}\xi_{n-1}. \quad (5.17)$$

D. Six-dimensional symmetry groups

1. Decomposition 6=3+1+1+1

a. $V(a_{i,n})=a_{i,n}(u_n\partial_{u_n}+v_n\partial_{v_n})$, $i=1,2,3$. The invariant interaction is as in Eq. (5.3), but h_n and k_n are functions of t only. We see that the coefficients $a_{i,n}$ do not figure in the interaction.

Hence, we can add an arbitrary number of vector fields $V(a_{i,n})$, $i\in\mathbb{Z}$ to the symmetry algebra. In other words, the symmetry algebra for the interaction (5.3) with h_n and k_n depending on t alone is infinite dimensional.

b. $V(a_{i,n})=a_{i,n}(u_n\partial_{u_n}+v_n\partial_{v_n})$, $i=1,2$, $T(b_n)=\partial_t+b_n(u_n\partial_{u_n}+v_n\partial_{v_n})$. The invariant interaction is as in Eq. (5.3), but h_n and k_n depend on one variable only, namely,

$$\omega=\eta e^{-2t|M|},\quad M=\begin{pmatrix} b_{n-1} & b_n & b_{n+1} \\ a_{1,n-1} & a_{1,n} & a_{1,n+1} \\ a_{2,n-1} & a_{2,n} & a_{2,n+1}\end{pmatrix}, \tag{5.18}$$

with η as in Eq. (5.8).

2. Decomposition 6=3+2+1

a. $V(a_n)=a_n(u_n\partial_{u_n}+v_n\partial_{v_n})$, $T(0)=\partial_t$, $D(c_n)=t\partial_t+(\frac{1}{2}+c_n)(u_n\partial_{u_n}+v_n\partial_{v_n})$. We start from Eq. (5.3). The presence of $T(0)=\partial_t$ implies that h_n and k_n do not depend on t. We first notice that if we have

$$\gamma_n=c_n+\tfrac{1}{2}=0\quad\text{or}\quad \gamma_n=c_n+\tfrac{1}{2}=\lambda a_n, \tag{5.19}$$

then we must have $h_n=k_n=0$ (no invariant interaction). In all other cases, invariance under $V(a_n)$ and $D(c_n)$ implies

$$\begin{gathered} h_n=(\xi_n)^{\mu}(\xi_{n+1})^{\nu}(\xi_{n-1})^{\rho}p_n(\omega),\quad k_n=(\xi_n)^{\mu}(\xi_{n+1})^{\nu}(\xi_{n-1})^{\rho}q_n(\omega),\\ \omega=(\xi_{n-1})^{A}(\xi_{n+1})^{B}(\xi_n)^{C}, \end{gathered}\tag{5.20}$$

with A, B, and C as in Eq. (5.9), with the substitutions

$$a_{1,n}\to c_n+\tfrac{1}{2},\quad a_{2,n}\to a_n.$$

The constants μ, ν, and ρ in Eq. (5.20) satisfy

$$\begin{gathered}(a_{n+1}+a_{n-1})\mu+(a_{n+1}+a_n)\nu+(a_{n-1}+a_n)\rho=0,\\ (\gamma_{n+1}+\gamma_{n-1})\mu+(\gamma_{n+1}+\gamma_n)\nu+(\gamma_{n-1}+\gamma_n)\rho=-2.\end{gathered}\tag{5.21}$$

Thus, for $C\neq 0$ we can put

$$\mu=0,\quad \nu=2\,\frac{a_n+a_{n-1}}{C},\quad \rho=-2\,\frac{a_n+a_{n+1}}{C}.$$

For $C=0$, $A\neq 0$,

$$\mu=2\,\frac{a_n+a_{n+1}}{A},\quad \nu=-2\,\frac{a_{n+1}+a_{n-1}}{A},\quad \rho=0.$$

For $C=A=0$, $B\neq 0$,

$$\mu=-2\,\frac{a_{n-1}+a_n}{B},\quad \nu=0,\quad \rho=2\,\frac{a_{n+1}+a_{n-1}}{B}.$$

The case $A=B=C=0$ corresponds to Eq. (5.19) and hence to the absence of an invariant interaction.

