



An exactly solvable supersymmetric spin chain of BC_N type

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Abstract

We construct a new exactly solvable supersymmetric spin chain related to the BC_N extended root system, which includes as a particular case the BC_N version of the Polychronakos–Frahm spin chain. We also introduce a supersymmetric spin dynamical model of Calogero type which yields the new chain in the large coupling limit. This connection is exploited to derive two different closed-form expressions for the chain’s partition function by means of Polychronakos’s freezing trick. We establish a boson–fermion duality relation for the new chain’s spectrum, which is in fact valid for a large class of (not necessarily integrable) spin chains of BC_N type. The exact expressions for the partition function are also used to study the chain’s spectrum as a whole, showing that the level density is normally distributed even for a moderately large number of particles. We also determine a simple analytic approximation to the distribution of normalized spacings between consecutive levels which fits the numerical data with remarkable accuracy. Our results provide further evidence that spin chains of Haldane–Shastry type are exceptional integrable models, in the sense that their spacings distribution is not Poissonian, as posited by the Berry–Tabor conjecture for “generic” quantum integrable systems.

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1. Introduction

In the last few years exactly solvable (or integrable) supersymmetric spin chains and their associated dynamical models have been the subject of extensive research in connection with different topics of current interest, such as the theory of strongly correlated systems [1,2] or the AdS-CFT correspondence [3]. Among these models, the supersymmetric versions of the celebrated Haldane–Shastry chain [4,5] and its rational counterpart proposed by Polychronakos [6] and Frahm [7] occupy a distinguished position, due to the rich mathematical structures at the heart of their highly solvable character. The Haldane–Shastry (HS) chain was introduced in an attempt to construct a simple one-dimensional model whose ground state coincided with Gutzwiller’s variational wave function for the Hubbard model in the limit of large on-site interaction [8–10]. It can also be obtained in this limit from the Hubbard model with long-range hopping studied in Ref. [11], in the half-filling regime. The original HS chain describes N spin $1/2$ particles equally spaced on a circle, with pairwise interactions inversely proportional to the square of the chord distance. Its Hamiltonian can be written as

$$\mathcal{H} = J_0 \sum_{i \neq j} \sin(\vartheta_i - \vartheta_j)^{-2} \mathbf{J}_i \cdot \mathbf{J}_j, \quad \vartheta_i \equiv \frac{i\pi}{N}, \tag{1}$$

where $\mathbf{J}_i \equiv \frac{1}{2}(\sigma_i^x, \sigma_i^y, \sigma_i^z)$, σ_i^α is a Pauli matrix at site i , and the summation indices run from 1 to N (as always hereafter, unless otherwise stated). A natural generalization of this chain to $\mathfrak{su}(m)$ spin [12,13] is obtained by taking $\mathbf{J}_i = (J_i^1, \dots, J_i^{m^2-1})$, where $\{J_i^\alpha\}$ are the generators of the fundamental representation of $\mathfrak{su}(m)$ at site i . In fact, with the usual normalization $\text{tr}(J_k^\alpha J_k^\gamma) = \frac{1}{2}\delta^{\alpha\gamma}$ we have

$$\mathbf{J}_i \cdot \mathbf{J}_j = \frac{1}{2} \left(S_{ij} - \frac{1}{m} \right),$$

where S_{ij} is the operator permuting the i th and j th spins, so that the $\mathfrak{su}(m)$ spin Hamiltonian (1) is a linear combination of spin permutation operators.

The spectrum of the spin $1/2$ HS Hamiltonian was numerically analyzed in the original papers of Haldane and Shastry. The more general $\mathfrak{su}(m)$ spin case was studied in a subsequent publication by Haldane et al. [14], who empirically found a complete description of the spectrum and explained its high degeneracy by an underlying $\mathcal{Y}(\mathfrak{sl}_m)$ Yangian symmetry. These results were rigorously established in Ref. [15] by constructing a transfer matrix using the Dunkl operators [16,17] of the (trigonometric) Sutherland dynamical model [18,19]. Although this approach yields an explicit formula for the energies in terms of the so-called *motifs*, the computation of their corresponding degeneracies becomes quite cumbersome when $m > 2$. This is probably the reason why the partition function of this chain was computed only very recently [20], using a procedure known as Polychronakos’s *freezing trick* which does not rely on the explicit knowledge of the spectrum [6,21].

The key idea behind the freezing trick is exploiting the connection between the HS chain and the Sutherland spin dynamical model in the strong coupling limit. Indeed, in this limit the particles in the latter model “freeze” at the coordinates of the (unique) equilibrium of the scalar part of the potential, which are exactly the chain sites. Thus, in this limit the dynamical and the spin degrees of freedom decouple, so that the spectrum of the spin Sutherland model becomes approximately the sum of the spectra of the scalar Sutherland model and the HS chain. This observation yields an explicit formula for the partition function of the HS spin chain as the strong

coupling limit of the quotient of the partition functions of the spin and the scalar Sutherland models, both of which can be computed in this limit. In fact, this method was first applied by Polychronakos to evaluate the partition function of the $su(m)$ spin chain associated with the spin Calogero (rational) model [22,23], usually known in the literature as the Polychronakos–Frahm (PF) chain, whose Hamiltonian (in the antiferromagnetic case) is given by

$$\mathcal{H} = \sum_{i < j} (\zeta_i - \zeta_j)^{-2} (1 + S_{ij}). \quad (2)$$

The sites ζ_i of this chain are the coordinates of the equilibrium of the scalar part of the Calogero potential, which coincide with the zeros of the Hermite polynomial of degree N [24].

Both the HS and the PF chains admit a natural $su(m|n)$ supersymmetric version, in which there are m bosonic and n fermionic degrees of freedom at each site [25–29]. In practice, the Hamiltonians of these chains are obtained by replacing the spin permutation operators S_{ij} by suitable supersymmetric generalizations thereof (cf. Eq. (4) below). The partition functions of the $su(m|n)$ PF and HS chains have also been computed in closed form using the freezing trick method outlined above [25,27]. These partition functions were then used to establish a remarkable boson–fermion duality relation for the spectrum of each of these chains [25,28].

The original HS and PF chains (and their supersymmetric versions) are associated with the A_{N-1} root system, since the interaction between any two spins depends only on the difference of their site coordinates. It is also of interest to consider generalizations of these chains to the BC_N extended root system, in which the spins interact not only among themselves but also with their mirror images with respect to a reflecting end located at the origin. In the non-supersymmetric case, the HS chain of BC_N type was first discussed in Ref. [30], and its partition function was evaluated by means of the freezing trick in Ref. [31]. Likewise, the partition function of the $su(m)$ PF chain of BC_N type introduced in [32] was recently computed in closed form by the authors [33]. To the best of our knowledge, however, the supersymmetric counterparts of both the HS and PF chains of BC_N type have not been studied so far. One of the main purposes of this paper is precisely to fill this gap in the case of the PF chain. More precisely, we shall apply the freezing trick to compute the partition function of this chain in closed form, relating it to the corresponding partition function of the $su(m|n)$ PF chain of type A . We shall also extend the boson–fermion duality mentioned above to a large class of supersymmetric spin chains of BC_N type which includes the PF chain as a particular case.

The spectra of the non-supersymmetric spin chains of Haldane–Shastry type whose partition function has been explicitly computed share some statistical properties that we shall now discuss. In the first place, the density of energy levels follows the Gaussian law with remarkable accuracy even for a moderately large number of particles [20,31,33,34]. Secondly (except for the HS chain of BC_N type, which has not yet been studied in this respect), the density of spacings $p(s)$ between consecutive (normalized) levels follows a characteristic distribution essentially different from Poisson’s law $p(s) = e^{-s}$. This fact is somewhat surprising, in view of the integrability of these chains [6,32] and the long-standing conjecture of Berry and Tabor [35], according to which the spacings distribution of a “generic” quantum integrable system should be Poissonian. The validity of this conjecture has been verified for a number of important integrable systems, including the Heisenberg chain, the t – J model, the Hubbard model [36], and the chiral Potts model [37]. On the other hand, the non-Poissonian character of the spacings distribution has recently been observed for the $su(m|n)$ HS chain of A_{N-1} type [27], whose density of spacings is qualitatively similar to that of the non-supersymmetric HS and PF chains. Another important aim of this paper is the study of the level density and the spacings distribution for the $su(m|n)$ PF

chain of BC_N type. Our main conclusion in this respect is that this chain behaves in essentially the same way as other chains of Haldane–Shastry type previously studied. In other words, when the number of spins is sufficiently large the level density follows with great accuracy the Gaussian distribution, and the spacings distribution also deviates substantially from Poisson’s law. As a matter of fact, we shall see that the spacings density is well approximated by the same “square root of a logarithm” distribution as the original HS and PF chains, and the $su(m)$ PF chain of BC_N type [33,34]. This lends further support to the fact that spin chains of Haldane–Shastry type are exceptional integrable systems from the point of view of the Berry–Tabor conjecture.

The paper is organized as follows. In Section 2 we recall the definition of the supersymmetric spin permutation operators [25], and introduce new supersymmetric spin reversal operators. With the help of these operators, we construct the $su(m|n)$ PF chain of BC_N type and explain how the freezing trick can be used to compute its partition function from that of its associated spin dynamical model. In Section 3 we compute the spectrum of the latter model, which is then used in Section 4 to evaluate the chain’s partition function. Using the grand canonical partition function of the spin dynamical model, in Section 5 we obtain an alternative expression for the chain’s partition function in terms of that of the $su(m|n)$ PF chain of type A . This expression turns out to be particularly efficient for the computation of the chain’s spectrum for a relatively large number of spins. Section 6 is devoted to generalizing to spin chains of BC_N type the boson–fermion duality uncovered in Refs. [25,28]. In the last section we use the explicit formulas for the partition function obtained in Section 5 to study the asymptotic behavior of the level density and the spacings distribution when the number of particles is very large. The paper also includes two appendices, the first one being a collection of several useful q -number identities, and the second a detailed derivation of the exact formulas for the mean and standard deviation of the chain’s energy used in Section 7.

