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Exactly solvable D_N-type quantum spin models with long-range interaction

B. Basu-Mallick [a], F. Finkel [b], A. González-López [b,*]

[a] *Theory Group, Saha Institute of Nuclear Physics, 1/AF Bidhan Nagar, Kolkata 700 064, India*
[b] *Depto. de Física Teórica II, Universidad Complutense, 28040 Madrid, Spain*

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Abstract

We derive the spectra of the D_N-type Calogero (rational) su(m) spin model, including the degeneracy factors of all energy levels. By taking the strong coupling limit of this model, in which its spin and dynamical degrees of freedom decouple, we compute the exact partition function of the su(m) Polychronakos–Frahm spin chain of D_N type. In particular, we show that this partition function cannot be obtained as a limiting case of its BC_N counterpart. With the help of the partition function we study several statistical properties of the chain's spectrum, such as the density of energy levels and the distribution of spacings between consecutive levels.

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1. Introduction

Recent studies of quantum integrable dynamical models and spin chains with long-range interactions [1–8] have not only enriched our understanding of strongly correlated many-particle systems in one dimension, but also influenced several branches of mathematics in a significant way. In particular, it is found that this class of quantum integrable systems have close connections with apparently diverse subjects like generalized exclusion statistics [8–10], quantum Hall effect [11], quantum electric transport in mesoscopic systems [12,13], random matrix theory [14],

* Corresponding author.
E-mail address: artemio@fis.ucm.es (A. González-López).

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multivariate orthogonal polynomials [15–17] and Yangian quantum groups [18–20]. The interest in quantum integrable models with long-range interaction was initiated by a seminal work of Calogero [1], where the exact spectrum of an N-particle system on a line with two-body interactions inversely proportional to the square of the distance and subject to a confining harmonic potential was computed in closed form. An exactly solvable trigonometric variant of the rational model introduced by Calogero was proposed shortly afterwards by Sutherland [2,3]. The particles in this so-called Sutherland model move on a circle, with two-body interactions proportional to the inverse square of their chord distances. Subsequently, Olshanetsky and Perelomov established the existence of an underlying A_{N-1} root system structure for both the Calogero and Sutherland models, and constructed generalizations thereof associated with other classical (extended) root systems like B_N, C_N and BC_N [4].

In a parallel development, Haldane and Shastry found an exactly solvable quantum spin-$\frac{1}{2}$ chain with long-range interactions, whose ground state coincides with the $U \to \infty$ limit of Gutzwiller's variational wave function for the Hubbard model, and provides a one-dimensional realization of the resonating valence bond state [5,6]. The lattice sites of this su(2) Haldane–Shastry (HS) spin chain are equally spaced on a circle, all spins interacting with each other through pairwise exchange interactions inversely proportional to the square of their chord distances. A close relation between the HS chain and the su(m) spin generalization of the original (type A) Sutherland model [21–23], which leads to many quantitative predictions, was subsequently established through the so-called "freezing trick" [7,24]. More precisely, it is found that in the strong coupling limit the particles in the spin Sutherland model "freeze" at the coordinates of the equilibrium position of the scalar part of the potential, and the dynamical and spin degrees of freedom decouple. The equilibrium coordinates coincide with the equally spaced lattice points of the HS spin chain, so that the decoupled spin degrees of freedom are governed by the Hamiltonian of the su(m) HS model. Moreover, in this freezing limit the conserved quantities of the spin Sutherland model immediately yield those of the HS spin chain, thereby explaining its complete integrability. By applying the freezing trick to the type A rational Calogero model with spin degrees of freedom, a new integrable spin chain with long-range interaction was constructed in Ref. [7]. The sites of this chain—commonly known in the literature as the Polychronakos or Polychronakos–Frahm (PF) spin chain—are unequally spaced on a line, and in fact coincide with the zeros of the Hermite polynomial of degree N [25]. BC_N versions of both the HS and PF chains were later discussed by several authors [26,27], mainly in connection with their complete integrability.

The powerful technique of the freezing trick was subsequently used to compute the exact partition functions of both the su(m) PF spin chain [28] and the su(m) HS chain [29], the BC_N counterparts of these chains [30,31], and their supersymmetric extensions [32–34]. The exact computation of the partition functions of these quantum integrable spin chains has opened up the exciting possibility of studying various statistical properties of their energy spectra. Indeed, it is found that for a large number of lattice sites the energy level density of such chains follows the Gaussian distribution with a high degree of accuracy [29–31,33–35]. It has also been observed that the distribution of the (normalized) spacings between consecutive energy levels of these chains is not of Poisson type, as might be expected in view of a well-known conjecture of Berry and Tabor [36]. An analytical expression, which explains the unexpected distribution of spacings between consecutive energy levels in the above mentioned chains, has recently been derived using only a few simple properties of their spectra [30].

Our aim in this article is first of all to analyze the spectrum of the su(m) spin Calogero model of D_N type. We shall then apply the freezing trick to compute the exact partition function of the

D_N version of the PF spin chain, and use this partition function to study various statistical properties of the chain's spectrum. It should be stressed that, although the Hamiltonian of the D_N-type su(m) spin Calogero model can be obtained by setting to zero one of the coupling constants of their BC_N counterparts, this fact does not allow one to find out all physically relevant properties of the D_N model as a limiting case of its BC_N version. For example, as will be explained in Section 2, the configuration space of the D_N-type spin Calogero model differs quite significantly from its BC_N counterpart. A more drastic change occurs in the Hilbert space of the D_N model, which gets "doubled" in comparison with the BC_N one. More precisely, the Hilbert space of the D_N spin Calogero model can be expressed as a direct sum of the Hilbert spaces associated to two different BC_N models with opposite "chiralities". As a result, the spectrum of the rational spin chain of D_N type differs essentially from that of its BC_N analog of either chirality. These remarkable properties of the D_N spin Calogero model and its associated spin chain indicate that they are "singular limits" of their BC_N counterparts, and are thus new solvable models worthy of consideration in their own right.

The paper is organized as follows. In Section 2 we introduce the su(m) spin Calogero model of D_N type and construct its associated (antiferromagnetic) spin chain by means of the freezing trick, discussing their relation with their BC_N counterparts. Section 3 is devoted to the evaluation of the spectrum of the spin Calogero model of D_N type, which is then used to compute in closed form the partition function of its associated spin chain by applying the freezing trick. We also show how to express this partition function in terms of those of the PF chains of types A and B. In Section 4 we make use of the closed-form expressions for the partition function of the PF chain of D_N type to analyze several statistical properties of its spectrum. We show that—as is the case with other chains of HS type—the level density follows with great accuracy the Gaussian law when the number of lattice sites is sufficiently large. We also prove that the cumulative distribution of spacings between consecutive levels follows the same "square root of a logarithm" law obeyed by the PF chains of types A and B and by the original HS chain. This provides further confirmation of the fact that spin chains of HS type are exceptional integrable systems from the point of view of the Berry–Tabor conjecture. In Section 5 we outline the generalization of the above results to the ferromagnetic chain and its associated spin dynamical model. In the last section we present our conclusions and outline some future developments.

2. The model

Since the su(m) spin Calogero model of D_N type is closely related to its BC_N counterpart, we shall start by briefly reviewing the latter model, whose Hamiltonian is given by [30,37]

$$H^{(\mathrm{B})} = -\sum_i \partial_{x_i}^2 + a\sum_{i\neq j}\left[\frac{a+S_{ij}}{(x_{ij}^-)^2} + \frac{a+\tilde{S}_{ij}}{(x_{ij}^+)^2}\right] + b\sum_i \frac{b-\epsilon S_i}{x_i^2} + \frac{a^2}{4}r^2. \tag{1}$$

Here the sums run from 1 to N (as always hereafter, unless otherwise stated), $a > 1/2$, $b > 0$, $\epsilon = \pm 1$, $x_{ij}^{\pm} = x_i \pm x_j$, $r^2 = \sum_i x_i^2$, S_{ij} is the operator which permutes the ith and jth spins, S_i is the operator reversing the ith spin, and $\tilde{S}_{ij} = S_i S_j S_{ij}$. Note that the spin operators S_{ij} and S_i can be expressed in terms of the fundamental su(m) spin generators J_k^α at the site k (with the normalization $\mathrm{tr}(J_k^\alpha J_k^\gamma) = \frac{1}{2}\delta^{\alpha\gamma}$) as

$$S_{ij} = \frac{1}{m} + 2\sum_{\alpha=1}^{m^2-1} J_i^\alpha J_j^\alpha, \quad S_i = \sqrt{2m}\, J_i^1.$$

The configuration space of the Hamiltonian (1) can be taken as one of the Weyl chambers of the BC_N root system, i.e., one of the maximal open subsets of $\mathbb{R}^N$ on which the linear functionals $x_i \pm x_j$ and x_i have constant signs. We shall choose as configuration space the principal Weyl chamber

$$C^{(\mathrm{B})} = \{\mathbf{x} \equiv (x_1, \ldots, x_N) : 0 < x_1 < x_2 < \cdots < x_N\}. \tag{2}$$

The spectrum of the BC_N spin Calogero model, including the degeneracy factors of all energy levels, has been determined by constructing a (non-orthonormal) basis of the Hilbert space in which the Hamiltonian (1) is triangular [30]. By setting $b = \beta a$ and taking the limit $a \to \infty$ in the Hamiltonian (1), one can obtain the su(m) PF spin chain of BC_N type, with Hamiltonian given by

$$\mathcal{H}^{(\mathrm{B})} = \sum_{i \neq j} \left[\frac{1 + S_{ij}}{(\xi_i - \xi_j)^2} + \frac{1 + \tilde{S}_{ij}}{(\xi_i + \xi_j)^2} \right] + \beta \sum_i \frac{1 - \epsilon S_i}{\xi_i^2}. \tag{3}$$

Here β is a positive real parameter, and the lattice sites ξ_i can be expressed in terms of the zeros y_i of the Laguerre polynomial $L_N^{\beta-1}$ as $y_i = \xi_i^2/2$. The exact partition function of the spin model (3) has also been recently computed with the help of the freezing trick [30].

