PHYSICAL REVIEW LETTERS

VOLUME 50 | 24 JANUARY 1983 | NUMBER 4

New Bäcklund Transformations and Superposition Principle for Gravitational Fields with Symmetries

F. J. Chinea
Departamento de Métodos Matemáticos de la Física, Facultad de Ciencias Físicas, Universidad de Madrid, Madrid-3, Spain
(Received 29 November 1982)

Vector Bäcklund transformations which relate solutions of the vacuum Einstein equations having two commuting Killing fields are introduced. Such transformations generalize those found by Pohlmeyer in connection with the nonlinear σ model. A simple algebraic superposition principle, which permits the combination of Bäcklund transforms in order to get new solutions, is given. The superposition preserves the asymptotic flatness condition, and the whole scheme is manifestly O(2,1) invariant.

PACS numbers: 04.20.Cv, 04.20.Jb, 11.10.Lm, 02.30.+g

In recent years, several methods have been proposed in order to construct physically meaningful exact solutions of the Einstein equations of general relativity. Much work has been done in the special case of vacuum gravitational fields possessing two Killing vector fields which commute[1-7]; this includes, in particular, stationary systems with axial symmetry. A technique to produce solutions with an appropriate asymptotic behavior, and a prescribed mass and angular momentum multipole structure, was found in Ref. 7. Alternative approaches to the subject include the application of the inverse scattering technique to this case,[8] and the use of Bäcklund transformations.[9] It is the purpose of this Letter to introduce a new type of vector Bäcklund transformation, which admits a rather simple, closed expression, and an associated superposition principle. The latter provides an easy way of algebraically combining two Bäcklund transforms of a given solution in order to get a new solution. It is found that, in the case of stationary axisymmetric systems, such a superposition preserves the asymptotic flatness condition automatically. The computations involved in getting a new solution through this method are quite simple, and the whole scheme is manifestly O(2,1) invariant.

If the gravitational field admits a two-parameter Abelian group of isometries, and if one assumes the existence of two-surfaces orthogonal to the group orbits ("orthogonal transitivity"), then the Einstein equations $R_{\mu\nu}=0$ separate into two sets: One of them is a system of integrable equations for a single scalar function, and the remaining set may be expressed as a single partial differential equation for a three-vector q. In case the two Killing fields are spacelike (e.g., cylindrical symmetry), the resulting equation is[6]

$$q_{uv} + (\tau_u/2\tau)q_v + (\tau_v/2\tau)q_u = (q_u\cdot q_v)q, \qquad q^2=-1 \tag{1}$$

with

$$\tau(u,v) = U(u) + V(v), \tag{2}$$

where U and V are arbitrary. [Here and in the sequel, subscripts denote ordinary partial derivatives; the scalar product of two vectors $q=(q^1,$

$q^2, q^3)$ and $p=(p^1,p^2,p^3)$ is $p\cdot q=(p^1q^1+p^2q^2-p^3q^3)$; $u=\frac{1}{2}(\rho+t)$, and $v=\frac{1}{2}(\rho-t)$.] It should be stressed that the components of the vector q appearing in Eq. (1) are *algebraic* combinations of the coefficients of the space-time metric.

The constraint $q^2=-1$ in Eq. (1) may be solved by means of a single complex function f, parametrizing q as

$$q=(f+f^*)^{-1}(i(f-f^*),1-ff^*,1+ff^*). \tag{3}$$

With use of (3), Eq. (1) reduces to

$$f_{uv}+\frac{\tau_u}{2\tau}f_v+\frac{\tau_v}{2\tau}f_u=\frac{2f_uf_v}{f+f^*}. \tag{4}$$

Equation (4) is formally identical to the Ernst equation[1] in the hyperbolic case. It should be noticed, however, that f is *not* the Ernst potential ϵ of the system. The relation between f and ϵ is easily worked out by writing the metric in the Lewis-Papapetrou form[10]

$$ds^2=h(dz+\omega d\varphi)^2+S^2h^{-1}d\varphi^2+e^{\nu}(d\rho^2-dt^2),$$

where h, ω, and ν are functions of ρ and t, and S may be taken as $S=\rho=u+v$. The functions f and ϵ are then

$$f=Sh^{-1}+i\omega,\quad \epsilon=h+i\psi,$$

where ψ is related to ω by

$$\psi_u=S^{-1}h^2\omega_u,\quad \psi_v=-S^{-1}h^2\omega_v.$$

For stationary axisymmetric systems, the relevant equation [analogous to Eq. (1)] may be shown to be[6]