2. The model

Let us begin by defining the Hilbert space of the internal degrees of freedom and the action therein of the $su(m|n)$ permutation and spin reversal operators $S_{ij}^{(m|n)}$ and $S_i^{\epsilon\epsilon'}$. There are two species of particles, “bosons” and “fermions” (the reason for this terminology shall be apparent later), with m and n denoting respectively the number of bosonic and fermionic degrees of freedom. More formally, the internal Hilbert space $\Sigma^{(m|n)} \approx (\mathbb{C}^{m+n})^{\otimes N}$ is spanned by states of the form $|s_1\rangle \otimes \cdots \otimes |s_N\rangle \equiv |s_1, \dots, s_N\rangle \equiv |\mathbf{s}\rangle$, where s_i is a two-vector (s_i^1, s_i^2) with

$$\begin{aligned} \text{(i)} \quad s_i^2 &\equiv \pi(s_i) = \begin{cases} 0, & \text{bosons,} \\ 1, & \text{fermions,} \end{cases} \\ \text{(ii)} \quad s_i^1 &\in \begin{cases} \{-\frac{m-1}{2}, -\frac{m-1}{2} + 1, \dots, \frac{m-1}{2}\}, & \pi(s_i) = 0, \\ \{-\frac{n-1}{2}, -\frac{n-1}{2} + 1, \dots, \frac{n-1}{2}\}, & \pi(s_i) = 1. \end{cases} \end{aligned}$$

We shall say that s_i^1 is the *value* of the “spin” s_i , while $s_i^2 \equiv \pi(s_i)$ is its *type* (bosonic or fermionic). As usual, the inner product on $\Sigma^{(m|n)}$ is defined in such a way that the set of vectors $|\mathbf{s}\rangle$ forms an orthonormal basis.

The action of the *spin reversal operators* $S_i^{\epsilon\epsilon'}$ on this basis is given by

$$S_i^{\epsilon\epsilon'} |s_1, \dots, s_i, \dots, s_N\rangle = \rho(s_i) |s_1, \dots, s_i^-, \dots, s_N\rangle, \tag{3}$$

where $s_i^- = (-s_i^1, s_i^2)$,

$$\rho(s_i) = \begin{cases} \epsilon, & \pi(s_i) = 0, \\ \epsilon', & \pi(s_i) = 1, \end{cases}$$

and $\epsilon, \epsilon' = \pm 1$ are two independent signs. In other words, $S_i^{\epsilon\epsilon'}$ reverses the value of the i th spin without affecting its type, and multiplies the state by the sign $\rho(s_i)$.

The definition of the spin permutation operators $S_{ij}^{(m|n)} \equiv S_{ji}^{(m|n)}$ is a bit more involved, namely (assuming that $i < j$)

$$S_{ij}^{(m|n)} | \dots, s_i, \dots, s_j, \dots \rangle = (-1)^{\alpha_{ij}(\mathbf{s})} | \dots, s_j, \dots, s_i, \dots \rangle, \tag{4}$$

where

$$\alpha_{ij}(\mathbf{s}) = \pi(s_i)\pi(s_j) + (\pi(s_i) + \pi(s_j)) f_{ij}(\mathbf{s}),$$

and

$$f_{ij}(\mathbf{s}) = \sum_{k=i+1}^{j-1} \pi(s_k)$$

is the number of fermions in between the i th and j th spins. In other words, the sign $(-1)^{\alpha_{ij}(\mathbf{s})}$ is -1 when either both s_i and s_j are fermionic, or s_i and s_j are spins of different type with an odd number of fermionic spins between them. An equivalent way of defining the action of $S_{ij}^{(m|n)}$ is by requiring that $S_{i,i+1}^{(m|n)}$ introduce a factor of -1 precisely when s_i and s_{i+1} are fermionic. This clearly implies the previous (apparently more general) rule, since an arbitrary permutation is a product of permutations of consecutive elements.

The Hamiltonian of the supersymmetric $su(m|n)$ PF chain of BC_N type is defined by

$$\mathcal{H}_{\epsilon\epsilon'}^{(m|n)} = \sum_{i \neq j} \left[\frac{1 - S_{ij}^{(m|n)}}{(\xi_i - \xi_j)^2} + \frac{1 - \tilde{S}_{ij}^{(m|n)}}{(\xi_i + \xi_j)^2} \right] + \beta \sum_i \frac{1 - S_i^{\epsilon\epsilon'}}{\xi_i^2}, \tag{5}$$

where¹ $\tilde{S}_{ij}^{(m|n)} = S_i^{\epsilon\epsilon'} S_j^{\epsilon'\epsilon} S_{ij}^{(m|n)}$ and $\beta > 0$. The chain sites ξ_i are the same as those of the ordinary PF chain of BC_N type studied in Ref. [33], namely the coordinates of the (unique) equilibrium of the scalar potential

$$U(\mathbf{x}) = \sum_{i \neq j} \left[\frac{1}{(x_i - x_j)^2} + \frac{1}{(x_i + x_j)^2} \right] + \sum_i \frac{\beta^2}{x_i^2} + \frac{r^2}{4} \tag{6}$$

in the open set $C = \{\mathbf{x} \equiv (x_1, \dots, x_N) \mid 0 < x_1 < \dots < x_N\}$. It can be shown [32] that $\xi_i = \sqrt{2y_i}$, where y_i is the i th zero of the Laguerre polynomial $L_N^{\beta-1}$. Note also that the convention for the sign of the SUSY spin permutation operators in the definition of $\mathcal{H}_{\epsilon\epsilon'}^{(m|n)}$ is the opposite to that of Ref. [27], but (as we shall see in a moment) our choice is more consistent with the use of the names ‘‘bosonic’’ and ‘‘fermionic’’ for the two species of spins. Since

$$S_{ij}^{(m|0)} = S_{ij}, \quad S_{ij}^{(0|n)} = -S_{ij}, \quad S_i^{\epsilon\epsilon} = \epsilon S_i, \tag{7}$$

¹ Note that the operators $\tilde{S}_{ij}^{(m|n)}$ actually depend on ϵ and ϵ' , although for simplicity we have chosen not to make this dependence explicit in the notation.

where S_i is the operator reversing the i th spin, the purely bosonic (respectively purely fermionic) spin chain $\mathcal{H}_\epsilon^{(m|0)}$ (respectively $\mathcal{H}_\epsilon^{(0|n)}$) coincides with the ordinary ferromagnetic (respectively antiferromagnetic) PF chain of BC_N type [33].

The spin chain (5) is naturally related to the $su(m|n)$ spin dynamical model

$$H_{\epsilon\epsilon'}^{(m|n)} = -\Delta + a \sum_{i \neq j} \left[\frac{a - S_{ij}^{(m|n)}}{(x_{ij}^-)^2} + \frac{a - \tilde{S}_{ij}^{(m|n)}}{(x_{ij}^+)^2} \right] + b \sum_i \frac{b - S_i^{\epsilon\epsilon'}}{x_i^2} + \frac{a^2}{4} r^2, \tag{8}$$

where $x_{ij}^\pm = x_i \pm x_j$, $b = \beta a$ and $a > 1/2$, and to its scalar counterpart

$$H_{sc} = -\Delta + a(a - 1) \sum_{i \neq j} \left[\frac{1}{(x_{ij}^-)^2} + \frac{1}{(x_{ij}^+)^2} \right] + b(b - 1) \sum_i \frac{1}{x_i^2} + \frac{a^2}{4} r^2. \tag{9}$$

Indeed, we have

$$H_{\epsilon\epsilon'}^{(m|n)} = H_{sc} + a \tilde{\mathcal{H}}_{\epsilon\epsilon'}^{(m|n)}, \tag{10}$$

where $\tilde{\mathcal{H}}_{\epsilon\epsilon'}^{(m|n)}$ is obtained from the spin chain Hamiltonian (5) by replacing the sites ξ_i by the coordinates x_i . Since

$$H_{\epsilon\epsilon'}^{(m|n)} = -\Delta + a^2 U + O(a),$$

as a grows to infinity the particles tend to concentrate at the chain sites ξ_i , so that the dynamical and internal degrees of freedom decouple. By Eq. (10), in this limit the energies of the dynamical spin model (8) are approximately given by

$$E_{ij} \simeq E_i^{sc} + a \mathcal{E}_j, \quad a \gg 1, \tag{11}$$

where E_i^{sc} and \mathcal{E}_j are any two energies of the scalar Hamiltonian (9) and the spin chain (5). The latter formula cannot be used directly to deduce the spectrum of $\mathcal{H}_{\epsilon\epsilon'}^{(m|n)}$ from those of $H_{\epsilon\epsilon'}^{(m|n)}$ and H_{sc} , since it is not known *a priori* which eigenvalues of $H_{\epsilon\epsilon'}^{(m|n)}$ and H_{sc} combine to yield an eigenvalue of $\mathcal{H}_{\epsilon\epsilon'}^{(m|n)}$. However, Eq. (11) immediately leads to the exact “freezing trick” formula [21]

$$\mathcal{Z}_{\epsilon\epsilon'}^{(m|n)}(T) = \lim_{a \rightarrow \infty} \frac{\mathcal{Z}_{\epsilon\epsilon'}^{(m|n)}(aT)}{\mathcal{Z}_{sc}(aT)} \tag{12}$$

relating the partition functions $\mathcal{Z}_{\epsilon\epsilon'}^{(m|n)}$, $\mathcal{Z}_{\epsilon\epsilon'}^{(m|n)}$ and \mathcal{Z}_{sc} of $\mathcal{H}_{\epsilon\epsilon'}^{(m|n)}$, $H_{\epsilon\epsilon'}^{(m|n)}$ and H_{sc} , respectively. The partition function \mathcal{Z}_{sc} of the scalar model (9) was computed in Ref. [33]. We shall see in the next section that the spectrum of $H_{\epsilon\epsilon'}^{(m|n)}$ is quite simple, a fact that shall be exploited in Section 4 to explicitly compute its partition function. Using Eq. (12) we shall then obtain the partition function $\mathcal{Z}_{\epsilon\epsilon'}^{(m|n)}$ in closed form.