3. Decomposition 6=3+3

a. $\mathrm{sl}(2,\mathbb{R})_1 \oplus \mathrm{sl}(2,\mathbb{R})_2$. The algebra $\mathrm{sl}(2,\mathbb{R})_2$ is as in Eq. (4.8) and the invariant interaction is

$$F_n = \frac{1}{(\xi_n)^2}\left[u_{n+1}\frac{\xi_{n-1}}{\xi_n}p_n(\chi_{n+1},\chi_{n-1}) + u_n q_n(\chi_{n+1},\chi_{n-1})\right],$$

$$G_n = \frac{1}{(\xi_n)^2}\left[v_{n+1}\frac{\xi_{n-1}}{\xi_n}p_n(\chi_{n+1},\chi_{n-1}) + v_n q_n(\chi_{n+1},\chi_{n-1})\right], \tag{5.22}$$

$$\chi_{n+1} = \frac{\xi_{n+1}}{\xi_n}, \quad \chi_{n-1} = \frac{\xi_{n-1}}{\xi_n}.$$

4. Decomposition 6=5+1

a. $\mathrm{saff}(2) \oplus A_1$. We have

$$Y_u(e^{a_n t}) = e^{a_n t}\partial_{u_n}, \quad Y_v(e^{a_n t}) = e^{a_n t}\partial_{v_n}, \quad T(a_n) = \partial_t + a_n(u_n\partial_{u_n} + v_n\partial_{v_n}).$$

The invariant interaction will be as in Eq. (5.4) with $\lambda_n = e^{a_n t}$. The functions ϕ_n and ψ_n will satisfy

$$\phi_n = e^{(a_n - a_{n-1} - a_{n+1})t}K_n(\omega), \quad \psi_n = e^{-a_{n+1}t}L_n(\omega),$$
$$\omega = e^{-(a_n + a_{n+1})t}\xi_{n+1} - e^{-(a_{n+1}+a_{n-1})t}\xi_n - e^{-(a_{n-1}+a_n)t}\xi_{n-1}. \tag{5.23}$$

5. Indecomposable symmetry algebras

It was shown in Sec. IV that two inequivalent gaff(2) symmetry algebras exist.

a. $\mathrm{gaff}(2,\mathbb{R})_1$.

$$Y_u(\lambda_n) = \lambda_n(t)\partial_{u_n}, \quad Y_v(\lambda_n) = \lambda_n(t)\partial_{v_n}, \quad V(1) = u_n\partial_{u_n} + v_n\partial_{v_n}.$$

The interaction is as in Eq. (5.4), however, ϕ_n and ψ_n depend only on t. This means that the equations are linear and, moreover, the equations (1.1) for u_n and v_n are decoupled.

b. $\mathrm{gaff}(2,\mathbb{R})_2$. The algebra is as in Eq. (4.22) [or (4.33)], the interaction as in Eq. (5.4) with $\lambda_n(t) = e^{(a_n - 1)t}$. The functions ϕ_n and ψ_n satisfy

$$\phi_n = e^{-(a_{n+1} + a_{n-1} - a_n - 1)t}K_n(\omega), \quad \psi_n = e^{(-a_{n+1}+1)t}L_n(\omega), \tag{5.24}$$

with ω as in Eq. (5.23).

E. Seven-dimensional symmetry groups

1. Decomposition 7=3+1+1+1+1

We exclude the case

$$V(a_{i,n}) = a_{i,n}(u_n\partial_{u_n} + v_n\partial_{v_n}), \quad i = 1,...,4,$$

since the only invariant interaction is (5.3) with h_n and k_n functions of t. We already know that the symmetry algebra is infinite dimensional.

a. $V(a_{i,n}) = a_{i,n}(u_n\partial_{u_n} + v_n\partial_{v_n})$, $i = 1,2,3$, $T(b_n) = \partial_t + b_n(u_n\partial_{u_n} + v_n\partial_{v_n})$. The interaction is as in Eq. (5.3) with h_n and k_n constants (depending on n). The algebra is actually infinite dimensional: we can take any number of operators $V(a_{i,n})$.