$$q_{\zeta\zeta^*}+\frac{\tau_\zeta}{2\tau}q_{\zeta^*}+\frac{\tau_{\zeta^*}}{2\tau}q_\zeta=-(q_\zeta\cdot q_{\zeta^*})q,\quad q^2=1, \tag{5}$$

where $\zeta=\frac{1}{2}(\rho+iz)$ and $\tau(\zeta,\zeta^*)=\eta(\zeta)+\eta^*(\zeta^*)$, with $\eta(\zeta)$ arbitrary. In the present case, a solution of the constraint $q^2=1$, closely resembling (3), is

$$q=(f+g)^{-1}(-f+g,1+fg,1-fg), \tag{6}$$

where f and g are now *real* functions. Equation (5) reduces to the following pair of equations:

$$f_{\zeta\zeta^*}+\frac{\tau_\zeta}{2\tau}f_{\zeta^*}+\frac{\tau_{\zeta^*}}{2\tau}f_\zeta=\frac{2f_\zeta f_{\zeta^*}}{f+g}, \tag{7a}$$

$$g_{\zeta\zeta^*}+\frac{\tau_\zeta}{2\tau}g_{\zeta^*}+\frac{\tau_{\zeta^*}}{2\tau}g_\zeta=\frac{2g_\zeta g_{\zeta^*}}{f+g}. \tag{7b}$$

With the chosen parametrization (3) and (6), the O(2,1) invariance of Eqs. (1) and (5) is reflected in the following SL(2,R) invariance transformation of Eq. (4):

$$f\to(af+ib)/(-icf+d)$$

and of Eqs. (7a) and (7b):

$$f\to(af+b)/(cf+d),\quad g\to(ag-b)/(-cg+d),$$

where a, b, c, and d are real constants such that $ad-bc=1$.

Bäcklund transformations have been introduced for the Ernst equations corresponding to gravitational fields of the type considered in this Letter,[9] and solutions of the Einstein field equations have been generated through their use.[11,12] On the other hand, it seems desirable to have a solution-generating technique directly applicable to the vector equations (1) and (5), as q may be simply read off the metric. By the exploiting of the close connection between such equations and the field equation for an O(2,1)-invariant nonlinear σ model in two dimensions, vector Bäcklund transformations generalizing those introduced by Pohlmeyer in Ref. 13 may be found. For Eq. (1), the transformation is

$$q_u+p_u=[1/2(U+\lambda)][(U+V)q_u\cdot p-U_u]q+[1/2(U+\lambda)][(U+V)q\cdot p_u-U_u]p, \tag{8a}$$

$$q_v-p_v=-[1/2(V-\lambda)][(U+V)q_v\cdot p+V_v]q+[1/2(V-\lambda)][(U+V)q\cdot p_v+V_v]p, \tag{8b}$$

together with the compatible constraints

$$q^2=p^2=-1,\quad p\cdot q=(U-V+2\lambda)/(U+V), \tag{9}$$

where λ is a real constant, with U and V the functions appearing in (2). Given a solution q of Eq. (1), a new solution p may be found by integrating Eqs. (8a) and (8b). [The integrability condition for (8a) and (8b), considered as a system of partial differential equations for p, is satisfied by virtue of Eq. (1).] The corresponding transformation for Eq. (5) is

$$q_\zeta+ip_\zeta=[1/2(\eta+i\lambda)][(\eta+\eta^*)p_\zeta\cdot q-i\eta_\zeta]p-[i/2(\eta+i\lambda)][(\eta+\eta^*)q_\zeta\cdot p-i\eta_\zeta]q \tag{10}$$

with

$$q^2=1,\quad p^2=-1,\quad q\cdot p=[i(\eta-\eta^*)-2\lambda]/(\eta+\eta^*). \tag{11}$$

Equations (8)–(11) are manifestly invariant under $p \to Rp$, $q \to Rq$, with a constant $R \in O(2,1)$. Notice that, while Eqs. (8a) and (8b) transform a solution of Eq. (1) into another solution of the same equation, Eq. (10) carries a solution of (5) into a solution of the equation

$$p_{\zeta\zeta^*} + (\tau_\zeta/2\tau)p_{\zeta^*} + (\tau_{\zeta^*}/2\tau)p_\zeta = (p_\zeta \cdot p_{\zeta^*})p, \quad p^2 = -1. \tag{12}$$

Thus, in order to go back to solutions of Eq. (5), a second Bäcklund transformation (10) must be used. Fortunately, no further integrations are necessary, because of the existence of a permutability property,[14] by means of which the second Bäcklund transform may be computed algebraically: Suppose Eq. (10) has been integrated for two different values λ and μ, giving the transformed vectors p and s, respectively. It may be easily shown that there exists a vector w, such that w is simultaneously a Bäcklund transform of p with parameter μ, and of s with parameter λ, and such that w satisfies Eq. (5); furthermore, there exists a simple algebraic relation among q, p, s, and w:

$$w = q + \frac{2(\lambda - \mu)}{\eta + \eta^*}\,\frac{1}{1 + p\cdot s}\,(s - p). \tag{13}$$

Equation (13) may be considered as a kind of nonlinear superposition principle. A detailed derivation of Eq. (13) will be published elsewhere. A similar relation holds for Eqs. (8a) and (8b).