3. Spectrum of the spin dynamical model

The first step in the evaluation of the partition function of the spin chain (5) by means of the freezing trick relation (12) consists in determining the spectrum of the spin dynamical model (8).

To this end, we introduce the auxiliary “exchange Hamiltonian”

$$H' = -\Delta + \sum_{i \neq j} \left[\frac{a}{(x_{ij}^-)^2} (a - K_{ij}) + \frac{a}{(x_{ij}^+)^2} (a - \tilde{K}_{ij}) \right] + \sum_i \frac{b}{x_i^2} (b - K_i) + \frac{a^2}{4} r^2, \quad (13)$$

where K_{ij} is the operator permuting the i th and j th spatial coordinates, K_i reverses the sign of the i th coordinate, and $\tilde{K}_{ij} \equiv K_i K_j K_{ij}$. The rationale for introducing this operator is the fact that its restriction to states symmetric under the action of the operators

$$\Pi_{ij}^{(m|n)} = K_{ij} S_{ij}^{(m|n)}, \quad \Pi_i^{\epsilon\epsilon'} = K_i S_i^{\epsilon\epsilon'}, \quad i, j = 1, \dots, N, \quad (14)$$

coincides with that of the Hamiltonian $H_{\epsilon\epsilon'}^{(m|n)}$, as we shall next discuss. To begin with, note that all three sets of operators $\{K_i, K_{ij}\}$, $\{S_i^{\epsilon\epsilon'}, S_{ij}^{(m|n)}\}$ and $\{\Pi_i^{\epsilon\epsilon'}, \Pi_{ij}^{(m|n)}\}$ generate a realization of the Weyl group of BC_N type, namely they satisfy the commutation relations

$$\begin{aligned} K_{ij}^2 &= 1, & K_{ij} K_{jk} &= K_{ik} K_{ij} = K_{jk} K_{ik}, & K_{ij} K_{kl} &= K_{kl} K_{ij}, \\ K_i^2 &= 1, & K_i K_j &= K_j K_i, & K_{ij} K_k &= K_k K_{ij}, & K_{ij} K_j &= K_i K_{ij} \end{aligned} \quad (15)$$

(where i, j, k, l are distinct) and similarly for the other two sets. We shall denote by $\Lambda_{\epsilon\epsilon'}^{(m|n)}$ the projector on states totally symmetric under the action of the operators $\{\Pi_i^{\epsilon\epsilon'}, \Pi_{ij}^{(m|n)}\}$. For instance, if $N = 3$ we have

$$\begin{aligned} \Lambda_{\epsilon\epsilon'}^{(m|n)} &= \frac{1}{2^3} (1 + \Pi_1^{\epsilon\epsilon'}) (1 + \Pi_2^{\epsilon\epsilon'}) (1 + \Pi_3^{\epsilon\epsilon'}) \\ &\quad \times \frac{1}{3!} (1 + \Pi_{12}^{(m|n)} + \Pi_{13}^{(m|n)} + \Pi_{23}^{(m|n)} + \Pi_{12}^{(m|n)} \Pi_{13}^{(m|n)} + \Pi_{12}^{(m|n)} \Pi_{23}^{(m|n)}). \end{aligned}$$

In particular, $\Lambda_{\epsilon\epsilon}^{(m|0)}$ (respectively $\Lambda_{\epsilon\epsilon}^{(0|n)}$) projects onto states totally symmetric (respectively antisymmetric) under particle permutations and with parity ϵ with respect to simultaneous sign reversals of spatial coordinates and spins. From the definition of the total symmetrizer $\Lambda_{\epsilon\epsilon'}^{(m|n)}$ it follows that

$$\Pi_{ij}^{(m|n)} \Lambda_{\epsilon\epsilon'}^{(m|n)} = \Lambda_{\epsilon\epsilon'}^{(m|n)} \Pi_{ij}^{(m|n)} = \Lambda_{\epsilon\epsilon'}^{(m|n)}, \quad \Pi_i^{\epsilon\epsilon'} \Lambda_{\epsilon\epsilon'}^{(m|n)} = \Lambda_{\epsilon\epsilon'}^{(m|n)} \Pi_i^{\epsilon\epsilon'} = \Lambda_{\epsilon\epsilon'}^{(m|n)},$$

so that

$$S_{ij}^{(m|n)} \Lambda_{\epsilon\epsilon'}^{(m|n)} = K_{ij} \Lambda_{\epsilon\epsilon'}^{(m|n)}, \quad S_i^{\epsilon\epsilon'} \Lambda_{\epsilon\epsilon'}^{(m|n)} = K_i \Lambda_{\epsilon\epsilon'}^{(m|n)}. \quad (16)$$

Hence

$$H_{\epsilon\epsilon'}^{(m|n)} \Lambda_{\epsilon\epsilon'}^{(m|n)} = H' \Lambda_{\epsilon\epsilon'}^{(m|n)}, \quad (17)$$

which, as we shall see, is precisely the relation needed to deduce the spectrum of $H_{\epsilon\epsilon'}^{(m|n)}$ from that of H' .

Indeed, recall [33] that the operator H' is represented by an upper triangular matrix in the (non-orthonormal) basis with elements

$$\phi_{\mathbf{k}} = e^{-\frac{a}{4} r^2} \prod_i |x_i|^b \cdot \prod_{i < j} |x_i^2 - x_j^2|^a \cdot \prod_i x_i^{k_i}, \quad (18)$$

where

$$\mathbf{k} \equiv (k_1, \dots, k_N) \in (\mathbb{N} \cup \{0\})^N \equiv \mathbb{N}_0^N,$$

ordered according to the degree $|\mathbf{k}| \equiv \sum_i k_i$. In other words,

$$H' \phi_{\mathbf{k}} = E'_{\mathbf{k}} \phi_{\mathbf{k}} + \sum_{|\mathbf{k}'| < |\mathbf{k}|} c_{\mathbf{k}'\mathbf{k}} \phi_{\mathbf{k}'},$$

where the eigenvalues are given by

$$E'_{\mathbf{k}} = a|\mathbf{k}| + E_0,$$

with $E_0 = Na(b + a(N - 1) + \frac{1}{2})$.

Due to the impenetrable nature of the singularities of the Hamiltonian (8), its Hilbert space is the set $L^2_0(C) \otimes \Sigma^{(m|n)}$ of spin functions square integrable on the open set C and vanishing sufficiently fast on the singular hyperplanes $x_i \pm x_j = 0$. However, it can be shown that $H_{\epsilon\epsilon'}^{(m|n)}$ is equivalent to its symmetric extension (under $\Pi_{ij}^{(m|n)}$ and $\Pi_i^{\epsilon\epsilon'}$) to the space $L^2_0(\mathbb{R}^N) \otimes \Sigma^{(m|n)}$. The states

$$\psi_{\mathbf{k},\mathbf{s}} = \Lambda_{\epsilon\epsilon'}^{(m|n)}(\phi_{\mathbf{k}}(\mathbf{x})|\mathbf{s}), \tag{19}$$

form a basis of this Hilbert space provided that the following conditions hold:

- (i) $k_1 \geq \dots \geq k_N$,
- (ii) if $k_i = k_j$ and $i < j$, then:
 - (ii.a) $\pi(s_i) \leq \pi(s_j)$,
 - (ii.b) $s_i^1 \geq s_j^1 + \pi(s_j)$ whenever $\pi(s_i) = \pi(s_j)$,
- (iii) $s_i^1 \geq 0$ for all i , and $s_i^1 > 0$ if $(-1)^{k_i} \rho(s_i) = -1$.

The last two conditions, which differ from the analogous ones for the $su(m)$ case [33], deserve further remark. Condition (ii.a) simply states that acting with appropriate generalized total permutation operators $\Pi_{ij}^{(m|n)}$ we can first reorder the spins so that within each *sector* of \mathbf{s} (spin components corresponding to a maximal sequence of equal components of \mathbf{k}) bosons precede fermions. Furthermore (condition (ii.b)), the bosonic spin components can be arranged in non-increasing order, while for the fermionic ones we can enforce strictly decreasing order since in this case the state vanishes if $s_i = s_j$ by antisymmetry. (This is, by the way, the justification for the names “bosonic” and “fermionic” used for the two species of spins.) Finally, condition (iii) stems from the fact that the spin functions (19) must be even under $\Pi_i^{\epsilon\epsilon'}$, while if $s_i^1 = 0$ we have

$$\Pi_i^{\epsilon\epsilon'} \psi_{\mathbf{k},\mathbf{s}} = \Lambda_{\epsilon\epsilon'}^{(m|n)}(\Pi_i^{\epsilon\epsilon'} \phi_{\mathbf{k}}(\mathbf{x})|\mathbf{s}) = (-1)^{k_i} \rho(s_i) \psi_{\mathbf{k},\mathbf{s}},$$

so that $(-1)^{k_i} \rho(s_i) = 1$. A direct consequence of this condition is that the number of bosonic spin values of s_j compatible with rule (iii) is given by

$$\bar{m}(k_j) = \begin{cases} \frac{m}{2}, & m \text{ even,} \\ \frac{m+\epsilon}{2}, & m \text{ odd, } k_j \text{ even,} \\ \frac{m-\epsilon}{2}, & m \text{ odd, } k_j \text{ odd.} \end{cases} \tag{20}$$

Similarly, the number $\bar{n}(k_j)$ of fermionic spin values of s_j compatible with the third rule is given by the previous expression, with m replaced by n and ϵ by ϵ' .