The Hamiltonian of the su(m) spin Calogero model of D_N type is obtained by setting $b = 0$ in its BC_N counterpart (1), namely

$$H = -\sum_i \partial_{x_i}^2 + \frac{a^2}{4} r^2 + a \sum_{i \neq j} \left[\frac{a + S_{ij}}{(x_{ij}^-)^2} + \frac{a + \tilde{S}_{ij}}{(x_{ij}^+)^2} \right]. \tag{4}$$

As configuration space of the Hamiltonian (4) we can take again one of the Weyl chambers of the D_N root system. For instance, the choice $x_1 < \cdots < x_N$ determines all the differences $x_i - x_j$. If we also require that $x_1 + x_2 > 0$ the sign of all the sums $x_i + x_j$ is determined as well. Indeed, $|x_1| < x_2$ implies that $|x_1| < x_j$ for all $j = 2, \ldots, N$, so that $x_1 + x_j > 0$ for $j > 1$, while the sums $x_i + x_j$ with $i, j > 2$ and $i \neq j$ are clearly positive on account of the positivity of x_k with $k > 1$. Thus we can take as configuration space of H the open set

$$C = \{\mathbf{x} \equiv (x_1, \ldots, x_N) : |x_1| < x_2 < \cdots < x_N\}, \tag{5}$$

which is just the principal Weyl chamber of the D_N root system. It is interesting to observe that this configuration space contains its BC_N counterpart (2) as a subset.

The Hamiltonian of the su(m) PF chain of D_N type can be obtained from the spin Hamiltonian (4) in the limit $a \to \infty$ by means of the freezing trick. More precisely, since

$$H = -\sum_i \partial_{x_i}^2 + a^2 U + O(a),$$

with

$$U(\mathbf{x}) = \sum_{i \neq j} \left[\frac{1}{(x_{ij}^-)^2} + \frac{1}{(x_{ij}^+)^2} \right] + \frac{r^2}{4}, \tag{6}$$

when the coupling constant a tends to infinity the particles in the spin dynamical model (4) concentrate at the coordinates ξ_i of the minimum $\boldsymbol{\xi}$ of the potential U in C. From the identity

$$H = H_{\mathrm{sc}} + a\tilde{\mathcal{H}}(\mathbf{x}), \tag{7}$$

where

$$H_{\rm sc} = -\sum_i \partial_{x_i}^2 + \frac{a^2}{4} r^2 + a(a-1) \sum_{i \neq j} \left[\frac{1}{(x_{ij}^-)^2} + \frac{1}{(x_{ij}^+)^2} \right] \tag{8}$$

and

$$\tilde{\mathcal{H}}(\mathbf{x}) = \sum_{i \neq j} \left[\frac{1 + S_{ij}}{(x_i - x_j)^2} + \frac{1 + \tilde{S}_{ij}}{(x_i + x_j)^2} \right],$$

it follows that in the limit $a \to \infty$ the internal degrees of freedom of H are governed by the Hamiltonian $\mathcal{H} = \tilde{\mathcal{H}}(\boldsymbol{\xi})$, explicitly given by

$$\mathcal{H} = \sum_{i \neq j} \left[\frac{1 + S_{ij}}{(\xi_i - \xi_j)^2} + \frac{1 + \tilde{S}_{ij}}{(\xi_i + \xi_j)^2} \right]. \tag{9}$$

Eq. (9) is the Hamiltonian of the (antiferromagnetic) su(m) PF chain of D_N type, whose sites ξ_i are the coordinates of the unique minimum $\boldsymbol{\xi}$ of the scalar potential (6) in the open set (5). The existence of this minimum follows from the fact that U tends to $+\infty$ on the boundary of C and as $r \to \infty$, and its uniqueness was established in Ref. [38] by expressing the potential U in terms of the logarithm of the ground state ρ of the scalar D_N Calogero model $H_{\rm sc}$, given by

$$\rho(\mathbf{x}) = \mathrm{e}^{-\frac{a}{4} r^2} \prod_{i<j} |x_i^2 - x_j^2|^a. \tag{10}$$

As shown in the latter reference, the sites ξ_i coincide with the coordinates of the (unique) critical point of $\log \rho$ in C, and therefore satisfy the nonlinear system

$$\xi_i \left(\sum_{j;\, j \neq i} \frac{1}{\xi_i^2 - \xi_j^2} - \frac{1}{4} \right) = 0, \quad i = 1, \ldots, N.$$

The numbers ξ_i cannot be all different from zero, since in that case we would obtain the contradiction

$$0 = \sum_{j \neq i} \frac{1}{\xi_i^2 - \xi_j^2} - \frac{N}{4} = -\frac{N}{4}.$$

Hence $\xi_i = 0$ for some i, and since $(\xi_1, \ldots, \xi_N)$ lies in C we must have

$$\xi_1 = 0, \tag{11a}$$

while the remaining $N - 1$ sites should satisfy the condition

$$\sum_{j;\, j \neq i} \frac{1}{\xi_i^2 - \xi_j^2} = \frac{1}{4}, \quad i = 2, \ldots, N. \tag{11b}$$

Substituting Eq. (11a) into (11b) one obtains

$$\sum_{\substack{j=2 \\ j \neq i}}^{N} \frac{1}{\xi_i^2 - \xi_j^2} = \frac{1}{4} - \frac{1}{\xi_i^2}, \quad i = 2, \ldots, N. \tag{12}$$

It is interesting to compare the above condition satisfied by the nonzero ξ_i's with the relation

$$\sum_{\substack{j=1\\ j\neq i}}^{M} \frac{1}{(y_i - y_j)} = \frac{1}{2} - \frac{\beta}{2y_i}, \tag{13}$$

obeyed by the zeros y_i of the Laguerre polynomial $L_M^{\beta-1}$ [39]. It is evident that Eq. (12) reduces to Eq. (13) when $M = N-1$, $\beta = 2$ and $y_i = \xi_{i-1}^2/2$. We therefore conclude that the sites $\xi_2 < \cdots < \xi_N$ are expressed in terms of the $N-1$ zeros $y_1 < \cdots < y_{N-1}$ of the Laguerre polynomial L_{N-1}^1 by $\xi_i = \sqrt{2y_{i-1}}$. On the other hand, it has already been mentioned that the lattice sites of the PF model of BC_N type (3) are expressed in terms of the zeros of the Laguerre polynomial $L_N^{\beta-1}$ by $y_i = \xi_i^2/2$. Since the potential U in Eq. (6) is obtained from its BC_N counterpart in the limit $\beta \to 0$, we could also have argued that the lattice site ξ_i of the D_N-type PF model is the square root of twice the ith zero of L_N^{-1} for $i = 1, \ldots, N$. The equivalence of both characterizations is substantiated by the well-known identity $L_N^{-1}(y) = -yL_{N-1}^1(y)/N$, cf. [38].

It is worth pointing out that, even though the lattice sites of the BC_N-type PF chain coincide with their D_N counterparts in the limit $\beta \to 0$, the Hamiltonian (3) of the PF chain of BC_N type does not reduce to its D_N variant (9) in the same limit. To establish this fact, note first that all roots of the equation $L_N^{\beta-1}(y) = 0$ except the smallest one tend to a finite nonzero value in the limit $\beta \to 0$. As a result, terms like $\beta(1-\epsilon S_i)/\xi_i^2$, which appear in the r.h.s. of Eq. (3), vanish for $i = 2, \ldots, N$. We next examine the behavior of the smallest root ξ_1 of the equation $L_N^{\beta-1}(y) = 0$. It can be shown [30] that the zeros of $L_N^{\beta-1}$ satisfy the relation

$$\beta \sum_{j=2}^{N} \frac{1}{y_j} = N - \frac{\beta}{y_1}. \tag{14}$$

Since the l.h.s. of this equation vanishes in the limit $\beta \to 0$, the r.h.s. yields $\lim_{\beta\to 0}(2\beta/\xi_1^2) = N$. Substituting this limiting value in Eq. (3) we find that the Hamiltonians of the BC_N- and D_N-type PF spin chains are related by

$$\lim_{\beta\to 0} \mathcal{H}^{(\mathrm{B})} = \mathcal{H} + \frac{N}{2}(1 - \epsilon S_1). \tag{15}$$

It is interesting to observe that the second term in the r.h.s. of the previous equation may be interpreted as an "impurity" interaction at the left end of the BC_N spin chain.