2. Decomposition 7=3+2+1+1

a. $V(a_{i,n})=a_{i,n}(u_n\partial_{u_n}+v_n\partial_{v_n})$, $i=1,2$, $T(0)=\partial_t$, $D(c_n)=t\partial_t+(\frac{1}{2}+c_n)(u_n\partial_{u_n}+v_n\partial_{v_n})$. We put $\gamma_n=c_n+\frac{1}{2}$. An invariant interaction exists if and only if we have

$$\Delta=\det\begin{pmatrix}\gamma_n & \gamma_{n+1} & \gamma_{n-1}\\ a_{1,n} & a_{1,n+1} & a_{1,n-1}\\ a_{2,n} & a_{2,n+1} & a_{2,n-1}\end{pmatrix}\neq 0. \tag{5.25}$$

The invariant interaction is that of Eq. (5.3), with

$$h_n=\eta^k p_n, \quad k_n=\eta^k q_n, \quad k=-\frac{2}{\Delta}. \tag{5.26}$$

The variable η is as in Eq. (5.8); p_n and q_n are constants.

3. Decomposition 7=3+3+1

a. $\mathrm{sl}(2,\mathbb{R})_1\oplus\mathrm{sl}(2,\mathbb{R})_2\oplus A_1$. We have $A_1=\{V(a_n)\}$. The invariant interaction can be obtained from Eq. (5.22). The additional invariance implied by the presence of $V(a_n)$ restricts p_n and q_n to

$$\begin{aligned} p_n&=\left(\frac{\xi_{n+1}}{\xi_n}\right)^{2(a_{n+1}+a_{n-1})/(a_n-a_{n-1})} r_n(\omega),\\ q_n&=\left(\frac{\xi_{n+1}}{\xi_n}\right)^{2(a_{n+1}+a_{n-1})/(a_n-a_{n-1})} s_n(\omega),\\ \omega&=(\xi_{n+1})^{a_{n+1}-a_n}(\xi_{n-1})^{a_n-a_{n-1}}(\xi_n)^{a_{n-1}-a_{n+1}}, \end{aligned} \tag{5.27}$$

and we must impose $a_n\neq a_{n-1}$ (otherwise we have $F_n=G_n=0$).

4. Decomposition 7=6+1

The algebra $\mathrm{gaff}(2,\mathbb{R})_1$ does not allow any nonlinear interactions. Let us consider $\mathrm{gaff}(2,\mathbb{R})_2$ of Eq. (4.22).

a. $\mathrm{gaff}(2,\mathbb{R})_2\oplus\{U=u_n\partial_{u_n}+v_n\partial_{v_n}\}$. The interaction is as in Eq. (5.4), with ϕ_n and ψ_n as in Eq. (5.24). Invariance under the dilations corresponding to U implies that ϕ_n and ψ_n do not depend on ω. Hence, the interaction is linear and decoupled.

5. Indecomposable Lie algebras

a. $Y_u(\lambda_n)=\lambda_n(t)\partial_{u_n}$, $Y_v(\lambda_n)=\lambda_n(t)\partial_{v_n}$, $Y_u(\mu_n)=\mu_n(t)\partial_{u_n}$, $Y_v(\mu_n)=\mu_n(t)\partial_{v_n}$. We start from Eq. (5.4) with ϕ_n and ψ_n functions of t and ω as in Eq. (5.17). If ϕ_n and ψ_n do not depend on ω, the interaction is already linear and decoupled. Hence, ω must be invariant under the transformations corresponding to $Y_u(\mu_n)$ and $Y_v(\mu_n)$. This implies that λ_n and μ_n are independent of n. Further, invariance implies

$$\frac{\ddot{\lambda}_n}{\lambda_n}=\frac{\ddot{\mu}_n}{\mu_n}=\tilde{k}, \tag{5.28}$$

with $\tilde{k}=\text{const}$. Equation (5.28) allows solutions,

$$\begin{pmatrix}\lambda_n\\ \mu_n\end{pmatrix}=\begin{pmatrix}\sin kt\\ \cos kt\end{pmatrix}, \quad \begin{pmatrix}\sinh kt\\ \cosh kt\end{pmatrix}, \quad \begin{pmatrix}1\\ t\end{pmatrix}. \tag{5.29}$$