The Hamiltonian $H_{\epsilon\epsilon'}^{(m|n)}$ is represented by an upper triangular matrix in the basis (19), ordered according to the degree $|\mathbf{k}|$. Indeed, taking into account that H' commutes with $\Lambda_{\epsilon\epsilon'}^{(m|n)}$, by Eq. (17) we have

$$H_{\epsilon\epsilon'}^{(m|n)} \psi_{\mathbf{k},\mathbf{s}} = H' \psi_{\mathbf{k},\mathbf{s}} = \Lambda_{\epsilon\epsilon'}^{(m|n)} ((H' \phi_{\mathbf{k}}) | \mathbf{s}) = E_{\mathbf{k}}' \psi_{\mathbf{k},\mathbf{s}} + \sum_{|\mathbf{k}'| < |\mathbf{k}|} c_{\mathbf{k}'\mathbf{k}} \psi_{\mathbf{k}',\mathbf{s}}.$$

Hence the eigenvalues of $H_{\epsilon\epsilon'}^{(m|n)}$ are given by

$$E_{\mathbf{k},\mathbf{s}} = a|\mathbf{k}| + E_0, \tag{21}$$

where \mathbf{k} and \mathbf{s} satisfy conditions (i)–(iii) above. Note that, although the value of $E_{\mathbf{k},\mathbf{s}}$ is independent of \mathbf{s} , the degeneracy of each level clearly depends on the spin through the latter conditions.

From Eqs. (7)–(9) it follows that the Hamiltonian H_{sc} of the scalar Calogero model is simply $H_+^{(1|0)}$, so that its spectrum is also given by the RHS of Eq. (21), with all k_i even by condition (iii). We can thus subtract the ground energy E_0 from the spectra of both H_{sc} and $H_{\epsilon\epsilon'}^{(m|n)}$ without altering the partition function of the chain (5). With this proviso, the energies of both models are proportional to the coupling constant a , which implies that $Z_{sc}(aT)$ and $Z_{\epsilon\epsilon'}^{(m|n)}(aT)$ are independent of a . Hence, in the calculation of the partition function of the chain (5) we can take without loss of generality $E_0 = 0$ and $a = 1$, and write the freezing trick formula (12) simply as

$$Z_{\epsilon\epsilon'}^{(m|n)}(T) = \frac{Z_{\epsilon\epsilon'}^{(m|n)}(T)}{Z_{sc}(T)}. \tag{22}$$

4. Evaluation of the partition function

We are now ready to compute the partition function of the spin chain (5) using the freezing trick formula (22). As for the ordinary (purely bosonic or fermionic) chain, the evaluation of the partition function $Z_{\epsilon\epsilon'}^{(m|n)}$ of the spin dynamical model (8) depends on the parity of the integers m and n , so that one has to consider four different cases. In the calculations that follow, we shall use the fact that the number of ways into which we can arrange the spin components s_i in a sector of \mathbf{s} corresponding to a certain component κ of \mathbf{k} repeated ν times is given by

$$d(\nu, \kappa) = \sum_{i=0}^{\min(\bar{n}(\kappa), \nu)} \binom{\bar{n}(\kappa)}{i} \binom{\bar{m}(\kappa) + \nu - i - 1}{\nu - i}. \tag{23}$$

Clearly, this number depends on κ only through its parity, cf. (20) and its fermionic analogue. Note also that when $\bar{m}(\kappa) = 0$ the second binomial coefficient in (23) should be interpreted as $\delta_{i\nu}$.

Case 1: m, n even

Recall, to begin with, that after setting $E_0 = 0$ and $a = 1$ the partition function of the scalar model (9) reads [33]

$$Z_{sc}(T) = \sum_{k_1 \geq \dots \geq k_N \geq 0} q^{2|\mathbf{k}|} = \prod_i (1 - q^{2i})^{-1}, \quad q \equiv e^{-1/(k_B T)}. \tag{24}$$

By Eq. (21), the partition function of the Hamiltonian (8) can be written as

$$Z_{\epsilon\epsilon'}^{(m|n)}(T) = \sum_{k_1 \geq \dots \geq k_N \geq 0} d_{\mathbf{k}} q^{|\mathbf{k}|}, \tag{25}$$

where the *spin degeneracy factor* $d_{\mathbf{k}}$ is the number of spin states $|\mathbf{s}\rangle$ satisfying conditions (ii) and (iii) for a given multi-index \mathbf{k} . Since both m and n are even, by Eq. (20) and its fermionic analogue we have $\bar{m}(k_j) = m/2$ and $\bar{n}(k_j) = n/2$. If

$$\mathbf{k} = (\overbrace{\kappa_1, \dots, \kappa_1}^{v_1}, \dots, \overbrace{\kappa_r, \dots, \kappa_r}^{v_r}), \quad \kappa_1 > \dots > \kappa_r \geq 0, \tag{26}$$

from Eq. (23) it follows that

$$d_{\mathbf{k}} = \prod_{j=1}^r d(v_j) \equiv d(\mathbf{v}), \tag{27}$$

where $\mathbf{v} = (v_1, \dots, v_r)$ and

$$d(v_j) = \sum_{i=0}^{\min(\frac{n}{2}, v_j)} \binom{\frac{n}{2}}{i} \binom{\frac{m}{2} + v_j - i - 1}{v_j - i}. \tag{28}$$

Since $\sum_{i=1}^r v_i = N$, the multi-index \mathbf{v} is an element of the set \mathcal{P}_N of partitions of N with order taken into account. Inserting Eqs. (26)–(28) into (25) we obtain

$$Z_{\epsilon\epsilon'}^{(m|n)} = \sum_{\mathbf{v} \in \mathcal{P}_N} d(\mathbf{v}) \sum_{\kappa_1 > \dots > \kappa_r \geq 0} q^{\sum_{i=1}^r v_i \kappa_i}.$$

The RHS of the previous equation can be evaluated as in [33, Eq. (24)], with the result

$$Z_{\epsilon\epsilon'}^{(m|n)} = q^{-N} \sum_{\mathbf{v} \in \mathcal{P}_N} d(\mathbf{v}) \prod_{j=1}^{\ell(\mathbf{v})} \frac{q^{N_j}}{1 - q^{N_j}}, \tag{29}$$

where $\ell(\mathbf{v}) = r$ is the number of components of the multi-index \mathbf{v} and

$$N_j = \sum_{i=1}^j v_i.$$

From Eqs. (22), (24) and (29) it follows that in this case the partition function of the $su(m|n)$ PF chain of BC_N type is given by

$$Z_{\epsilon\epsilon'}^{(m|n)} = \prod_i (1 + q^i) \cdot \sum_{\mathbf{v} \in \mathcal{P}_N} d(\mathbf{v}) q^{\sum_{j=1}^{\ell(\mathbf{v})-1} N_j} \prod_{j=1}^{N-\ell(\mathbf{v})} (1 - q^{N'_j}), \quad m, n \in 2\mathbb{N}_0, \tag{30}$$

where the positive integers N'_j are defined by

$$\{N'_1, \dots, N'_{N-\ell(\mathbf{v})}\} = \{1, \dots, N - 1\} - \{N_1, \dots, N_{\ell(\mathbf{v})-1}\}.$$

Note that, as for the purely bosonic/fermionic model, in this case the partition function does not depend on ϵ, ϵ' . We shall therefore drop from now on the subscripts ϵ, ϵ' from $Z_{\epsilon\epsilon'}^{(m|n)}$ when both m and n are even. The first product in Eq. (30) is simply the partition function of the $su(2|0)$ chain

(5) (cf. Eq. (34) in Ref. [33]), whereas the remaining sum coincides with the partition function $\mathcal{Z}_{(A)}^{(\frac{m}{2}|\frac{n}{2})}$ of the $\mathfrak{su}(\frac{m}{2}|\frac{n}{2})$ PF chain of type A_{N-1} computed in [29]. We thus obtain the remarkable relation

$$\mathcal{Z}^{(m|n)} = \mathcal{Z}^{(2|0)} \mathcal{Z}_{(A)}^{(\frac{m}{2}|\frac{n}{2})}, \quad m, n \in 2\mathbb{N}_0. \tag{31}$$

Recall [25] that the partition function of the $\mathfrak{su}(k|l)$ PF chain of type A is explicitly given by

$$\mathcal{Z}_{(A)}^{(k|l)} = \sum_{M_1+\dots+M_{k+l}=N} q^{\frac{1}{2} \sum_{j=k+1}^{k+l} M_j(M_j-1)} \frac{(q)_N}{(q)_{M_1} \cdots (q)_{M_{k+l}}}, \tag{32}$$

where the notation $(q)_k$ is defined in Eq. (A.1) of Appendix A. For instance, from Eq. (32) with $k = l = 1$ and Eqs. (A.1)–(A.3) one obtains

$$\mathcal{Z}_{(A)}^{(1|1)} = \sum_{M=0}^N \begin{bmatrix} N \\ M \end{bmatrix}_q q^{\binom{M}{2}} = \prod_{i=0}^{N-1} (1 + q^i),$$

so that

$$\mathcal{Z}^{(2|2)} = \mathcal{Z}^{(2|0)} \mathcal{Z}_{(A)}^{(1|1)} = 2(1 + q^N) \prod_{i=1}^{N-1} (1 + q^i)^2. \tag{33}$$