3. Spectrum and partition function

We shall start by deriving the spectra and partition functions of the D_N-type su(m) spin Calogero model (4) and its scalar counterpart (8). Since the spin and dynamical degrees of freedom of the Hamiltonian (4) decouple in the freezing limit $a \to \infty$, by Eq. (7) its eigenvalues are approximately given by

$$E_{ij} \simeq E_i^{\mathrm{sc}} + a\mathcal{E}_j, \quad a \gg 1, \tag{16}$$

where E_i^{sc} and $\mathcal{E}_j$ are two arbitrary eigenvalues of H_{sc} and $\mathcal{H}$, respectively. The asymptotic relation (16) immediately yields the following exact formula for the partition function $\mathcal{Z}$ of the

D_N-type PF spin chain (9):

$$\mathcal{Z}(T) = \lim_{a\to\infty} \frac{Z(aT)}{Z_{\rm sc}(aT)}, \tag{17}$$

where Z and $Z_{\rm sc}$ are the partition functions of H and $H_{\rm sc}$, respectively. Inserting the expressions for the partition functions Z and $Z_{\rm sc}$ in the latter equation we shall obtain an explicit formula for the partition function $\mathcal{Z}$ of the chain (9).

In order to determine the spectra of the corresponding Hamiltonians H and $H_{\rm sc}$ in Eqs. (4) and (8), following Ref. [30] we introduce the auxiliary operator

$$H' = -\sum_i \partial_{x_i}^2 + \frac{a^2}{4} r^2 + \sum_{i\neq j}\left[\frac{a}{(x_{ij}^-)^2}(a - K_{ij}) + \frac{a}{(x_{ij}^+)^2}(a - \tilde{K}_{ij})\right], \tag{18}$$

where K_{ij} and K_i are coordinate permutation and sign reversing operators, defined by

$$(K_{ij} f)(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_N) = f(x_1, \ldots, x_j, \ldots, x_i, \ldots, x_N),$$
$$(K_i f)(x_1, \ldots, x_i, \ldots, x_N) = f(x_1, \ldots, -x_i, \ldots, x_N),$$

and $\tilde{K}_{ij} = K_i K_j K_{ij}$. We then have the obvious relations

$$H = H'|_{K_{ij}\to -S_{ij}, K_i \to \epsilon S_i}, \tag{19a}$$
$$H_{\rm sc} = H'|_{K_{ij}\to 1, K_i \to \epsilon}, \tag{19b}$$

where ϵ can take *both* values ± 1. On the other hand, the spectrum of H' is easily computed by noting that this operator can be written in terms of the rational Dunkl operators of D_N type [40]

$$J_i^- = \partial_{x_i} + a\sum_{j\neq i}\left[\frac{1}{x_{ij}^-}(1 - K_{ij}) + \frac{1}{x_{ij}^+}(1 - \tilde{K}_{ij})\right], \tag{20}$$

$i = 1, \ldots, N$, as follows [41]:

$$H' = \rho\left[-\sum_i \left(J_i^-\right)^2 + a\sum_i x_i \partial_{x_i} + E_0\right]\rho^{-1}, \tag{21}$$

where

$$E_0 = Na\left(a(N-1) + \frac{1}{2}\right). \tag{22}$$

Since the Dunkl operators (20) map any monomial $\prod_i x_i^{n_i}$ into a polynomial of total degree $n_1 + \cdots + n_N - 1$, by Eq. (21) the operator H' is represented by an upper triangular matrix in the (non-orthonormal) basis with elements

$$\phi_{\mathbf{n}} = \rho\prod_i x_i^{n_i}, \quad \mathbf{n} \equiv (n_1, \ldots, n_N) \in \left(\mathbb{N}\cup\{0\}\right)^N, \tag{23}$$

ordered according to the total degree $|\mathbf{n}| \equiv n_1 + \cdots + n_N$ of the monomial part. More precisely,

$$H'\phi_{\mathbf{n}} = E'_{\mathbf{n}}\phi_{\mathbf{n}} + \sum_{|\mathbf{m}|<|\mathbf{n}|} c_{\mathbf{mn}}\phi_{\mathbf{m}}, \tag{24}$$

where

$$E'_{\mathbf{n}} = a|\mathbf{n}| + E_0 \tag{25}$$

and the coefficients $c_{\mathbf{mn}}$ are real constants.

We shall now construct a basis of the Hilbert space of the Hamiltonian H in which this operator is also represented by an upper triangular matrix. To this end, let us denote by $\Sigma \approx (\mathbb{C}^m)^{\otimes N}$ the Hilbert space of the su(m) internal degrees of freedom, and let

$$|\mathbf{s}\rangle \equiv |s_1, \ldots, s_N\rangle, \quad s_i = -M, -M+1, \ldots, M \equiv \frac{m-1}{2},$$

be an arbitrary element of the canonical (orthonormal) basis in this space. Due to the impenetrable nature of the singularities of the Hamiltonian (4), its Hilbert space is the set $L_0^2(C) \otimes \Sigma$ of spin wave functions square integrable on the open set C which vanish sufficiently fast on the singular hyperplanes $x_i \pm x_j = 0$, $1 \leqslant i < j \leqslant N$. It can be shown, however, that H is equivalent to its natural extension to the subspace of $L_0^2(\mathbb{R}^N) \otimes \Sigma$ consisting of spin wave functions antisymmetric under particle permutations and symmetric under sign reversals of an *even* number of coordinates and spins. (This is essentially due to the fact that any point in $\mathbb{R}^N$ not lying on the singular subset $x_i \pm x_j = 0$, $1 \leqslant i < j \leqslant N$, can be mapped in a unique way to a point in C via a suitable element of the D_N Weyl group, which is generated by coordinate permutations and sign reversals of an even number of coordinates [42].) We can therefore assume without loss of generality that the Hilbert space of H is the closure of the subspace spanned by the functions

$$\psi^{\epsilon}_{\mathbf{n},\mathbf{s}}(\mathbf{x}) = \Lambda^{\epsilon}\big(\phi_{\mathbf{n}}(\mathbf{x})|\mathbf{s}\rangle\big), \quad \epsilon = \pm 1, \tag{26}$$

where Λ^{ϵ} denotes the projector on states antisymmetric under simultaneous permutations of spatial and spin coordinates, and with parity ϵ under sign reversals of coordinates and spins. The latter functions are linearly independent, and hence form a (non-orthonormal) basis of the Hilbert space of H, provided that the quantum numbers $\mathbf{n}$ and $\mathbf{s}$ satisfy the following conditions:

(i) $n_1 \geqslant \cdots \geqslant n_N$.
(ii) $s_i > s_j$ whenever $n_i = n_j$ and $i < j$.
(iii) $s_i \geqslant 0$ for all i, and $s_i > 0$ if $(-1)^{n_i} = -\epsilon$.

The first two conditions are a consequence of the antisymmetry of the states (26) under particle permutations, while the last one is due to the fact that $\psi^{\epsilon}_{\mathbf{n},\mathbf{s}}$ must have parity ϵ under sign reversals. It should be noted that the Hilbert space $\mathbf{V}$ of the Hamiltonian H just defined can be written as the direct sum

$$\mathbf{V} = \mathbf{V}_+ \oplus \mathbf{V}_-, \tag{27}$$

where the subspace $\mathbf{V}_{\epsilon}$ is the closure of the span of the basis vectors $\psi^{\epsilon}_{\mathbf{n},\mathbf{s}}(\mathbf{x})$. Within each subspace $\mathbf{V}_{\epsilon}$, a partial ordering among these basis vectors may again be defined by the degree $|\mathbf{n}|$. We shall now show that the Hamiltonian H is represented by an upper triangular matrix in this basis (and thus by a direct sum of two upper triangular matrices in the total Hilbert space $\mathbf{V}$). Indeed, since $K_{ij}\Lambda^{\epsilon} = -S_{ij}\Lambda^{\epsilon}$ and $K_i\Lambda^{\epsilon} = \epsilon S_i \Lambda^{\epsilon}$, it follows that $H\Lambda^{\epsilon} = H'\Lambda^{\epsilon}$. Using this identity, Eq. (24), and the fact that H' obviously commutes with Λ^{ϵ}, we have

$$H\psi^{\epsilon}_{\mathbf{n},\mathbf{s}} = H'\psi^{\epsilon}_{\mathbf{n},\mathbf{s}} = \Lambda^{\epsilon}\big((H'\phi_{\mathbf{n}})|\mathbf{s}\rangle\big) = E'_{\mathbf{n}}\psi^{\epsilon}_{\mathbf{n},\mathbf{s}} + \sum_{|\mathbf{m}|<|\mathbf{n}|} c_{\mathbf{mn}}\psi^{\epsilon}_{\mathbf{m},\mathbf{s}}. \tag{28}$$