These solutions are all equivalent under allowed transformations. We choose $\lambda_n=1$, $\mu_n=t$, i.e.,

$$Y_u(1)=\partial_{u_n},\quad Y_v(1)=\partial_{v_n},\quad Y_u(t)=t\partial_{u_n},\quad Y_v(t)=t\partial_{v_n}. \tag{5.30}$$

The invariant interaction is

$$\begin{aligned} F_n&=(u_{n+1}-u_{n-1})\phi_n(\omega,t)+(u_n-u_{n+1})\psi_n(\omega,t),\\ G_n&=(v_{n+1}-v_{n-1})\phi_n(\omega,t)+(v_n-v_{n+1})\psi_n(\omega,t), \end{aligned} \tag{5.31}$$

with

$$\omega=\xi_{n+1}-\xi_{n-1}-\xi_n. \tag{5.32}$$

b. $Y_u(1)=\partial_{u_n}$, $Y_v(1)=\partial_{v_n}$, $T(0)=\partial_t$, $D(b)=t\partial_t+(\frac{1}{2}+b)(u_n\partial_{u_n}+v_n\partial_{v_n})$, $b\neq-\frac{1}{2}$, $b=\text{const}$. The invariant interaction is as in Eq. (5.31), with

$$\phi_n=k_n\omega^{-2/(2b+1)},\quad \psi_n=p_n\omega^{-2/(2b+1)}, \tag{5.33}$$

with k_n and p_n constants, ω as in Eq. (5.32). For $b=-\frac{1}{2}$ there is no invariant interaction. For $b\neq-\frac{1}{2}$ the symmetry algebra is actually larger and includes $Y_u(t)=t\partial_{u_n}$ and $Y_v(t)=t\partial_{v_n}$.

F. Symmetry groups of dimensions 8, 9, and 10

By now, all invariant interactions have been specified up to arbitrary constants (depending on n), except those involving symmetry algebras containing the subalgebra $\mathrm{sl}(2,\mathbb{R})_1\oplus\mathrm{sl}(2,\mathbb{R})_2$, or the subalgebra $\{Y_u(1),Y_v(1),Y_u(t),Y_v(t)\}$ of Eq. (5.30). Let us consider the remaining nonlinear interactions.

1. $\mathbf{sl(2,\mathbb{R})_1\oplus sl(2,\mathbb{R})_2\oplus\{V(a_{1,n})\}\oplus\{V(a_{2,n})\}}$

The invariant interaction is obtained from Eq. (5.27) by specifying $r_n(\omega)$ and $s_n(\omega)$ to be specific powers of ω. The result is

$$\begin{aligned} F_n&=\xi_n^{-2}\left[u_{n+1}\frac{\xi_{n-1}}{\xi_n}p_n+u_nq_n\right](\xi_{n-1})^{-2A/D}(\xi_{n+1})^{-2B/D}(\xi_n)^{2[(A+B)/D]},\\ G_n&=\xi_n^{-2}\left[v_{n+1}\frac{\xi_{n-1}}{\xi_n}p_n+v_nq_n\right](\xi_{n-1})^{-2A/D}(\xi_{n+1})^{-2B/D}(\xi_n)^{2[(A+B)/D]}. \end{aligned} \tag{5.34}$$

Here p_n and q_n are constants, A and B are as in Eq. (5.9), and

$$D=a_{1,n}(a_{2,n+1}-a_{2,n-1})+a_{1,n+1}(a_{2,n-1}-a_{2,n})+a_{1,n-1}(a_{2,n}-a_{2,n+1}). \tag{5.35}$$

We assume $D\neq0$; otherwise there is no invariant interaction. In particular, we have $a_{1,n}\neq a_{1,n+1}$, $a_{2,n}\neq a_{2,n+1}$.