Case 2: m odd, n even

In this case, by Eq. (20) and its fermionic counterpart we have $\bar{n}(k_j) = n/2$, while $\bar{m}(k_j)$ depends on the parity of k_j . Hence it is convenient to modify condition (i) by first grouping separately the components of \mathbf{k} with the same parity and then ordering separately the even and odd components. In other words, we shall now write $\mathbf{k} = (\mathbf{k}_e, \mathbf{k}_o)$, where

$$\begin{aligned} \mathbf{k}_e &= (\overbrace{2\kappa_1, \dots, 2\kappa_1}^{v_1}, \dots, \overbrace{2\kappa_s, \dots, 2\kappa_s}^{v_s}), \\ \mathbf{k}_o &= (\overbrace{2\kappa_{s+1} + 1, \dots, 2\kappa_{s+1} + 1}^{v_{s+1}}, \dots, \overbrace{2\kappa_r + 1, \dots, 2\kappa_r + 1}^{v_r}), \end{aligned}$$

and

$$\kappa_1 > \dots > \kappa_s \geq 0, \quad \kappa_{s+1} > \dots > \kappa_r \geq 0.$$

By Eq. (20), $\bar{m}(2\kappa_j) = (m + \epsilon)/2$ for $j = 1, \dots, s$, while $\bar{m}(2\kappa_j + 1) = (m - \epsilon)/2$ for $j = s + 1, \dots, r$. From Eq. (23) it follows that the spin degeneracy factor $d_{\mathbf{k}}$ is now given by

$$d_{\mathbf{k}} = \prod_{j=1}^s d_{\epsilon}(v_j) \cdot \prod_{j=s+1}^r d_{-\epsilon}(v_j) \equiv d_{\epsilon}^s(\mathbf{v}), \tag{34}$$

where

$$d_{\epsilon}(v_j) = \sum_{i=0}^{\min(\frac{n}{2}, v_j)} \binom{\frac{n}{2}}{i} \binom{\frac{m+\epsilon}{2} + v_j - i - 1}{v_j - i}. \tag{35}$$

Eq. (25) for the partition function of the dynamical spin model (8) then becomes

$$Z_{\epsilon\epsilon'}^{(m|n)} = \sum_{\mathbf{v} \in \mathcal{P}_N} \sum_{s=0}^r d_{\epsilon}^s(\mathbf{v}) \sum_{\substack{\kappa_1 > \dots > \kappa_s \geq 0 \\ \kappa_{s+1} > \dots > \kappa_r \geq 0}} q^{\sum_{i=1}^s 2v_i \kappa_i + \sum_{i=s+1}^r v_i (2\kappa_i + 1)},$$

where the RHS can be evaluated as in [33, Eq. (29)]. We thus obtain

$$Z_{\epsilon\epsilon'}^{(m|n)} = \sum_{\mathbf{v} \in \mathcal{P}_N} \sum_{s=0}^{\ell(\mathbf{v})} d_{\epsilon}^s(\mathbf{v}) q^{-(N+N_s)} \prod_{j=1}^s \frac{q^{2N_j}}{1 - q^{2N_j}} \cdot \prod_{j=s+1}^{\ell(\mathbf{v})} \frac{q^{2\tilde{N}_j}}{1 - q^{2\tilde{N}_j}}, \tag{36}$$

where

$$\tilde{N}_j = \sum_{i=s+1}^j v_i, \quad j = s + 1, \dots, \ell(\mathbf{v}).$$

Substituting Eqs. (24) and (36) into (22) we arrive at the following explicit expression for the partition function of the spin chain (5) for odd m and even n :

$$Z_{\epsilon\epsilon'}^{(m|n)} = \prod_i (1 - q^{2i}) \cdot \sum_{\mathbf{v} \in \mathcal{P}_N} \sum_{s=0}^{\ell(\mathbf{v})} d_{\epsilon}^s(\mathbf{v}) q^{-(N+N_s)} \prod_{j=1}^s \frac{q^{2N_j}}{1 - q^{2N_j}} \cdot \prod_{j=s+1}^{\ell(\mathbf{v})} \frac{q^{2\tilde{N}_j}}{1 - q^{2\tilde{N}_j}}. \tag{37}$$

Note that in this case the partition function depends on ϵ but not on ϵ' .

As is the case for any finite system with integer energies, the partition function $Z_{\epsilon\epsilon'}^{(m|n)}$ should be a polynomial in q . We shall now prove this fact by simplifying Eq. (37) with the help of the identities in Appendix A. To this end, we define the two sets of integers

$$\begin{aligned} \{N'_1, \dots, N'_{N_s-s}\} &= \{1, \dots, N_s - 1\} - \{N_1, \dots, N_{s-1}\}, \\ \{\tilde{N}'_{N_s-s+1}, \dots, \tilde{N}'_{N-\ell(\mathbf{v})}\} &= \{1, \dots, N - N_s - 1\} - \{\tilde{N}_{s+1}, \dots, \tilde{N}_{\ell(\mathbf{v})-1}\}, \end{aligned}$$

in terms of which we can rewrite Eq. (37) as

$$\begin{aligned} Z_{\epsilon\epsilon'}^{(m|n)} &= \sum_{\mathbf{v} \in \mathcal{P}_N} \sum_{s=0}^{\ell(\mathbf{v})} d_{\epsilon}^s(\mathbf{v}) q^{N-N_s+2\sum_{j=1}^{s-1} N_j+2\sum_{j=s+1}^{\ell(\mathbf{v})-1} \tilde{N}_j} \begin{bmatrix} N \\ N_s \end{bmatrix}_{q^2} \\ &\times \prod_{j=1}^{N_s-s} (1 - q^{2N'_j}) \cdot \prod_{j=N_s-s+1}^{N-\ell(\mathbf{v})} (1 - q^{2\tilde{N}'_j}). \end{aligned} \tag{38}$$

From the discussion in Appendix A (cf. Eqs. (A.6)–(A.7)), it follows that the binomial coefficient $\begin{bmatrix} N \\ N_s \end{bmatrix}_{q^2}$ is an even polynomial of degree $2N_s(N - N_s)$ in q . Thus the RHS of Eq. (38) is also a polynomial in q , as expected.

Case 3: m even, n odd

The evaluation of the partition function $Z_{\epsilon\epsilon'}^{(m|n)}$ is performed as in the previous case, the only difference being that the spin degeneracy factor $d_{\epsilon}^s(\mathbf{v})$ in Eq. (38) should be replaced by

$$d_{\epsilon}^s(\mathbf{v}) = \prod_{j=1}^s d_{\epsilon'}(v_j) \cdot \prod_{j=s+1}^r d_{-\epsilon'}(v_j), \tag{39}$$

where now

$$d_{\epsilon'}(v_j) = \sum_{i=0}^{\min(\frac{n+\epsilon'}{2}, v_j)} \binom{\frac{n+\epsilon'}{2}}{i} \binom{\frac{m}{2} + v_j - i - 1}{v_j - i}. \tag{40}$$

In particular, the partition function $\mathcal{Z}_{\epsilon\epsilon'}^{(m|n)}$ does not depend on ϵ in this case.

Case 4: m, n odd

In this case both $\bar{m}(k_j)$ and $\bar{n}(k_j)$ depend on the parity of k_j , so that the computation of the partition function proceeds as in Case 2. The final result is still Eq. (38), with the spin degeneracy factor $d_{\epsilon}^s(\mathbf{v})$ replaced by

$$d_{\epsilon\epsilon'}^s(\mathbf{v}) = \prod_{j=1}^s d_{\epsilon\epsilon'}(v_j) \cdot \prod_{j=s+1}^r d_{-\epsilon, -\epsilon'}(v_j), \tag{41}$$

where

$$d_{\epsilon\epsilon'}(v_j) = \sum_{i=0}^{\min(\frac{n+\epsilon'}{2}, v_j)} \binom{\frac{n+\epsilon'}{2}}{i} \binom{\frac{m+\epsilon}{2} + v_j - i - 1}{v_j - i}. \tag{42}$$

Thus in this case the partition function depends on both ϵ and ϵ' .

5. Relation with the spin chain of type A

We have seen in the previous section that when the integers m and n are both even, the partition function $\mathcal{Z}^{(m|n)}$ of the chain (5) is related in a simple way to that of the $\text{su}(\frac{m}{2} | \frac{n}{2})$ PF chain of A_{N-1} type, cf. Eq. (31). The purpose of this section is to derive an analogous relation for odd values of m and/or n . To this end, we shall deduce an alternative expression for the partition function by means of the grand canonical partition function of the spin dynamical model (8), following an approach similar to that of Refs. [21,25] for the ordinary and supersymmetric PF chains of type A.

In order to compute the grand canonical partition function, it is convenient to replace conditions (i)–(iii) in Section 3 ordering the labels \mathbf{k}, \mathbf{s} of the basis states (19) by the following equivalent set of rules:

- (a) If $i < j$ then $\pi(s_i) \leq \pi(s_j)$;
- (b) If $i < j$ and $\pi(s_i) = \pi(s_j)$, then $s_i^1 \leq s_j^1$;
- (c) If $i < j$ and $s_i = s_j$, then $k_i \geq k_j + \pi(s_j)$;
- (d) $s_i \geq 0$, and $(-1)^{k_i} = \rho(s_i)$ if $s_i^1 = 0$.