Suppose now that both $\mathbf{n}$ and $\mathbf{s}$ satisfy conditions (i)–(iii) above, so that $\psi^{\epsilon}_{\mathbf{n},\mathbf{s}}$ belongs to the basis of $\mathbf{V}_{\epsilon}$ under consideration. Although a given pair of quantum numbers $(\mathbf{m}, \mathbf{s})$ in the r.h.s. of the previous equation need not satisfy these conditions, it is easy to see that there is a permutation $\pi_{\mathbf{m}}$ such that $\mathbf{m}' \equiv \pi_{\mathbf{m}}(\mathbf{m})$ and $\mathbf{s}' \equiv \pi_{\mathbf{m}}(\mathbf{s})$ do satisfy (i)–(iii). Since $\psi^{\epsilon}_{\mathbf{m},\mathbf{s}}$ differs from the basis

vector $\psi^{\epsilon}_{\mathbf{m}',\mathbf{s}'}$ at most by a sign, and $|\mathbf{m}'| = |\mathbf{m}| < |\mathbf{n}|$, our claim follows directly from Eq. (28). Moreover, the latter equation and Eq. (25) imply that the eigenvalues of the spin Calogero Hamiltonian (4) are given by

$$E^{\epsilon}_{\mathbf{n},\mathbf{s}} = a|\mathbf{n}| + E_0, \tag{29}$$

where $\epsilon = \pm 1$ and $\mathbf{n}, \mathbf{s}$ satisfy conditions (i)–(iii) above. Since the numerical value of $E^{\epsilon}_{\mathbf{n},\mathbf{s}}$ is independent of $\mathbf{s}$ and ϵ, the energy associated with a quantum number $\mathbf{n}$ will be in general highly degenerate. For any given $\mathbf{n}$, this degeneracy factor $d_{\mathbf{n}}$ can be found by counting the numbers $d^{\epsilon}_{\mathbf{n}}$ of independent spin states $|\mathbf{s}\rangle$ satisfying conditions (ii) and (iii) for each case $\epsilon = +1$ and $\epsilon = -1$, and finally taking the sum of these two numbers. Explicit expressions for such degeneracy factors will be given shortly when computing the partition function of the model.

It is important at this point to elucidate the connection between the Hilbert spaces of the D_N-type spin Calogero model and its BC_N counterpart. The key fact in this respect is that the D_N Hamiltonian (4) does not depend on the discrete parameter ϵ. Consequently, as shown in Eq. (26), we can use both projectors Λ^+ and Λ^- for constructing the Hilbert space. On the other hand, since ϵ appears explicitly in the Hamiltonian of the BC_N spin Calogero model (1), for any given value of ϵ only the corresponding projector Λ^{ϵ} can be used to construct the Hilbert space [30]. Moreover, when $b = 0$ this Hilbert space is essentially the subspace $\mathbf{V}_{\epsilon}$ of $\mathbf{V}$ in Eq. (27). Thus the presence of ϵ in the Hamiltonian of the BC_N spin Calogero model effectively introduces a "chirality" in this system. By Eq. (27), the Hilbert space of the D_N spin Calogero model is simply the direct sum of the two Hilbert spaces associated with two BC_N models with opposite chiralities (and $b = 0$).

Turning next to the scalar Hamiltonian H_{sc}, in view of Eq. (19b) we now need to consider scalar functions of the form

$$\psi^{\epsilon}_{\mathbf{n}}(\mathbf{x}) = \Lambda^{\epsilon}_{\mathrm{s}}\phi_{\mathbf{n}}(\mathbf{x}), \quad \epsilon = \pm 1, \tag{30}$$

where $\Lambda^{\epsilon}_{\mathrm{s}}$ is the projector onto states symmetric with respect to permutations and with parity ϵ under sign reversals. In fact, we can take as the Hilbert space of H_{sc} the space of symmetric functions in $L^2_0(\mathbb{R}^n)$ with even parity with respect to an even number of coordinate sign reversals. In other words, the Hilbert space of H_{sc} is the direct sum of its two subspaces $\mathbf{V}^{\mathrm{s}}_{\epsilon} \equiv \Lambda^{\epsilon}_{\mathrm{s}} L^2_0(\mathbb{R}^n)$, whose elements have parity ϵ under sign reversals. The functions (30) form a (non-orthonormal) basis of the corresponding subspace $\mathbf{V}^{\mathrm{s}}_{\epsilon}$ provided that either $n_i = 2k_i$ for all i (for $\epsilon = 1$), or $n_i = 2k_i + 1$ for all i (for $\epsilon = -1$), with $k_1 \geqslant \cdots \geqslant k_N$ in both cases. Just as before, if for each $\epsilon = \pm 1$ we order the basis functions $\psi^{\epsilon}_{\mathbf{n}}(\mathbf{x})$ according to the degree $|\mathbf{n}|$, the matrix of the scalar Hamiltonian H_{sc} in the basis (30) is expressed as a direct sum of two upper triangular matrices, with diagonal elements $E^{\mathrm{sc}}_{\mathbf{n}}$ also given by the r.h.s. of (29). However, due to the absence in this case of spin degrees of freedom, the degeneracy factor $d^{\epsilon}_{\mathbf{n}}$ of every quantum number $\mathbf{n}$ is one. Note also that from Eq. (29), its analog for the energies of the scalar Hamiltonian, and the freezing trick relation (16), it follows that all the energies of the spin chain (9) are integers.

Let us next compute the partition functions Z_{sc} and Z of the models (8) and (4). To begin with, from now on we shall drop the common ground state energy E_0 in both models, since by Eq. (17) it does not contribute to the partition function $\mathcal{Z}$. With this convention, the partition function of the scalar Hamiltonian H_{sc} is given by

$$Z_{\mathrm{sc}}(aT) = \left(1 + q^N\right) \sum_{k_1 \geqslant \cdots \geqslant k_N \geqslant 0} q^{2|\mathbf{k}|},$$

where $q = \mathrm{e}^{-1/(k_{\mathrm{B}}T)}$. The latter sum is easily recognized as the partition function

$$Z_{\mathrm{sc}}^{(\mathrm{B})}(aT) = \prod_{i=1}^{N} \bigl(1 - q^{2i}\bigr)^{-1}$$

of the scalar Calogero model of BC_N type evaluated in Ref. [30]. We thus have

$$Z_{\mathrm{sc}}(aT) = \bigl(1 + q^N\bigr) Z_{\mathrm{sc}}^{(\mathrm{B})}(aT) = \bigl(1 - q^N\bigr)^{-1} \prod_{i=1}^{N-1} \bigl(1 - q^{2i}\bigr)^{-1}. \tag{31}$$

We are now ready to compute the partition function of the spin Hamiltonian H in Eq. (4). As for the BC_N model [30], it is convenient to deal separately with the cases of even and odd m.

3.1. Even m

When m is even, condition (iii) above simplifies to

(iii′) $s_i > 0$ for all i.

By Eq. (29), after dropping E_0 the partition function of the Hamiltonian (4) can be written as

$$Z(aT) = \sum_{n_1 \geqslant \cdots \geqslant n_N \geqslant 0} d_{\mathbf{n}} q^{|\mathbf{n}|}, \tag{32}$$

where $d_{\mathbf{n}}$ is the spin degeneracy factor associated with the quantum number $\mathbf{n}$. Writing

$$\mathbf{n} = (\overbrace{k_1, \ldots, k_1}^{\nu_1}, \ldots, \overbrace{k_r, \ldots, k_r}^{\nu_r}), \quad k_1 > \cdots > k_r \geqslant 0, \tag{33}$$

and using the conditions (ii) and (iii′), we have

$$d_{\mathbf{n}} = 2 \prod_{i=1}^{r} \binom{m/2}{\nu_i} \equiv 2 d(\boldsymbol{\nu}), \quad \boldsymbol{\nu} = (\nu_1, \ldots, \nu_r), \tag{34}$$

where $d(\boldsymbol{\nu})$ is the corresponding degeneracy factor for the BC_N type of spin Calogero model (1) with even m, and the factor of 2 is due to the two values taken by ϵ in Eq. (26). Note that $\sum_{i=1}^{r} \nu_i = N$, so that the multi-index $\boldsymbol{\nu}$ can be regarded as an element of the set $\mathcal{P}_N$ of partitions of N (taking order into account). With the previous notation, Eq. (32) becomes

$$Z(aT) = 2 \sum_{\boldsymbol{\nu} \in \mathcal{P}_N} d(\boldsymbol{\nu}) \sum_{k_1 > \cdots > k_r \geqslant 0} q^{\sum_{i=1}^{r} \nu_i k_i} = 2 Z^{(\mathrm{B})}(aT), \tag{35}$$

where

$$Z^{(\mathrm{B})}(aT) = q^{-N} \sum_{\boldsymbol{\nu} \in \mathcal{P}_N} d(\boldsymbol{\nu}) \prod_{j=1}^{r} \frac{q^{N_j}}{1 - q^{N_j}}, \quad N_j \equiv \sum_{i=1}^{j} \nu_i,$$

is the partition function of the su(m) spin Calogero model of BC_N type with even m, cf. [30]. From Eqs. (17), (31), (35) and the latter expression we finally obtain the following explicit formula for the partition function of the su(m) PF chain of D_N type in the case of even m:

$$\mathcal{Z}(T) = 2 \prod_{i=1}^{N-1} \bigl(1 - q^{2i}\bigr) \sum_{\boldsymbol{\nu} \in \mathcal{P}_N} d(\boldsymbol{\nu}) \prod_{j=1}^{\ell(\boldsymbol{\nu})-1} \frac{q^{N_j}}{1 - q^{N_j}}, \tag{36}$$