2. Algebras containing $(Y_u(1),Y_v(1),Y_u(t),Y_v(t))$ *of (5.30) plus one additional operator* Z

The interaction is as in Eq. (5.31) with a restriction on ϕ_n and ψ_n.

(i) $Z=T(a)=\partial_t+a(u_n\partial_{u_n}+v_n\partial_{v_n})$, $a\equiv a_n=a_{n+1}$,

$$\phi_n=\phi_n(\eta),\quad \psi_n=\psi_n(\eta),\quad \eta=\omega e^{-2at}. \tag{5.36}$$

(ii) $Z=D(a)=t\partial_t+(\frac{1}{2}+a)(u_n\partial_{u_n}+v_n\partial_{v_n})$, $a\equiv a_n=a_{n+1}$,

$$\phi_n=\frac{1}{t^2}r_n(\eta),\quad \psi_n=\frac{1}{t^2}s_n(\eta),\quad \eta=\omega t^{-(2a+1)}. \tag{5.37}$$

(iii) $Z=R(b)=(t^2+1)\partial_t+(t+b)(u_n\partial_{u_n}+v_n\partial_{v_n}),\ b\equiv b_n=b_{n+1},$

$$\phi_n=\frac{1}{(t^2+1)^2}r_n(\eta),\quad \psi_n=\frac{1}{(t^2+1)^2}s_n(\eta),$$
$$\eta=\frac{\omega}{1+t^2}e^{-2b\arctan t}, \tag{5.38}$$

with ω as in Eq. (5.32) in all cases.

(iv) $Z=V(1)$. Then ϕ_n and ψ_n depend only on t and the interaction is linear.

We can add two operators to those of Eq. (5.30)

$$T(0)=\partial_t,\quad D(b)=t\partial_t+(\tfrac{1}{2}+b)(u_n\partial_{u_n}+v_n\partial_{v_n}).$$

The invariant interaction coincides with that of Eq. (5.33).

Finally, the interaction (5.31) is invariant under a ten-dimensional symmetry algebra of the form

$$(\mathrm{sl}(2,\mathbb{R})_1\oplus\mathrm{sl}(2,\mathbb{R})_2)\rhd\{Y_u(1),Y_v(1),Y_u(t),Y_v(t)\},$$

for

$$\phi_n=k_n\omega^{-2},\quad \psi_n=p_n\omega^{-2}, \tag{5.39}$$

i.e., $b=0$ in Eq. (5.33).

VI. SUMMARY AND CONCLUSIONS

Let us first sum up the results on invariant interactions and the corresponding symmetry algebras. We shall follow the summary of possible symmetry algebras outlined in Sec. IV E. The results are presented in the following tables.

Table I. The Series A of symmetry algebras. The algebra L_C of Eq. (4.27) consists entirely of $\mathrm{sl}(2,\mathbb{R})_1$ singlets. In the first column of Table I we list the symmetry algebras. The number in brackets [e.g., $A_1(3)$] denotes the dimension of the symmetry algebra. The notation for basis elements in column 2 are as in Eqs. (4.13)–(4.18). Note that if the functions h_n and k_n in the interaction (5.3) depend only on t or are constants, then the symmetry algebra is infinite dimensional, although the interaction is nonlinear.

The case $A_3(7)$ corresponds to an algebra L with $\dim L=7$ and the interaction is completely specified [see (5.3), (5.25)–(5.26)]. In other cases the functions h_n and k_n depend on one, two, or three variables involving u_k and v_k.

Table II. The Series B of symmetry algebras. The symmetry algebras are either five or six dimensional. The interactions are as in Eq. (5.4) and involve two arbitrary functions, ϕ_n and ψ_n. A B-type symmetry allows ϕ_n and ψ_n to depend on just one variable involving u_k and v_k. Any extension of the B-type algebras will restrict $\lambda_n(t)$ to be $\lambda_n=1$ and will involve a further pair with $\lambda_n=t$. This takes us into the series C of symmetry algebras.