By Eq. (21) (setting, as before, $E_0 = 0$ and $a = 1$) the grand canonical partition function $\mathbf{Z}_{\epsilon\epsilon'}^{(m|n)}$ of the spin dynamical model (8) is given by

$$\mathbf{Z}_{\epsilon\epsilon'}^{(m|n)}(T, \mu) = \sum_{N=0}^{\infty} \sum_{\mathbf{k}, \mathbf{s}} q^{|\mathbf{k}| - N\mu}, \tag{43}$$

where μ is the chemical potential and the inner sum runs over all values of $\mathbf{k} \in \mathbb{N}_0^N$ and \mathbf{s} compatible with conditions (a)–(d). As in the previous section, the computation of $\mathbf{Z}_{\epsilon\epsilon'}^{(m|n)}$ depends on the parity of m and n . Since the case of even m and n has already been dealt with above, we shall start with the case of m odd and n even:

Case 2: m odd, n even

Calling $\tilde{m} = (m - 1)/2$ and $\tilde{n} = n/2$, by conditions (a), (b), and (d) we can write

$$\mathbf{s} = \left(\overbrace{0, \dots, 0}^{v_0}, \dots, \overbrace{\frac{m-1}{2}, \dots, \frac{m-1}{2}}^{v_{\tilde{m}}}, \overbrace{\frac{1}{2}, \dots, \frac{1}{2}}^{v_{\tilde{m}+1}}, \dots, \overbrace{\frac{n-1}{2}, \dots, \frac{n-1}{2}}^{v_{\tilde{m}+\tilde{n}}} \right), \tag{44}$$

where $v_i \in \mathbb{N}_0$ and $\sum_{i=0}^{\tilde{m}+\tilde{n}} v_i = N$. With a slight abuse of notation, we have suppressed the second component s_i^2 of each spin s_i , since clearly $s_i^2 = 0$ for the first $v_0 + \dots + v_{\tilde{m}}$ spins and $s_i^2 = 1$ for the remaining ones. By condition (c), the multi-index \mathbf{k} is of the form

$$\mathbf{k} = (2\kappa_1^0 + p, \dots, 2\kappa_{v_0}^0 + p, \kappa_1^1, \dots, \kappa_{v_1}^1, \dots, \kappa_1^{\tilde{m}+\tilde{n}}, \dots, \kappa_{v_{\tilde{m}+\tilde{n}}}^{\tilde{m}+\tilde{n}}), \tag{45}$$

where

$$p = \frac{1}{2}(1 - \epsilon) \tag{46}$$

and the nonnegative integers κ_j^i satisfy

$$\begin{cases} \kappa_1^i \geq \dots \geq \kappa_{v_i}^i, & i = 0, \dots, \tilde{m}, \\ \kappa_1^i > \dots > \kappa_{v_i}^i, & i = \tilde{m} + 1, \dots, \tilde{m} + \tilde{n}. \end{cases}$$

By Eq. (43), the grand canonical partition function of the chain (5) is given by

$$\begin{aligned} \mathbf{Z}_{\epsilon\epsilon'}^{(m|n)} &= \sum_{v_0, \dots, v_{\tilde{m}+\tilde{n}} \geq 0} \sum_{\substack{\kappa_1^i \geq \dots \geq \kappa_{v_i}^i \\ i=0, \dots, \tilde{m}}} \sum_{\substack{\kappa_1^j > \dots > \kappa_{v_j}^j \\ j=\tilde{m}+1, \dots, \tilde{m}+\tilde{n}}} q^{\sum_{l=1}^{v_0} (2\kappa_l^0 + p) - v_0\mu} \prod_{l=1}^{\tilde{m}+\tilde{n}} q^{\sum_{r=1}^{v_l} \kappa_r^l - v_l\mu} \\ &= \left[\sum_{\substack{\kappa_1^0 \geq \dots \geq \kappa_{v_0}^0 \\ v_0 \geq 0}} q^{\sum_{l=1}^{v_0} (2\kappa_l^0 + p) - v_0\mu} \right] \left[\prod_{i=1}^{\tilde{m}} \sum_{\substack{\kappa_1^i \geq \dots \geq \kappa_{v_i}^i \\ v_i \geq 0}} q^{\sum_{r=1}^{v_i} \kappa_r^i - v_i\mu} \right] \\ &\quad \times \left[\prod_{j=\tilde{m}+1}^{\tilde{m}+\tilde{n}} \sum_{\substack{\kappa_1^j > \dots > \kappa_{v_j}^j \\ v_j \geq 0}} q^{\sum_{r=1}^{v_j} \kappa_r^j - v_j\mu} \right]. \end{aligned}$$

The first factor in the last equality is clearly the grand canonical partition function $\mathbf{Z}_{\epsilon}^{(1|0)}$ of the purely bosonic spinless dynamical model (8) (which coincides with the scalar model (9) for $\epsilon = 1$). On the other hand, in Ref. [25] it was shown that

$$\sum_{v=0}^{\infty} \sum_{\kappa_1 \geq \dots \geq \kappa_v} q^{\sum_{r=1}^v \kappa_r - v\mu} = \mathbf{Z}_{(A)}^{(1|0)}, \quad \sum_{v=0}^{\infty} \sum_{\kappa_1 > \dots > \kappa_v} q^{\sum_{r=1}^v \kappa_r - v\mu} = \mathbf{Z}_{(A)}^{(0|1)},$$

where $\mathbf{Z}_{(A)}^{(1|0)}$ (respectively $\mathbf{Z}_{(A)}^{(0|1)}$) denotes the grand canonical partition function of the purely bosonic (respectively fermionic) spinless Calogero model of type A . We thus have

$$\mathbf{Z}_{\epsilon\epsilon'}^{(m|n)} = \mathbf{Z}_{\epsilon}^{(1|0)} (\mathbf{Z}_{(A)}^{(1|0)})^{\frac{m-1}{2}} (\mathbf{Z}_{(A)}^{(0|1)})^{\frac{n}{2}}.$$

Using again the results in Ref. [25], we can write the previous formula as

$$\mathbf{Z}_{\epsilon\epsilon'}^{(m|n)} = \mathbf{Z}_{\epsilon}^{(1|0)} \mathbf{Z}_{(A)}^{(\frac{m-1}{2}|\frac{n}{2})}, \tag{47}$$

where $\mathbf{Z}_{(A)}^{(\frac{m-1}{2}|\frac{n}{2})}$ denotes the grand canonical partition function of the $\text{su}(\frac{m-1}{2}|\frac{n}{2})$ spin Calogero model of type A . On the other hand, the grand canonical partition function $\mathbf{Z}(T, \mu)$ can be expressed in terms of the N -particle canonical partition function $Z_N(T)$ as

$$\mathbf{Z}(T, \mu) = \sum_{N=0}^{\infty} y^N Z_N(T),$$

where $y \equiv q^{-\mu}$ and $Z_0(T) \equiv 1$. Equating the coefficient of y^N in both sides of Eq. (47) we readily obtain

$$Z_{\epsilon\epsilon', N}^{(m|n)} = \sum_{M=0}^N Z_{\epsilon, M}^{(1|0)} Z_{(A), N-M}^{(\frac{m-1}{2}|\frac{n}{2})},$$

where $Z_{(A), N-M}^{(\frac{m-1}{2}|\frac{n}{2})}$ is the partition function of the $\text{su}(\frac{m-1}{2}|\frac{n}{2})$ spin Calogero model of type A with $N - M$ particles. By Eq. (22), the partition function of the N -particle $\text{su}(m|n)$ PF chain of BC_N type is thus given by

$$\mathcal{Z}_{\epsilon\epsilon', N}^{(m|n)} = \sum_{M=0}^N \frac{Z_{\epsilon, M}^{(1|0)}}{Z_{\text{sc}, N}} Z_{(A), N-M}^{(\frac{m-1}{2}|\frac{n}{2})}. \tag{48}$$

The partition function $Z_{+, M}^{(1|0)}$ is simply given by

$$Z_{+, M}^{(1|0)} = Z_{\text{sc}, M} = \frac{1}{(q^2)_M}, \tag{49}$$

where we have used (24). On the other hand, from conditions (i) and (iii) in Section 3 we have

$$Z_{-, M}^{(1|0)} = \sum_{k_1 \geq \dots \geq k_M \geq 0} q^{\sum_{i=1}^M (2k_i + 1)} = q^M Z_{\text{sc}, M} = \frac{q^M}{(q^2)_M}. \tag{50}$$

The partition function of the $\text{su}(\frac{m-1}{2}|\frac{n}{2})$ spin chain of type A with $N - M$ particles is related to the partition function of its corresponding spin dynamical model by [25]

$$\mathcal{Z}_{(A), N-M}^{(\frac{m-1}{2}|\frac{n}{2})} = (q)_{N-M} Z_{(A), N-M}^{(\frac{m-1}{2}|\frac{n}{2})}. \tag{51}$$

Inserting Eqs. (49)–(51) into (48) we finally obtain

$$\begin{aligned} \mathcal{Z}_{\epsilon\epsilon',N}^{(m|n)} &= \sum_{M=0}^N q^{\frac{1}{2}(1-\epsilon)M} \frac{(q^2)_N}{(q^2)_M (q)_{N-M}} \mathcal{Z}_{(A),N-M}^{(\frac{m-1}{2}|\frac{n}{2})} \\ &= \sum_{M=0}^N q^{\frac{1}{2}(1-\epsilon)M} \prod_{i=M+1}^N (1+q^i) \cdot \left[\begin{matrix} N \\ M \end{matrix} \right]_q \mathcal{Z}_{(A),N-M}^{(\frac{m-1}{2}|\frac{n}{2})}, \end{aligned} \tag{52}$$

where we have used the identity (A.2). Alternatively, from the first equality in (52) and Eq. (31) we can also derive the relation

$$\mathcal{Z}_{\epsilon\epsilon',N}^{(m|n)} = \sum_{M=0}^N q^{\frac{1}{2}(1-\epsilon)M} \left[\begin{matrix} N \\ M \end{matrix} \right]_{q^2} \mathcal{Z}_{N-M}^{(m-1|n)}.$$