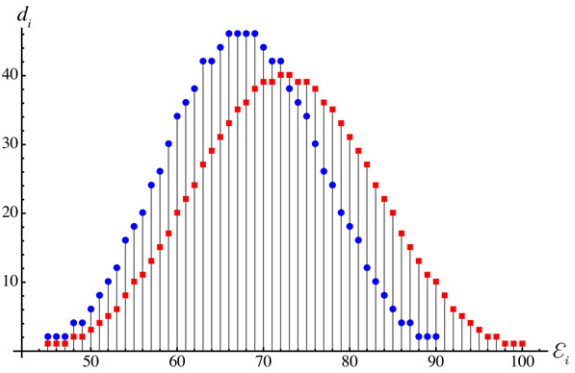

Fig. 1. Plot of the degeneracies d_i versus the energies $\mathcal{E}_i$ of the D_N-type spin chain (9) with $N = 10$ and $m = 2$ (blue circles), compared to its BC_N counterpart (3) (red squares). (Recall that the spectrum of the latter chain is independent of β.) (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

where $\ell(\boldsymbol{\nu}) = r$ is the number of components of the multi-index $\boldsymbol{\nu}$. The latter equation can be also written as

$$\mathcal{Z}(T) = 2\prod_{i=1}^{N-1}\bigl(1+q^i\bigr)\sum_{\boldsymbol{\nu}\in\mathcal{P}_N} d(\boldsymbol{\nu})q^{\sum_{j=1}^{\ell(\boldsymbol{\nu})-1}N_j}\prod_{j=1}^{N-\ell(\boldsymbol{\nu})}\bigl(1-q^{N'_j}\bigr), \tag{37}$$

where the positive integers N'_j are defined by

$$\{N'_1,\ldots,N'_{N-\ell(\boldsymbol{\nu})}\} = \{1,\ldots,N-1\} - \{N_1,\ldots,N_{\ell(\boldsymbol{\nu})-1}\}.$$

Note also that from the freezing trick relation (17), its analog for the BC_N models, and Eqs. (31)–(35) one easily obtains the identity

$$\mathcal{Z}(T) = 2\bigl(1+q^N\bigr)^{-1}\mathcal{Z}^{(\mathrm{B})}(T) \quad (\text{even } m), \tag{38}$$

where $\mathcal{Z}^{(\mathrm{B})}(T)$ is the partition function of the su(m) PF chain (3) of BC_N type. The latter equation shows that even if the spectra of the D_N and BC_N chains are related in a simple way, they are essentially different; see, e.g., Fig. 1 for a comparison of these spectra in the case $m = 2$ and $N = 10$.

For the simplest case of the spin-$\frac{1}{2}$ chain, we have $v_i = 1$ for all i, and therefore $\ell(\boldsymbol{\nu}) = N$, $d(\boldsymbol{\nu}) = 1$ and $N_j = j$, so that Eq. (37) simplifies to

$$\mathcal{Z}(T) = 2q^{\frac{1}{2}N(N-1)}\prod_{i=1}^{N-1}\bigl(1+q^i\bigr), \quad m = 2. \tag{39}$$

Thus, for spin $\frac{1}{2}$ the spectrum of the chain (9) is given by

$$\mathcal{E}_j = \frac{1}{2}N(N-1) + j, \quad j = 0,1,\ldots,\frac{1}{2}N(N-1), \tag{40}$$

and the degeneracy of the energy $\mathcal{E}_j$ is twice the number $Q_{N-1}(j)$ of partitions of the integer j into distinct parts no larger than $N-1$ (with $Q_{N-1}(0) \equiv 1$).

3.2. Odd m

Let us consider now the case of odd m. As for the BC_N chain, in this case it is convenient to slightly modify condition (i) above by first grouping the components of $\mathbf{n}$ with the same parity and then ordering separately the even and odd components. In other words, we shall write $\mathbf{n} = (\mathbf{n}_e, \mathbf{n}_o)$, where

$$\mathbf{n}_e = (\overbrace{2k_1, \ldots, 2k_1}^{\nu_1}, \ldots, \overbrace{2k_s, \ldots, 2k_s}^{\nu_s}),$$
$$\mathbf{n}_o = (\overbrace{2k_{s+1}+1, \ldots, 2k_{s+1}+1}^{\nu_{s+1}}, \ldots, \overbrace{2k_r+1, \ldots, 2k_r+1}^{\nu_r}),$$

and

$$k_1 > \cdots > k_s \geqslant 0, \qquad k_{s+1} > \cdots > k_r \geqslant 0.$$

The spin degeneracy factor is now

$$d_{\mathbf{n}} = d_s^-(\boldsymbol{\nu}) + d_s^+(\boldsymbol{\nu}) \equiv d_s(\boldsymbol{\nu}), \tag{41}$$

where $d_s^{\pm}(\boldsymbol{\nu})$ is the number of independent spin states $|s\rangle$ satisfying conditions (ii) and (iii) with $\epsilon = \pm 1$, namely (cf. [30, Eq. (28)])

$$d_s^{\epsilon}(\boldsymbol{\nu}) = \prod_{i=1}^{s} \binom{\frac{m+\epsilon}{2}}{\nu_i} \cdot \prod_{i=s+1}^{r} \binom{\frac{m-\epsilon}{2}}{\nu_i}. \tag{42}$$

Calling

$$\tilde{N}_j = \sum_{i=s+1}^{j} \nu_i, \quad j = s+1, \ldots, r,$$

and proceeding as before, we obtain

$$\begin{aligned} Z(aT) &= \sum_{\boldsymbol{\nu} \in \mathcal{P}_N} \sum_{s=0}^{r} d_s(\boldsymbol{\nu}) \sum_{\substack{k_1 > \cdots > k_s \geqslant 0 \\ k_{s+1} > \cdots > k_r \geqslant 0}} q^{\sum_{i=1}^{s} 2\nu_i k_i} q^{\sum_{i=s+1}^{r} \nu_i (2k_i+1)} \\ &= Z_+^{(\mathrm{B})}(aT) + Z_-^{(\mathrm{B})}(aT), \end{aligned} \tag{43}$$

where $Z_{\pm}^{(\mathrm{B})}$ denote the partition functions of the su(m) spin Calogero models of BC_N type (1) with odd m and $\epsilon = \pm 1$. Using the expressions of $Z_{\pm}^{(\mathrm{B})}$ derived in Ref. [30] we finally obtain

$$Z(aT) = \sum_{\boldsymbol{\nu} \in \mathcal{P}_N} \sum_{s=0}^{\ell(\boldsymbol{\nu})} d_s(\boldsymbol{\nu}) q^{-(N+N_s)} \prod_{j=1}^{s} \frac{q^{2N_j}}{1-q^{2N_j}} \cdot \prod_{j=s+1}^{\ell(\boldsymbol{\nu})} \frac{q^{2\tilde{N}_j}}{1-q^{2\tilde{N}_j}}. \tag{44}$$

Substituting the previous expression and (31) into (17), we immediately deduce the following explicit formula for the partition function of the su(m) PF chain of D_N type for odd m:

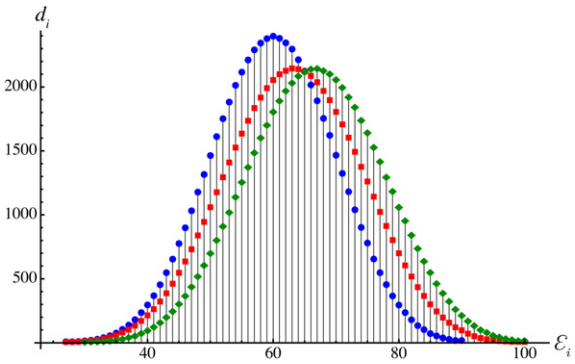

Fig. 2. Plot of the degeneracies d_i versus the energies $\mathcal{E}_i$ of the D_N-type spin chain (9) with $N = 10$ and $m = 3$ (blue circles), compared to its BC_N counterparts (3) with $\epsilon = 1$ (red squares) and $\epsilon = -1$ (green rhombuses). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

$$\mathcal{Z}(T) = \bigl(1-q^N\bigr)\prod_{i=1}^{N-1}\bigl(1-q^{2i}\bigr)\sum_{\boldsymbol{\nu}\in\mathcal{P}_N}\sum_{s=0}^{\ell(\boldsymbol{\nu})} d_s(\boldsymbol{\nu})q^{-(N+N_s)}\prod_{j=1}^{s}\frac{q^{2N_j}}{1-q^{2N_j}}\cdot\prod_{j=s+1}^{\ell(\boldsymbol{\nu})}\frac{q^{2\tilde{N}_j}}{1-q^{2\tilde{N}_j}}. \tag{45}$$

Equivalently (cf. Eqs. (31) and (43))

$$\mathcal{Z}(T) = \bigl(1+q^N\bigr)^{-1}\bigl(\mathcal{Z}_+^{(\mathrm{B})}(T)+\mathcal{Z}_-^{(\mathrm{B})}(T)\bigr), \tag{46}$$

where $\mathcal{Z}_\pm^{(\mathrm{B})}(T)$ are the partition functions of the su(m) PF chains (3) of BC_N type for odd m. Note that the latter formula is also valid for even m, since in that case $\mathcal{Z}_+^{(\mathrm{B})} = \mathcal{Z}_-^{(\mathrm{B})} \equiv \mathcal{Z}^{(\mathrm{B})}$. As in the case of even m, it is clear from Eq. (46) that the spectrum of the D_N chain (9) with odd m differs in a significant way from that of its BC_N analog of either chirality; see Fig. 2 for a comparison of the spectra of these chains for $m = 3$ and $N = 10$.