The algebras B_2, B_6, and B_7 of Eqs. (4.32), (4.36), and (4.37) lead to linear interactions. Any interaction invariant with respect to B_5 will be invariant under a larger group, corresponding to a Lie algebra in the series C. We do not include linear interactions in the tables and we list interactions together with their *maximal* symmetry algebras.

Table III. The Series C of symmetry algebras. The interaction will be as in Eq. (5.31), involving a variable ω as in Eq. (5.32). The algebras $C_5(8),C_7(9),C_8(9),C_9(9),C_{11}(10),C_{12}(11)$, absent in the table, lead to a linear interaction.

TABLE I. Series A of symmetry algebras. The interaction has the form (5.3).

No.	L_C	Restrictions on h_n and k_n	Variables and comments
$A_1(3)$	$\cdots$	$\cdots$	$t,\xi_{n+1},\xi_{n-1},\xi_n$ (5.2)
$A_1(4)$	$V(a_n)$	$\cdots$	$\begin{cases} t,\eta_{n+1},\eta_{n-1} \text{ (5.5)} \\ t,\xi_n,\eta \text{ (5.6)} \end{cases}$
$A_1(5)$	$V(a_{1,n}),V(a_{2,n})$	$\cdots$	$t,\ \eta$ (5.8)
$A_1(\infty)$	$V(a_{i,n}),i\in\mathbb{Z}^>$	$\cdots$	t
$A_2(4)$	$T(b_n)$	$\cdots$	$\zeta_{n+1},\zeta_{n-1},\zeta_n$ (5.7)
$A_2(5)$	$T(b_n),V(a_n)$	$\cdots$	$\begin{cases} \rho_{n-1},\rho_{n+1} \text{ (5.10)} \\ \rho_n,\sigma_n \text{ (5.11)} \end{cases}$
$A_2(6)$	$T(b_n),V(a_{1,n}),V(a_{2,n})$	$\cdots$	η (5.18)
$A_2(\infty)$	$T(b_n),V(a_{k,n}),k\in\mathbb{Z}^>$	h_n,k_n constants	None
$A_3(5)$	$T(0),D(b_n)$	(5.12) or (5.14)	(5.13) or (5.15)
$A_3(6)$	$T(0),D(c_n),V(a_n)$	(5.20)	ω (5.20)
$A_3(7)$	$T(0),D(c_n),V(a_{1,n})V(a_{2,n})$	(5.26)	None

For $C_6(9)$ and $C_{10}(10)$ the interactions are specified up to constants (that can depend on n). In all other cases, the arbitrary functions depend on one variable, involving u_k and v_k.

Table IV. The Series D of symmetry algebras. There are three such algebras of dimension 6, 7, and 8, respectively. They all lead to nontrivial invariant interactions of the form (5.22). For $D_3(8)$, the interaction is completely specified. We do not list $D_4(10)$ in Table IV since it coincides with $C_{10}(10)$ of Table III. The algebra $D_5(11)$ corresponds to a linear interaction.

For each interaction we have verified that the given symmetry algebra is the maximal one.

A few words about the interpretation of the invariant interactions. From Eq. (5.1) and the variables (5.2) we see that invariance under $\mathrm{sl}(2,\mathbb{R})_1$ imposes very strong restrictions.

(1) In particular, if the interaction is linear and $\mathrm{sl}(2,\mathbb{R})_1$ invariant, we must have

$$F_n=\sum_{k=n-1}^{n+1} A_k(t)u_k, \quad G_n=\sum_{k=n-1}^{n+1} A_k(t)v_k, \tag{6.1}$$

i.e., the equations (1.1) for u_k and v_k decouple (into identical equations for u_n and v_n separately).