Case 3: *m even, n odd*

This case is very similar to the previous one, the main difference being that Eq. (47) now becomes

$$\mathbf{Z}_{\epsilon\epsilon'}^{(m|n)} = \mathbf{Z}_{\epsilon'}^{(0|1)} \mathbf{Z}_{(A)}^{(\frac{m}{2}|\frac{n-1}{2})}, \tag{53}$$

which leads to the following expression for the chain’s partition function:

$$\mathcal{Z}_{\epsilon\epsilon',N}^{(m|n)} = \sum_{M=0}^N \frac{\mathbf{Z}_{\epsilon',M}^{(0|1)}}{\mathbf{Z}_{sc,N}} \mathcal{Z}_{(A),N-M}^{(\frac{m}{2}|\frac{n-1}{2})}. \tag{54}$$

By conditions (i)–(iii) in Section 3, the partition function $\mathbf{Z}_{+,M}^{(0|1)}$ of the spinless fermionic model (8) with even parity is given by

$$\begin{aligned} \mathbf{Z}_{+,M}^{(0|1)} &= \sum_{k_1 > \dots > k_M \geq 0} q^{2\sum_{i=1}^M k_i} = \sum_{l_1 \geq \dots \geq l_M \geq 0} q^{2\sum_{i=1}^M (l_i + M - i)} \\ &= q^{M(M-1)} \mathbf{Z}_{sc,M} = \frac{q^{M(M-1)}}{(q^2)_M}. \end{aligned} \tag{55}$$

Similarly,

$$\mathbf{Z}_{-,M}^{(0|1)} = \sum_{k_1 > \dots > k_M \geq 0} q^{\sum_{i=1}^M (2k_i + 1)} = q^M \mathbf{Z}_{+,M}^{(0|1)} = \frac{q^{M^2}}{(q^2)_M}. \tag{56}$$

Inserting (55) and (56) into Eq. (54) and proceeding as before, we obtain the following two expressions for the partition function of the chain (5) with even *m* and odd *n*:

$$\mathcal{Z}_{\epsilon\epsilon',N}^{(m|n)} = \sum_{M=0}^N q^{M(M-\frac{1}{2}(1+\epsilon'))} \prod_{i=M+1}^N (1+q^i) \cdot \left[\begin{matrix} N \\ M \end{matrix} \right]_q \mathcal{Z}_{(A),N-M}^{(\frac{m}{2}|\frac{n-1}{2})} \tag{57a}$$

$$= \sum_{M=0}^N q^{M(M-\frac{1}{2}(1+\epsilon'))} \left[\begin{matrix} N \\ M \end{matrix} \right]_{q^2} \mathcal{Z}_{N-M}^{(m|n-1)}. \tag{57b}$$

Case 4: m, n odd

In this case both the bosonic and fermionic spin components can take the zero value, so that instead of Eq. (47) we now have

$$\mathcal{Z}_{\epsilon\epsilon'}^{(m|n)} = \mathcal{Z}_{\epsilon}^{(1|0)} \mathcal{Z}_{\epsilon'}^{(0|1)} \mathcal{Z}_{(A)}^{\left(\frac{m-1}{2} \mid \frac{n-1}{2}\right)}. \tag{58}$$

In particular, setting $m = n = 1$ in the previous this formula we get

$$\mathcal{Z}_{\epsilon\epsilon'}^{(1|1)} = \mathcal{Z}_{\epsilon}^{(1|0)} \mathcal{Z}_{\epsilon'}^{(0|1)}, \tag{59}$$

and therefore

$$\mathcal{Z}_{\epsilon\epsilon'}^{(m|n)} = \mathcal{Z}_{\epsilon\epsilon'}^{(1|1)} \mathcal{Z}_{(A)}^{\left(\frac{m-1}{2} \mid \frac{n-1}{2}\right)}. \tag{60}$$

Proceeding again as in Case 2, we easily obtain

$$\mathcal{Z}_{\epsilon\epsilon',N}^{(m|n)} = \sum_{M=0}^N \frac{\mathcal{Z}_{\epsilon\epsilon',M}^{(1|1)}}{\mathcal{Z}_{sc,N}} \mathcal{Z}_{(A),N-M}^{\left(\frac{m-1}{2} \mid \frac{n-1}{2}\right)} = \sum_{M=0}^N \frac{(q^2)_N}{(q^2)_M (q)_{N-M}} \mathcal{Z}_{\epsilon\epsilon',M}^{(1|1)} \mathcal{Z}_{(A),N-M}^{\left(\frac{m-1}{2} \mid \frac{n-1}{2}\right)}. \tag{61}$$

As before, this expression can be simplified in two alternative ways, leading to the following remarkable identities for the partition function of the chain (5) with odd m and n :

$$\mathcal{Z}_{\epsilon\epsilon',N}^{(m|n)} = \sum_{M=0}^N \prod_{i=M+1}^N (1 + q^i) \cdot \left[\begin{matrix} N \\ M \end{matrix} \right]_q \mathcal{Z}_{\epsilon\epsilon',M}^{(1|1)} \mathcal{Z}_{(A),N-M}^{\left(\frac{m-1}{2} \mid \frac{n-1}{2}\right)} \tag{62a}$$

$$= \sum_{M=0}^N \left[\begin{matrix} N \\ M \end{matrix} \right]_{q^2} \mathcal{Z}_{\epsilon\epsilon',M}^{(1|1)} \mathcal{Z}_{N-M}^{(m-1|n-1)}. \tag{62b}$$

Let us finally evaluate the partition function $\mathcal{Z}_{\epsilon\epsilon'}^{(1|1)}$ appearing in Eqs. (62a)–(62b). To begin with, from Eq. (59) for the grand canonical partition function and the freezing trick formula (22) we obtain

$$\mathcal{Z}_{\epsilon\epsilon'}^{(1|1)} = \sum_{M=0}^N \frac{\mathcal{Z}_{\epsilon,N-M}^{(1|0)} \mathcal{Z}_{\epsilon',M}^{(0|1)}}{\mathcal{Z}_{sc,N}} = \sum_{M=0}^N q^{p(N-M)+M(M-1+p')} \left[\begin{matrix} N \\ M \end{matrix} \right]_{q^2}, \tag{63}$$

where p is given by (46) and $p' = (1 - \epsilon')/2$. We thus have

$$\mathcal{Z}_{++}^{(1|1)} = \sum_{M=0}^N q^{2\binom{M}{2}} \left[\begin{matrix} N \\ M \end{matrix} \right]_{q^2} = 2 \prod_{i=1}^{N-1} (1 + q^{2i}), \tag{64}$$

where we have used Eq. (A.3) with $x = 1$ and q replaced by q^2 . On the other hand, if $\epsilon = \epsilon' = -1$ Eq. (63) becomes

$$\mathcal{Z}_{--}^{(1|1)} = q^N \sum_{M=0}^N q^{2\binom{M}{2}} \left[\begin{matrix} N \\ M \end{matrix} \right]_{q^2} = q^N \mathcal{Z}_{++}^{(1|1)}. \tag{65}$$

Next, if $\epsilon = -\epsilon' = 1$ we have

$$\mathcal{Z}_{+-}^{(1|1)} = \sum_{M=0}^N q^{M^2} \left[\begin{matrix} N \\ M \end{matrix} \right]_{q^2} = \prod_{i=0}^{N-1} (1 + q^{2i+1}), \tag{66}$$

where we have used Eq. (A.4). Finally, when $\epsilon = -\epsilon' = -1$, Eqs. (63) and (A.5) yield

$$\begin{aligned} \mathcal{Z}_{-+}^{(1|1)} &= q^N \sum_{M=0}^N q^{M(M-2)} \begin{bmatrix} N \\ M \end{bmatrix}_{q^2} = q^N \prod_{i=0}^{N-1} (1 + q^{2i-1}) \\ &= q^{N-1} (1 + q)^2 \prod_{i=2}^{N-1} (1 + q^{2i-1}). \end{aligned} \tag{67}$$

6. Boson–fermion duality

In Ref. [28] it was uncovered a remarkable boson–fermion duality satisfied by the spectrum of any spin chain of A_{N-1} type with global $\text{su}(m|n)$ symmetry. We shall now extend these ideas to the PF chain of BC_N type (5), although our results will in fact be valid for any chain of BC_N type possessing global $\text{su}(m|n)$ symmetry.