Eq. (46) can be used to verify that the expression (45) for the partition function of the su(m) PF spin chain of D_N type is a polynomial in q, as should be the case for a finite system with integer energies. To this end, recall from Ref. [34] that the partition function $\mathcal{Z}_\epsilon^{(\mathrm{B})}$ can be written as

$$\mathcal{Z}_\epsilon^{(\mathrm{B})}(T) = \sum_{K=0}^{N} q^{K(K-\frac{1}{2}(1+\epsilon))}\prod_{i=K+1}^{N}\bigl(1+q^i\bigr)\cdot\begin{bmatrix}N\\K\end{bmatrix}_q\mathcal{Z}_{N-K}^{(\mathrm{A})}\Bigl(q;\frac{m-1}{2}\Bigr), \tag{47}$$

where $\mathcal{Z}_{N-K}^{(\mathrm{A})}(q;\frac{m-1}{2})$ is the partition function of the su($\frac{m-1}{2}$) PF spin chain of type A with $N-K$ particles, and

$$\begin{bmatrix}N\\K\end{bmatrix}_q = \frac{(q)_N}{(q)_K(q)_{N-K}}, \qquad (q)_j \equiv \prod_{i=1}^{j}\bigl(1-q^i\bigr).$$

It can be shown that both the q-binomial coefficient $\left[\begin{smallmatrix}N\\K\end{smallmatrix}\right]_q$ and the partition function $\mathcal{Z}_{N-K}^{(\mathrm{A})}$ are polynomials in q, cf. Refs. [32,43]. Since all the terms in the sum in the r.h.s. of Eq. (47) contain a factor of $1+q^N$ except for $K = N$, the partition function $\mathcal{Z}_\epsilon^{(\mathrm{B})}$ can be expressed as

$$\mathcal{Z}_\epsilon^{(\mathrm{B})}(T) = \bigl(1+q^N\bigr)\mathcal{P}_\epsilon(q) + q^{N(N-1)}q^{\frac{N}{2}(1-\epsilon)},$$

where

$$\mathcal{P}_\epsilon(q) = \sum_{K=0}^{N-1} q^{K(K-\frac{1}{2}(1+\epsilon))} \prod_{i=K+1}^{N-1} \left(1+q^i\right) \cdot \begin{bmatrix} N \\ K \end{bmatrix}_q \mathcal{Z}^{(\mathrm{A})}_{N-K}$$

is a polynomial in q. Inserting the latter equations into (46) we immediately conclude that

$$\mathcal{Z}(T) = \sum_{K=0}^{N} q^{K(K-1)} \prod_{i=K}^{N-1} \left(1+q^i\right) \cdot \begin{bmatrix} N \\ K \end{bmatrix}_q \mathcal{Z}^{(\mathrm{A})}_{N-K}\left(q; \frac{m-1}{2}\right) \tag{48}$$

is a polynomial in q, as claimed.

4. Statistical analysis of the spectrum

In this subsection we shall take advantage of the explicit expressions for the partition function of the su(m) PF chain of D_N type (9) just derived to check that its spectrum shares the global properties of the spectra of other spin chains of Haldane–Shastry type mentioned in the Introduction. In practice, in order to compute the spectrum for given values of N and m it is more efficient to use Eq. (48) for odd m and its analog for even m

$$\mathcal{Z}(T) = 2\mathcal{Z}^{(\mathrm{A})}_N\left(q; \frac{m}{2}\right) \prod_{i=1}^{N-1} \left(1+q^i\right), \tag{49}$$

obtained from Eq. (38) using Eq. (31) in Ref. [34], together with the explicit expression

$$\mathcal{Z}^{(\mathrm{A})}_K(q; n) = \sum_{M_1+\cdots+M_n=K} q^{\frac{1}{2}\sum_{j=1}^n M_j(M_j-1)} \frac{(q)_K}{(q)_{M_1}\cdots(q)_{M_n}}$$

derived in Ref. [32]. With the help of the previous formulas it is possible to determine the chain's spectrum for relatively large values of N and m; for instance, using MATHEMATICA™ on a personal computer it takes less than 10 seconds to evaluate the partition function in the case $N = 50$ and $m = 3$.

In the first place, our calculations of the spectrum for a wide range of values of m and N show that the energies of the D_N chain (9) form a set of consecutive integers, as is the case for all the previously studied (non-supersymmetric) rational chains, of both types A and B [28,30]. As to the (normalized) level density

$$f(\mathcal{E}) = m^{-N} \sum_{i=1}^{L} d_i \delta(\mathcal{E} - \mathcal{E}_i), \tag{50}$$

where $\mathcal{E}_1 < \cdots < \mathcal{E}_L$ are the distinct energy levels and d_i is the degeneracy of $\mathcal{E}_i$, we have verified that when N is sufficiently large it can be approximated with great accuracy by the Gaussian law

$$g(\mathcal{E}) = \frac{1}{\sqrt{2\pi}\sigma} \mathrm{e}^{-\frac{(\mathcal{E}-\mu)^2}{2\sigma^2}} \tag{51}$$

with parameters μ and σ given by the mean and standard deviation of the chain's spectrum. Since the energy levels are consecutive integers, this means that

$$\frac{d_i}{m^N} \simeq g(\mathcal{E}_i) \quad (N \gg 1). \tag{52}$$

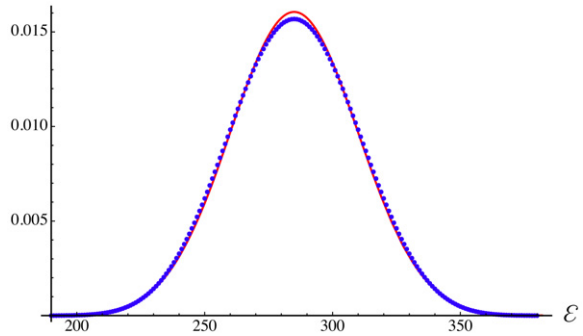

Fig. 3. Plot of the Gaussian distribution (51) (continuous red line) versus the l.h.s. of Eq. (52) (blue dots) in the case $m = 2$ and $N = 20$. The root mean square error (normalized to the mean) of the adjustment is 3.01×10^{-2}. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

As an illustration, in Fig. 3 we have plotted both sides of the latter equation in the case $m = 2$ and $N = 20$.

In view of the approximate relation (52), it is of interest to evaluate the mean and standard deviation of the energy in closed form for arbitrary values of N and m. This can be done in essentially the same way as for the BC_N chain (3), using the formulas for the traces of the spin operators S_{ij}, S_i and $\tilde{S}_{ij}$ in Ref. [31]. Indeed, setting

$$h_{ij} = (\xi_i - \xi_j)^{-2}, \qquad \tilde{h}_{ij} = (\xi_i + \xi_j)^{-2},$$

the mean energy is given by

$$\mu = m^{-N} \operatorname{tr} \mathcal{H} = \left(1 + \frac{1}{m}\right) \sum_{i \neq j} (h_{ij} + \tilde{h}_{ij}).$$

The sum in the r.h.s. of the previous equation is clearly half the maximum energy $\mathcal{E}_{\max}$ of the Hamiltonian (9), so that by Eq. (61) below we have

$$\mu = \frac{1}{2}\left(1 + \frac{1}{m}\right) N(N-1). \tag{53}$$

Similarly, the variance of the energy is given by

$$\sigma^2 = \frac{\operatorname{tr}(\mathcal{H}^2)}{m^N} - \mu^2 = 2\left(1 - \frac{1}{m^2}\right) \sum_{i \neq j} \left(h_{ij}^2 + \tilde{h}_{ij}^2\right) - \frac{4}{m^2}(1-p) \sum_{i \neq j} h_{ij} \tilde{h}_{ij},$$

where p is the parity of m, and we have used Eq. (A6) in Ref. [30]. From Eqs. (A8), (A9) and (A12) of the latter reference with $\beta = 0$, one easily obtains

$$\sigma^2 = \frac{1}{36}\left(1 - \frac{1}{m^2}\right) N(N-1)(4N+1) - \frac{1}{4m^2}(1-p) N(N-1). \tag{54}$$

With the help of the above expressions for μ and σ, we can show that Eq. (46) is compatible with the fact that the level densities of the three chains $\mathcal{H}$ and $\mathcal{H}^{(\mathrm{B})}$ with $\epsilon = \pm 1$ are approximately Gaussian for large N. Indeed, writing (46) as

$$\left(1 + q^N\right) \mathcal{Z}(T) = \mathcal{Z}_+^{(\mathrm{B})}(T) + \mathcal{Z}_-^{(\mathrm{B})}(T) \tag{55}$$

we see that the l.h.s. of (55) represents the superposition of the spectrum of the D_N chain (9) and its translation by N, whose level density tends to the sum of the Gaussian $g(\mathcal{E})$ in (51) and its translate $g(\mathcal{E}-N)$ as $N\to\infty$. But in this limit we have $N\ll\sigma=O(N^{3/2})$, so that $g(\mathcal{E})+g(\mathcal{E}-N)\simeq 2g(\mathcal{E})$. Similarly, the r.h.s. of Eq. (55) is the partition function of the superposition of the spectra of the chain Hamiltonians (3) with $\epsilon=\pm 1$, whose level density for large N is approximately the sum of two Gaussians with the same standard deviation as (51) and mean equal to $\mu+\frac{N}{2}(1-\frac{\epsilon p}{m})$, cf. Ref. [30]. Hence as $N\to\infty$ the level density of the r.h.s. of (55) is approximately given by

$$g\left(\mathcal{E}-\frac{N}{2}\left(1+\frac{p}{m}\right)\right)+g\left(\mathcal{E}-\frac{N}{2}\left(1-\frac{p}{m}\right)\right)\simeq 2g(\mathcal{E}),$$

as the l.h.s.