(2) If the interaction terms F_n and G_n in Eq. (5.1) are nonlinear, they always involve many-body forces. That is, they cannot be written as sums of terms of the type $h_n(u_n,v_n)$ or $h_n(u_n,v_{n+1})$, etc. Indeed, each invariant variable $\xi_n,\xi_{n+1},\xi_{n-1}$ itself involves four of the original variables u_i,v_i simultaneously. This many-body character becomes more pronounced when the invariance algebra is larger.

(3) The operators $V(a_n)$ correspond to site-depending dilations,

TABLE II. Series B of symmetry algebras. The algebra includes one pair $Y_u(\lambda_n)$, $Y_v(\lambda_n)$. The interaction has the form (5.4).

No.	Restrictions on λ_n, additional Elements of L_C	Restrictions on ϕ_n and ψ_n	Variables and comments
$B_1(5)$	$\cdots$	$\cdots$	$t,\ \omega$ as in (5.17)
$B_4(6)$	$\lambda_n=e^{a_n t},T(a_n)$	(5.23)	ω (5.23)
$B_3(6)$	$\lambda_n=e^{(a_n-1)t},T(a_n)$	(5.24)	ω (5.23)

TABLE III. Series C symmetry algebras. The algebras contain $\mathrm{sl}(2,\mathbb{R})_1, Y_u(1), Y_v(1), Y_u(t), Y_v(t)$, and possibly additional elements. The interaction is as in Eq. (5.31).

No.	Additional elements	Conditions on ϕ_n and ψ_n	Variables
$C_1(7)$	–	$\cdots$	ω, t (5.32)
$C_2(8)$	$T(a)$	$\cdots$	$\eta=\omega e^{-2at}$
$C_3(8)$	$D(a)$	$\phi_n=t^{-2}r_n(\eta),\ \psi_n=t^{-2}s_n(\eta)$	$\eta=\omega t^{-(2a+1)}$
$C_4(8)$	$R(b)$	$\phi_n=(t^2+1)^{-2}r_n(\eta)$, $\psi_n=(t^2+1)^{-2}s_n(\eta)$	$\eta=\omega(t^2+1)^{-1}$ $e^{-2b\arctan t}$
$C_6(9)$	$T(0),D(a)$	$\phi_n=k_n\omega^{-2/(2a+1)},\ \psi_n=p_n\omega^{-2(2a+1)}$ $k_n,\ p_n$ constants, $2a+1\neq 0$	None
$C_{10}(10)$	$T(0),D(0),P(0)$	$\phi_n=k_n\omega^{-2},\ \psi_n=p_n\omega^{-2}$	None

$$\tilde{u}_n=e^{\epsilon a_n}u_n,\quad \tilde{v}_n=e^{\epsilon a_n}v_n. \tag{6.2}$$

Invariance under two such one-dimensional symmetry groups, generated by $\{V(a_{1,n}),V(a_{2,n})\}$, where $a_{1,n}$ and $a_{2,n}$ are two linearly independent functions of n, introduces the symmetry variable

$$\omega_D\equiv(\xi_{n-1})^A(\xi_{n+1})^B(\xi_n)^C, \tag{6.3}$$

as in Eq. (5.8). Here all six variables are coupled together.

(4) The pair of operators $Y_u(\lambda_n),Y_v(\lambda_n)$ induces site-dependent (and time-dependent) shifts of the dependent variables,

$$\tilde{u}_n=u_n+\epsilon\lambda_n(t),\quad \tilde{v}_n=v_n+\epsilon\lambda_n(t). \tag{6.4}$$

The corresponding invariant variable again involves all six variables [see Eq. (5.17)],

$$\omega_T\equiv\lambda_{n-1}\xi_{n+1}-\lambda_n\xi_n-\lambda_{n+1}\xi_{n-1}. \tag{6.5}$$

A special case of the variable ω_T is obtained setting $\lambda_n=\lambda_{n-1}=\lambda_{n+1}=1$. This is the case of Eq. (5.32), where

$$\omega=\omega_S=\xi_{n+1}-\xi_n-\xi_{n-1} \tag{6.6}$$

is invariant with respect to two such translations:

$$\tilde{u}_n=u_n+\epsilon_1+\epsilon_2 t,\quad \tilde{v}_n=v_n+\epsilon_1+\epsilon_2 t \tag{6.7}$$

(ϵ_1 and ϵ_2 are group parameters and hence constants).