The following important aspect should now be noticed: As there exists a manifest duality between p and q in Eqs. (10) and (11), one may take the point of view that the equation to be solved is Eq. (12). This is motivated by the fact that the Ernst equation for the potential ϵ of a stationary axisymmetric system is[1]

$$\epsilon_{\zeta\zeta^*} + (\tau_\zeta/2\tau)\epsilon_{\zeta^*} + (\tau_{\zeta^*}/2\tau)\epsilon_\zeta = 2\epsilon_\zeta\epsilon_{\zeta^*}/(\epsilon + \bar{\epsilon}) \tag{14}$$

which may be cast in precisely the form (12) by defining

$$p = (\epsilon + \epsilon^*)^{-1}(i(\epsilon - \epsilon^*), 1 - \epsilon\epsilon^*, 1 + \epsilon\epsilon^*).$$

The starting point will now be the vector p; a new solution ϵ' of Eq. (14) may be obtained by means of Eq. (10). As a result of its symmetry, the superposition (13) is equally applicable in this case. It may now be written as

$$s = p + [2(\lambda - \mu)/(\eta + \eta^*)(w \cdot q - 1)]^{(w-q)}, \tag{15}$$

where q (respectively, w) is a Bäcklund transform of p with parameter λ (respectively, μ). The new Ernst potential is

$$\epsilon' = (1 - is^1)/(s^2 + s^3),$$

where $s = (s^1, s^2, s^3)$. The superposition formula (15) has the additional interest of preserving the right asymptotic conditions. In order to see this, it is convenient to find the general solution of (15) starting from Minkowski space, described by $p = (0,0,1)$. An asymptotically flat space will approach this value, and the conclusions that follow will also apply in that case. Integrating (10) with $\eta(\zeta) = \zeta$, so that $\tau = \rho$ (using Weyl canonical coordinates ρ and z), one gets

$$q = (\zeta + \zeta^*)^{-1}(2\cos\alpha(\zeta + i\lambda)^{1/2}(\zeta^* - i\lambda)^{1/2}, 2\sin\alpha(\zeta + i\lambda)^{1/2}(\zeta^* - i\lambda)^{1/2}, -i(\zeta - \zeta^*) + 2\lambda), \tag{16}$$

where α is a constant of integration. Similarly, we find w by substituting $\lambda \to \mu$, $\alpha \to \beta$ in (16). After computing s according to (15) we get the following expression:

$$\frac{1 + \epsilon'}{1 - \epsilon'} = \frac{s^2 - is^1}{s^3 - 1} = \frac{i}{a - b}\,(e^{i\beta}r_1 - e^{i\alpha}r_2), \tag{17}$$

where $r_1 = (r^2 + b^2 + 2br\cos\theta)^{1/2}$, $r_2 = (r^2 + a^2 + 2ar \times \cos\theta)^{1/2}$, $\rho = r\sin\theta$, $z = r\cos\theta$, $a = 2\lambda$, and $b = 2\mu$. From (17), one obtains the following expansion for ϵ' in terms of the radial coordinate r, showing the correct asymptotic behavior:

$$\epsilon' = 1 + \frac{2(a - b)}{i(e^{i\beta} - e^{i\alpha})}\,\frac{1}{r} + O\left(\frac{1}{r^2}\right).$$

In order to get a real coefficient in the $1/r$ term, we may either set $\beta = -\alpha$ or subject ϵ' to an Ehlers-type transformation,

$$\epsilon'' = \frac{\cos\sigma\,\epsilon' + i\sin\sigma}{i\sin\sigma\,\epsilon' + \cos\sigma},$$

with an appropriately chosen σ. Both procedures leave the zeroth-order term invariant. The Kerr solution, $(1 + \epsilon')(1 - \epsilon')^{-1} = x\cos\nu + iy\sin\nu$, is obtained as a special case of (16) by setting $a = -b = k$, $\alpha = -\beta = \nu - \pi/2$, and defining $x + y = k^{-1}(z^2 + \rho^2 + k^2 + 2kz)^{1/2}$, $x - y = k^{-1}(z^2 + \rho^2 + k^2 - 2kz)^{1/2}$.