Let us consider now the distribution of the spacings between consecutive levels in the "unfolded" spectrum. Recall [44], to begin with, that the unfolding of the levels $\mathcal{E}_i$ of a spectrum is the mapping $\mathcal{E}_i\mapsto\eta_i\equiv\eta(\mathcal{E}_i)$, where $\eta(\mathcal{E})$ is the continuous part of the cumulative level density

$$F(\mathcal{E})\equiv\int_{-\infty}^{\mathcal{E}} f(\mathcal{E}')\,\mathrm{d}\mathcal{E}'=m^{-N}\sum_{i;\mathcal{E}_i\leqslant\mathcal{E}} d_i.$$

The unfolding mapping makes it possible to compare different spectra in a consistent way, since the unfolded spectrum $\{\eta_i\}_{i=1}^{L}$ can be shown to be uniformly distributed regardless of the initial level density. In our case, by the above discussion we can take $\eta(\mathcal{E})$ as the cumulative Gaussian density (51), namely

$$\eta(\mathcal{E})=\int_{-\infty}^{\mathcal{E}} g(\mathcal{E}')\,\mathrm{d}\mathcal{E}'=\frac{1}{2}\left[1+\mathrm{erf}\left(\frac{\mathcal{E}-\mu}{\sqrt{2}\sigma}\right)\right]. \tag{56}$$

One then defines the normalized spacings

$$s_i=(\eta_{i+1}-\eta_i)/\Delta,\quad i=1,\ldots,L-1,$$

where $\Delta\equiv(\eta_L-\eta_1)/(L-1)$ is the mean spacing of the unfolded energies, so that $\{s_i\}_{i=1}^{L-1}$ has unit mean. According to a well-known conjecture of Berry and Tabor, for a quantum integrable system the density $p(s)$ of normalized spacings should be given by Poisson's law $p(s)=\mathrm{e}^{-s}$. By contrast, for a system whose classical counterpart is chaotic, it is generally believed that the spacings distribution follows instead Wigner's law $p(s)=(\pi s/2)\exp(-\pi s^2/4)$, typical of the Gaussian ensembles in random matrix theory [44].

We shall now see that the spacings distribution of the PF chain of D_N type (9) follows neither Poisson's nor Wigner's law, as is the case for all spin chains of HS type studied so far [29,30,33–35]. More precisely, we will show that the cumulative spacings distribution $P(s)\equiv\int_0^s p(s')\,\mathrm{d}s'$ is approximately given by

$$P(s)\simeq 1-\frac{2}{\sqrt{\pi}s_{\max}}\sqrt{\log\left(\frac{s_{\max}}{s}\right)}, \tag{57}$$

where $s_{\max}$ is the maximum spacing. In fact, as proved in Ref. [30], the previous approximation necessarily holds for *any* spectrum $\mathcal{E}_{\min}\equiv\mathcal{E}_1<\cdots<\mathcal{E}_L\equiv\mathcal{E}_{\max}$ satisfying the following conditions:

(i) The energies are equally spaced, i.e., $\mathcal{E}_{i+1} - \mathcal{E}_i = \delta\mathcal{E}$ for $i = 1, \ldots, L-1$.
(ii) The level density (normalized to unity) is approximately given by the Gaussian law (51).
(iii) $\mathcal{E}_{\max} - \mu, \mu - \mathcal{E}_{\min} \gg \sigma$.
(iv) $\mathcal{E}_{\min}$ and $\mathcal{E}_{\max}$ are approximately symmetric with respect to μ, namely $|\mathcal{E}_{\min} + \mathcal{E}_{\max} - 2\mu| \ll \mathcal{E}_{\max} - \mathcal{E}_{\min}$.

Moreover, when these conditions are satisfied the maximum spacing can be estimated with great accuracy as

$$s_{\max} = \frac{\mathcal{E}_{\max} - \mathcal{E}_{\min}}{\sqrt{2\pi}\sigma}. \tag{58}$$

It should also be noted that Eq. (57) is valid only for spacings $s \in [s_0, s_{\max}]$, where

$$s_0 = s_{\max} \mathrm{e}^{-\frac{\pi}{4} s_{\max}^2} \ll s_{\max} \tag{59}$$

is the unique zero of the r.h.s. of (57) (the inequality in (59) follows easily from condition (iii) and Eq. (58)).

We shall next check that conditions (i)–(iv) above are indeed satisfied by the spectrum of the chain (9) when $N \gg 1$. In fact, we already known that conditions (i) (with $\delta\mathcal{E} = 1$) and (ii) hold. In order to verify condition (iii), we first need to compute the maximum and minimum energies $\mathcal{E}_{\max}$ and $\mathcal{E}_{\min}$. The maximum energy is clearly

$$\mathcal{E}_{\max} = 2 \sum_{i \neq j} \big[(\xi_i - \xi_j)^{-2} + (\xi_i + \xi_j)^{-2} \big], \tag{60}$$

whose corresponding eigenvectors are the spin states symmetric under permutations and with parity ± 1 under spin reversals. Since $\mathcal{E}_{\max}$ is independent of m, it is most easily computed for the spin-$\frac{1}{2}$ chain, whose spectrum is explicitly given in Eq. (40). We thus obtain

$$\mathcal{E}_{\max} = N(N-1). \tag{61}$$

As to the minimum energy, Eq. (55) implies that

$$\mathcal{E}_{\min} = \min\big(\mathcal{E}^{(\mathrm{B})}_{\min,-}, \mathcal{E}^{(\mathrm{B})}_{\min,+}\big),$$

where the minimum energies $\mathcal{E}^{(\mathrm{B})}_{\min,\epsilon}$ of the BC_N chain (3) were computed in Ref. [30]. From Eqs. (B1)–(B2) of the latter reference it easily follows that $\mathcal{E}^{(\mathrm{B})}_{\min,+} \leqslant \mathcal{E}^{(\mathrm{B})}_{\min,-}$, so that

$$\mathcal{E}_{\min} = \frac{N^2}{m} - \frac{N}{2}\left(1 + \frac{p}{m}\right) + \frac{1}{2m}(m + p - 2l)\big(l - mp\theta(2l - m - 1)\big), \tag{62}$$

with

$$l = N \mod \frac{m}{2}(1+p).$$

From Eqs. (53), (54), (61) and (62) it immediately follows that $(\mathcal{E}_{\max} - \mu)/\sigma$ and $(\mathcal{E}_{\min} - \mu)/\sigma$ are both $O(N^{1/2})$ as $N \to \infty$, so that condition (iii) is also satisfied. Finally, from the latter equations it also follows that $\mathcal{E}_{\min} + \mathcal{E}_{\max} - 2\mu$ is at most $O(N)$ while $\mathcal{E}_{\max} - \mathcal{E}_{\min} = O(N^2)$, which proves condition (iv).

The previous argument shows that the cumulative spacings distribution of the D_N chain (9) should be well approximated by the r.h.s. of Eq. (57) when N is sufficiently large. We have

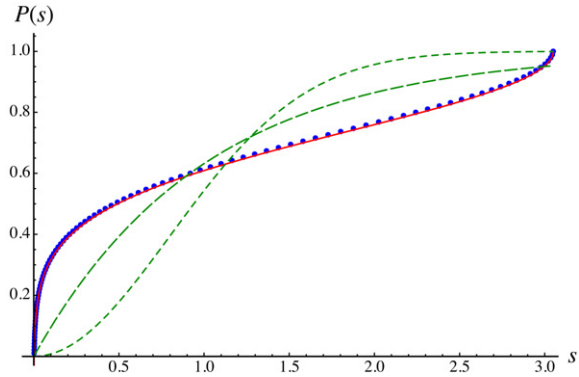

Fig. 4. Cumulative spacings distribution $P(s)$ and its approximation (57) (continuous red line) for $N = 20$ and $m = 2$. For convenience, we have also represented Poisson's (green, long dashes) and Wigner's (green, short dashes) cumulative distributions. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

verified that (57) is indeed in excellent agreement with the numerical data for many different values of N and m. For instance, in the case $N = 20$ and $m = 2$ presented in Fig. 4 the root mean square error (normalized to the mean) of the adjustment of $P(s)$ to the r.h.s. of Eq. (57) is 1.03×10^{-2}, and this error decreases to 4.69×10^{-4} when $N = 100$. It should be stressed that the approximation (57) contains no free parameters, since the maximum spacing $s_{\max}$ is completely determined as a function of N and m by Eqs. (54), (58), (61) and (62). In fact, from the latter equations it immediately follows that for large N the maximum spacing is asymptotically given by

$$s_{\max} \simeq \frac{3}{\sqrt{2\pi}} \sqrt{\frac{m-1}{m+1}} N^{1/2} + O\big(N^{-1/2}\big), \tag{63}$$

as for the (non-supersymmetric) PF chains of BC_N type [30].