A continuation of this study is in progress. It involves several aspects.

The first is a study of the integrability properties of the equations that are completely specified by their symmetries. These are, first of all, those with infinite-dimensional symmetry groups, namely

$$\ddot{u}_n=u_{n+1}\frac{\xi_{n-1}}{\xi_n}h_n+u_nk_n,\quad \ddot{v}_n=v_{n+1}\frac{\xi_{n-1}}{\xi_n}h_n+v_nk_n, \tag{6.8}$$

TABLE IV. Series D of symmetry algebras. The algebra contains $\mathrm{sl}(2,\mathbb{R})_1\oplus\mathrm{sl}(2,\mathbb{R})_2$. The interaction has the form (5.22).

No.	Additional elements in L_C	Conditions on p_n and q_n	Variables
$D_1(6)$	$\cdots$	$\cdots$	$\chi_{n+1},\ \chi_{n-1}$ as in (5.22)
$D_2(7)$	$V(a_n)$	(5.27)	η as in (5.27)
$D_3(8)$	$V(a_{1,n}),\ V(a_{2,n})$	(5.34)	$\cdots$

with h_n and k_n functions of t or constants [see $A_1(\infty)$ and $A_2(\infty)$ in Table I].

Completely specified equations with finite-dimensional symmetry algebras L are the following ones.

(i)

$$\ddot{u}_n=\left(u_{n+1}\frac{\xi_{n-1}}{\xi_n}p_n+u_nq_n\right)\omega_D^{-2/\Delta}, \quad \ddot{v}_n=\left(v_{n+1}\frac{\xi_{n-1}}{\xi_n}p_n+v_nq_n\right)\omega_D^{-2/\Delta}, \tag{6.9}$$

with ω_D as in Eq. (6.3), Δ as in Eq. (5.25). This is case $A_3(7)$ of Table I.

(ii)

$$\ddot{u}_n=[(u_{n+1}-u_{n-1})p_n+(u_n-u_{n+1})q_n]\omega_S^{-2/(2a+1)},$$

$$\ddot{v}_n=[(v_{n+1}-v_{n-1})p_n+(v_n-v_{n+1})q_n]\omega_S^{-2/(2a+1)}, \tag{6.10}$$

with ω_S as in Eq. (6.6), $p_n, q_n, a\neq -\frac{1}{2}$ const. This is case $C_6(9)$ of Table III.

(iii) For $a=0$, Eq. (6.10) is invariant under a ten-dimensional symmetry algebra, namely $C_{10}(10)$ of Table III.

(iv)

$$\ddot{u}_n=(\xi_{n-1})^{-2A/k}(\xi_{n+1})^{-2B/D}(\xi_n)^{[2(A+B-D)/D]}\left[u_{n+1}\frac{\xi_{n-1}}{\xi_n}p_n+u_nq_n\right],$$

$$\ddot{v}_n=(\xi_{n-1})^{-2A/D}(\xi_{n+1})^{-2B/D}(\xi_n)^{[2(A+B-D)/D]}\left[v_{n+1}\frac{\xi_{n-1}}{\xi_n}p_n+v_nq_n\right],$$

with p_n and q_n depending only on n. The constants A and B are given in Eq. (5.9), D in Eq. (5.35).

A further task is to complete the classification, that is, to treat the cases of other sl$(2,\mathbb{R})$ algebras and also of solvable symmetry algebras.

ACKNOWLEDGMENTS

The authors thank D. Levi and M. A. Rodriguez for helpful discussions. The research of S.L. and P.W. was partly supported by the NSERC of Canada and FCAR du Québec. S. L. would like to thank the Departamento de Física Teórica II de la Universidad Complutense for their hospitality during his stay in Madrid. D.G.U.'s work was partly supported by DGES Grant No. PB95-0401. He would like to express his gratitude to the Centre de Recherches Mathématiques for their kind hospitality.

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