Further work on other applications of the present method, and on its relations with existing techniques for generating solutions of the Einstein equations and of the equations for non-Abelian gauge fields, is currently in progress.

Financial support by Comisión Asesora de Investigación Científica y Técnica is gratefully acknowledged.

[1]F. J. Ernst, Phys. Rev. 167, 1175 (1968).

[2]D. Kramer and G. Neugebauer, Commun. Math. Phys. 10, 132 (1968).

[3]R. Geroch, J. Math. Phys. (N.Y.) 13, 394 (1972).

[4]W. Kinnersley, J. Math. Phys. (N.Y.) 18, 1529 (1977).

[5]W. Kinnersley and D. M. Chitre, Phys. Rev. Lett. 40, 1608 (1978).

[6]D. Maison, Phys. Rev. Lett. 41, 521 (1978), and J. Math. Phys. (N.Y.) 20, 871 (1979).

[7]C. Hoenselaers, W. Kinnersley, and B. C. Xanthopoulos, Phys. Rev. Lett. 42, 481 (1978), and J. Math. Phys. (N.Y.) 20, 2530 (1979).

[8]V. A. Belinskiĭ and V. E. Zakharov, Zh. Eksp. Teor. Fiz. 75, 1955 (1978) [Sov. Phys. JETP 48, 985 (1978)].

[9]Bäcklund transformations for the Ernst equation of general relativity were independently introduced for the first time in the following: B. K. Harrison, Phys. Rev. Lett. 41, 1197, 1835(E) (1978); G. Neugebauer, J. Phys. A12, L67 (1979). Other expressions for Bäcklund transformations of the same problem, and their relation with gauge transformations, may be seen in the following: M. Omote and M. Wadati, J. Math. Phys. (N.Y.) 22, 961 (1981); F. J. Chinea, Phys. Rev. D 24, 1053 (1981), and 26, 2175(E) (1982), and Physica (Utrecht) 114A, 151 (1982).

[10]T. Lewis, Proc. Roy. Soc. London, Ser. A 136, 176 (1932); A. Papapetrou, Ann. Phys. (Leipzig) 12, 309 (1953).

[11]G. Neugebauer and D. Kramer, Exp. Tech. Phys. 28, 3 (1980); G. Neugebauer, J. Phys. A 13, L19 (1980); D. Kramer and G. Neugebauer, Phys. Lett. 75A, 259 (1980).

[12]B. K. Harrison, Phys. Rev. D 21, 1695 (1980).

[13]K. Pohlmeyer, Commun. Math. Phys. 46, 207 (1976).

[14]A permutability property for Bäcklund transformations of the Ernst equation was found by G. Neugebauer (Ref. 9).

Data on the Gross–Llewellyn Smith Sum Rule as a Function of q^2

T. Bolognese

Centre d'Etudes Nucléaires, Saclay, F-91190 Gif-sur-Yvette, France

and

P. Fritze and J. Morfin[a]

Physikalisches Institut der Technischen Hochschule, D-5100 Aachen, Germany

and

D. H. Perkins and K. Powell

Department of Nuclear Physics, University of Oxford, Oxford OX1 3RH, United Kingdom

and

W. G. Scott[b]

CERN, European Organization for Nuclear Research, CH-1211 Geneva 23, Switzerland

(Received 22 June 1982)

Data are presented on the Gross–Llewellyn Smith sum rule obtained from combined narrow-band neon and Freon bubble-chamber neutrino-antineutrino experiments. Remarkably no significant deviation from the parton-model prediction for the sum rule is observed at very low values of $q^2 \lesssim 1$ GeV^2. Limits on the effective QCD scale parameter Λ and on the magnitude of the twist-4 correction are set. The best fit, neglecting higher-twist contributions, gives $\Lambda = 92^{+20}_{-36}$ MeV.

PACS numbers: 13.15.Em, 12.35.Eq

In the quark-parton model the neutrino-nucleon structure function F_3 measures the difference of the x distributions of the quarks and the antiquarks in the nucleon:

$$F_3(x) \underset{q^2 \to \infty}{=} \frac{dN_q}{dx} - \frac{dN_{\bar{q}}}{dx}, \qquad (1)$$

where x is the usual Bjorken scaling variable. The integral of F_3 measures the number of "valence quarks" per nucleon, equal to three in the quark model:

$$\int_0^1 F_3(x)\,dx \underset{q^2 \to \infty}{=} N_q - N_{\bar{q}} = 3 \text{ valence quarks}. \qquad (2)$$