5. The ferromagnetic case

The ferromagnetic spin chain of D_N type with Hamiltonian

$$\mathcal{H}_{\mathrm{F}} = \sum_{i \neq j} \left[\frac{1 - S_{ij}}{(\xi_i - \xi_j)^2} + \frac{1 - \tilde{S}_{ij}}{(\xi_i + \xi_j)^2} \right] \tag{64}$$

and its corresponding spin model

$$H_{\mathrm{F}} = -\sum_i \partial_{x_i}^2 + \frac{a^2}{4} r^2 + a \sum_{i \neq j} \left[\frac{a - S_{ij}}{(x_{ij}^-)^2} + \frac{a - \tilde{S}_{ij}}{(x_{ij}^+)^2} \right]$$

can be studied in much the same way as their antiferromagnetic versions (4)–(9). Since now

$$H_{\mathrm{F}} = H'|_{K_{ij} \to S_{ij}, K_i \to \epsilon S_i}, \tag{65}$$

we must replace the operator Λ^ϵ in Eq. (26) by the projector $\Lambda_{\mathrm{s}}^\epsilon$ onto states *symmetric* under simultaneous permutations of the particles' spatial and spin coordinates, and with parity ϵ under

sign reversal of coordinates and spin. Hence condition (ii) above for the new basis states

$$\tilde{\psi}^{\epsilon}_{\mathbf{n},\mathbf{s}} \equiv \Lambda^{\epsilon}_{\mathrm{s}}\big(\phi_{\mathbf{n}}(\mathbf{x})|\mathbf{s}\rangle\big), \quad \epsilon = \pm 1,$$

should now read

(ii′) $s_i \geqslant s_j$ whenever $n_i = n_j$ and $i < j$.

As a result, the degeneracy factors $d(\boldsymbol{\nu})$ and $d_s(\boldsymbol{\nu})$ in Eqs. (34) and (41) should be replaced by their "bosonic" versions

$$d_{\mathrm{F}}(\boldsymbol{\nu}) = \prod_{i=1}^{r} \binom{\frac{m}{2} + \nu_i - 1}{\nu_i}$$

and

$$d_{\mathrm{F},s}(\boldsymbol{\nu}) = d^{-}_{\mathrm{F},s}(\boldsymbol{\nu}) + d^{+}_{\mathrm{F},s}(\boldsymbol{\nu}),$$

where

$$d^{\epsilon}_{\mathrm{F},s}(\boldsymbol{\nu}) = \prod_{i=1}^{s} \binom{\frac{m+\epsilon}{2} + \nu_i - 1}{\nu_i} \cdot \prod_{i=s+1}^{r} \binom{\frac{m-\epsilon}{2} + \nu_i - 1}{\nu_i}.$$

Therefore the partition function of the ferromagnetic su(m) PF chain of D_N type (64) is still given by Eq. (36) (for even m) or (45) (for odd m), but with $d(\boldsymbol{\nu})$ and $d_s(\boldsymbol{\nu})$ replaced respectively by $d_{\mathrm{F}}(\boldsymbol{\nu})$ and $d_{\mathrm{F},s}(\boldsymbol{\nu})$.

On the other hand, the chains (9) and (64) are obviously related by

$$\mathcal{H}_{\mathrm{F}} + \mathcal{H} = 2\sum_{i\neq j}\big[(\xi_i - \xi_j)^{-2} + (\xi_i + \xi_j)^{-2}\big] = N(N-1), \tag{66}$$

where we have used Eqs. (60)–(61). Thus the partition functions $\mathcal{Z}$ and $\mathcal{Z}_{\mathrm{F}}$ of $\mathcal{H}$ and $\mathcal{H}_{\mathrm{F}}$ satisfy the remarkable identity

$$\mathcal{Z}_{\mathrm{F}}(q) = q^{N(N-1)}\mathcal{Z}\big(q^{-1}\big). \tag{67}$$

This is a manifestation of the boson–fermion duality discussed in detail in Ref. [45] for the su($m|n$) supersymmetric HS spin chain, since the ferromagnetic (respectively antiferromagnetic) chain can be regarded as purely bosonic (respectively fermionic). For instance, using the latter identity and Eq. (39) we easily obtain the following expression for the partition function of the ferromagnetic spin-$\frac{1}{2}$ chain:

$$\mathcal{Z}_{\mathrm{F}}(T) = 2\prod_{i=1}^{N-1}\big(1 + q^i\big), \quad m = 2. \tag{68}$$

With the help of the duality relation (67) and the elementary q-number identity

$$\big(q^{-1}\big)_K = (-1)^K q^{-\frac{1}{2}K(K+1)}(q)_K$$

it is straightforward to derive the analogs of Eqs. (48) and (49) for the ferromagnetic chain (64). Calling $\mathcal{Z}^{(A)}_{K,\mathrm{F}}(q;n)$ the partition function of the su(n) ferromagnetic PF chain of type A for

K spins, given by [32]

$$\mathcal{Z}^{(A)}_{K,\mathrm{F}}(q;n) = \sum_{M_1+\cdots+M_n=K} \frac{(q)_K}{(q)_{M_1}\cdots(q)_{M_n}},$$

we obtain in this way

$$\mathcal{Z}(T) = 2\mathcal{Z}^{(A)}_{N,\mathrm{F}}\left(q;\frac{m}{2}\right)\prod_{i=1}^{N-1}\left(1+q^i\right)$$

for even m, and

$$\mathcal{Z}(T) = \sum_{K=0}^{N}\prod_{i=K}^{N-1}\left(1+q^i\right)\cdot\begin{bmatrix}N\\K\end{bmatrix}_q \mathcal{Z}^{(A)}_{N-K,\mathrm{F}}\left(q;\frac{m-1}{2}\right)$$

for odd m. Finally, from the duality relation (66) it clearly follows that the statistical properties of the spectrum of $\mathcal{H}_{\mathrm{F}}$ are identical to those of $\mathcal{H}$, namely when N is large enough the level density is approximately Gaussian, and the spacings distribution follows Eq. (57) with great accuracy.

6. Concluding remarks

In this article we compute the spectrum and partition function of the su(m) spin Calogero model of D_N type and its associated PF spin chain. Even though the Hamiltonian of the D_N-type su(m) spin Calogero model (4) can be obtained by taking the $b \to 0$ limit of its BC_N counterpart (1), it is found that the Hilbert spaces of these two models differ in a significant way. More precisely, we establish that the Hilbert space of the spin Calogero model of D_N type can be expressed as the direct sum of the Hilbert spaces of two different BC_N models with opposite "chiralities". Consequently, the D_N-type spin Calogero model turns out to be a singular limit of its BC_N counterpart, which explains why the spectra of these models are different.

The distinction between the D_N- and BC_N-type models becomes even more apparent when we consider the freezing limit of the spin dynamical models, which leads to the associated PF spin chains. As shown in Eq. (15), in contrast to what happens with the dynamical models, the Hamiltonian of the PF chain of D_N type cannot be obtained as a limiting case of its BC_N counterpart. Therefore, it is natural to expect that the spectra of the PF spin chains associated with the D_N and BC_N root systems are essentially different. We show that this is indeed the case by exactly computing the partition function of the D_N-type chain with the help of the freezing trick. Using this partition function, we study several statistical properties of the chain's spectrum, like the level density and the distribution of spacings between consecutive energies. We also establish an interesting duality relation between the spectra of the ferromagnetic and the antiferromagnetic PF chains of D_N type.

From the above discussion it is clear that the connection between the Hamiltonians of the spin Calogero models associated with the D_N and BC_N root systems is a subtle one, and cannot be extended in a straightforward way to the corresponding Hilbert spaces. Indeed, it is well known that the Hilbert space of the spin Calogero model of either A_{N-1} or BC_N type can be constructed from the Hilbert space of an appropriate auxiliary operator by using a single projector. It turns out, however, that *two* independent projectors are needed in order to construct the Hilbert space of the D_N-type spin Calogero model from that of the corresponding auxiliary operator. This unique property of the D_N-type spin Calogero model plays a central role in our computation of

the spectra of this integrable system and of its associated spin chain. In fact, the relation between the dynamical models and associated chains of D_N and BC_N type is even more intriguing in the trigonometric case (i.e., for the spin Sutherland model and the Haldane–Shastry chain), as we shall discuss in a forthcoming paper.

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