Let us start by defining the *star operator* $*$: $\Sigma^{(m|n)} \rightarrow \Sigma^{(m|n)}$ by

$$|\mathbf{s}\rangle^* = (-1)^{\sum_i i\pi(s_i)} |\mathbf{s}\rangle, \tag{68}$$

and the *exchange operator* $\mathcal{X}^{(m|n)}$: $\Sigma^{(m|n)} \rightarrow \Sigma^{(n|m)}$ by

$$\mathcal{X}^{(m|n)} |s_1, \dots, s_N\rangle = |s'_1, \dots, s'_N\rangle, \quad s'_i \equiv (s_i^1, 1 - s_i^2). \tag{69}$$

In other words, s'_i has the same value as s_i but opposite type. Following Ref. [28], we next define the operator $\mathcal{U}^{(m|n)}$: $\Sigma^{(m|n)} \rightarrow \Sigma^{(n|m)}$ as the composition

$$\mathcal{U}^{(m|n)} = \mathcal{X}^{(m|n)} \circ *. \tag{70}$$

Since, clearly, the operators $* \circ *$ and $\mathcal{X}^{(n|m)} \circ \mathcal{X}^{(m|n)}$ are the identity in $\Sigma^{(m|n)}$, $\mathcal{U}^{(m|n)}$ is invertible and

$$\mathcal{U}^{(m|n)-1} = * \circ \mathcal{X}^{(n|m)} = (-1)^{\frac{1}{2}N(N+1)} \mathcal{U}^{(n|m)}.$$

Indeed, if $|\mathbf{s}\rangle \in \Sigma^{(n|m)}$, we have

$$\begin{aligned} |s'_1, \dots, s'_N\rangle^* &= (-1)^{\sum_i i\pi(s'_i)} |s'_1, \dots, s'_N\rangle = (-1)^{\sum_i i(1-\pi(s_i))} |s'_1, \dots, s'_N\rangle \\ &= (-1)^{\sum_i i} \mathcal{U}^{(n|m)} |\mathbf{s}\rangle = (-1)^{\frac{1}{2}N(N+1)} \mathcal{U}^{(n|m)} |\mathbf{s}\rangle. \end{aligned}$$

Moreover, it is also easy to see that the star operator is self-adjoint and $\mathcal{X}^{(m|n)\dagger} = \mathcal{X}^{(n|m)}$, so that $\mathcal{U}^{(m|n)\dagger} = * \circ \mathcal{X}^{(n|m)} = \mathcal{U}^{(m|n)-1}$, i.e., $\mathcal{U}^{(m|n)}$ is unitary. A straightforward computation [28] shows that $\mathcal{U}^{(m|n)} S_{ij}^{(m|n)} = -S_{ij}^{(n|m)} \mathcal{U}^{(m|n)}$, or equivalently

$$\mathcal{U}^{(m|n)-1} S_{ij}^{(n|m)} \mathcal{U}^{(m|n)} = -S_{ij}^{(m|n)}. \tag{71}$$

The above considerations are enough for proving duality in the A_N case. However, in the BC_N case we need also consider the behavior of the spin reversal operators $S_i^{\epsilon\epsilon'}$ with respect to conjugation by $\mathcal{U}^{(m|n)}$. For clarity’s sake, in the following discussion we shall use the more precise notation $S_i^{(m\epsilon|n\epsilon')}$ to denote the operator $S_i^{\epsilon\epsilon'} : \Sigma^{(m|n)} \rightarrow \Sigma^{(m|n)}$. Noting that $(s_i^-)' = (s_i^+)^-$ and $\pi(s_i^-) = \pi(s_i)$, if $|\mathbf{s}\rangle \in \Sigma^{(m|n)}$ we have

$$\begin{aligned}
 \mathcal{U}^{(m|n)} S_i^{(m\epsilon|n\epsilon')} |s\rangle &= \rho(s_i) \mathcal{U}^{(m|n)} |s_1, \dots, s_i^-, \dots, s_N\rangle \\
 &= \rho(s_i) (-1)^{\sum_i i\pi(s_i)} |s'_1, \dots, (s_i^-)', \dots, s'_N\rangle \\
 &= \rho(s_i) (-1)^{\sum_i i\pi(s_i)} |s'_1, \dots, (s_i^-)^-, \dots, s'_N\rangle \equiv S_i^{(n\epsilon'|m\epsilon)} \mathcal{U}^{(m|n)} |s\rangle,
 \end{aligned}$$

and therefore

$$\mathcal{U}^{(m|n)-1} S_i^{(n\epsilon'|m\epsilon)} \mathcal{U}^{(m|n)} = S_i^{(m\epsilon|n\epsilon')} = -S_i^{(m, -\epsilon|n, -\epsilon')}. \tag{72}$$

With the help of Eqs. (71) and (72) one easily obtains

$$\begin{aligned}
 \mathcal{H}_{-\epsilon, -\epsilon'}^{(m|n)} + \mathcal{U}^{(m|n)-1} \mathcal{H}_{\epsilon', \epsilon}^{(n|m)} \mathcal{U}^{(m|n)} \\
 = 2 \left(\sum_{i \neq j} [(\xi_i - \xi_j)^{-2} + (\xi_i + \xi_j)^{-2}] + \beta \sum_i \xi_i^{-2} \right) = N^2,
 \end{aligned}$$

where the last sum was evaluated in Ref. [33]. Since $\mathcal{U}^{(m|n)-1} \mathcal{H}_{\epsilon', \epsilon}^{(n|m)} \mathcal{U}^{(m|n)}$ and $\mathcal{H}_{\epsilon', \epsilon}^{(n|m)}$ are isospectral, from the previous equation we obtain the remarkable duality relation

$$\mathcal{Z}_{\epsilon', \epsilon}^{(n|m)}(q) = q^{N^2} \mathcal{Z}_{-\epsilon, -\epsilon'}^{(m|n)}(q^{-1}). \tag{73}$$

When $\epsilon' = \epsilon$, the previous formula relates the spectra of two chains which differ in the exchange of bosons with fermions and the action of the spin reversal operators. On the other hand, when $\epsilon' = -\epsilon$ Eq. (73) is a genuine duality relation with respect to the exchange of bosons and fermions, i.e., it relates the spectra of the chains $\mathcal{H}_{-\epsilon, \epsilon}^{(m|n)}$ and $\mathcal{H}_{-\epsilon, \epsilon}^{(n|m)}$. In addition, if $m = n$ Eq. (73) establishes that the spectrum of the chain $\mathcal{H}_{-\epsilon, \epsilon}^{(m|m)}$ is invariant under the transformation $\mathcal{E} \mapsto N^2 - \mathcal{E}$, i.e., it is symmetric about $N^2/2$. For instance, for $m = 1$ this property can be checked directly from Eqs. (66)–(67) for the partition function.

Similarly, if m and n are both even the partition function does not depend on ϵ and ϵ' , so that Eq. (73) is again a genuine boson–fermion duality relation. For the same reason, if $m = n \in 2\mathbb{N}_0$ the spectrum of the chain $\mathcal{H}_{\epsilon, \epsilon'}^{(m|m)}$ is invariant under $\mathcal{E} \mapsto N^2 - \mathcal{E}$. As an example, if $m = n = 2$ this property can be readily verified using the explicit formula Eq. (33).

As mentioned above, a duality relation of the form (73) clearly holds for an arbitrary chain with global $su(m|n)$ symmetry, with Hamiltonian

$$\hat{\mathcal{H}}_{\epsilon, \epsilon'}^{(m|n)} = \sum_{i \neq j} [c_{ij}(1 - S_{ij}^{(m|n)}) + \tilde{c}_{ij}(1 - \tilde{S}_{ij}^{(m|n)})] + \sum_i c_i(1 - S_i^{\epsilon, \epsilon'}),$$

where c_{ij} , \tilde{c}_{ij} and c_i are real constants. More precisely, from Eqs. (71) and (72) it now follows that

$$\hat{\mathcal{H}}_{-\epsilon, -\epsilon'}^{(m|n)} + \mathcal{U}^{(m|n)-1} \hat{\mathcal{H}}_{\epsilon', \epsilon}^{(n|m)} \mathcal{U}^{(m|n)} = 2 \left(\sum_{i \neq j} (c_{ij} + \tilde{c}_{ij}) + \sum_i c_i \right) \equiv C,$$

so that Eq. (73) becomes

$$\hat{\mathcal{Z}}_{\epsilon', \epsilon}^{(n|m)}(q) = q^C \hat{\mathcal{Z}}_{-\epsilon, -\epsilon'}^{(m|n)}(q^{-1}).$$

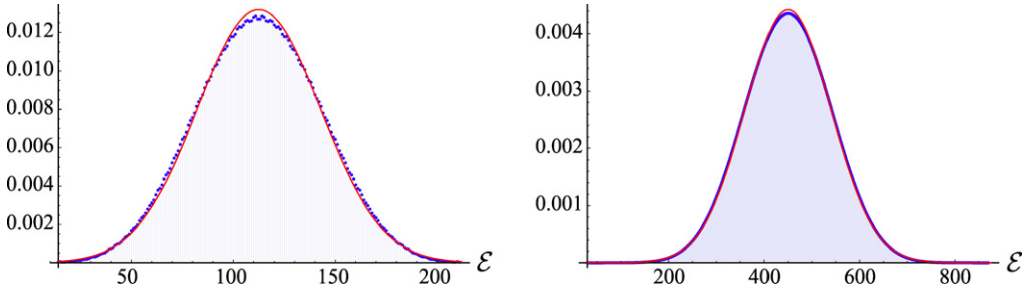


Fig. 1. Plot of the Gaussian distribution (75) (continuous red line) versus the LHS of Eq. (76) (blue dots) in the case $m = n = 1$, $\epsilon' = -\epsilon = 1$, for $N = 15$ (left) and $N = 30$ (right). The root mean square errors (normalized to the mean) of the adjustments are 4.0×10^{-2} and 2.2×10^{-2} , respectively. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this paper.)

7. Level density and spacings distribution

Having derived several explicit formulas for the partition function of the spin chain (5), it is natural to inquire on the global properties of its spectrum. In fact, the partition function (and hence the spectrum) can be computed in a very efficient way for relatively large values of N by expanding in powers of q Eqs. (31), (52), (57a), (62a) and (64)–(67), together with Eq. (32), with the help of a symbolic computation package. For instance, using MATHEMATICA™ on a personal computer it takes about five seconds to evaluate the partition function $\mathcal{Z}_{--}^{(2|1)}$ for $N = 40$ particles.

In the first place, our calculations for a wide range of values of N , m and n show that the energy levels of the chain (5) are equidistant, as already observed by us in the purely bosonic or fermionic case [33]. More precisely, we have found that the distance $\delta\mathcal{E}$ between consecutive levels is equal to one, except for the $\text{su}(1|1)$ chain with $\epsilon = \epsilon'$, for which $\delta\mathcal{E} = 2$ by Eqs. (64) and (65).

Secondly, our numerical calculations evidence that for sufficiently large N ($N \gtrsim 20$) the normalized level density

$$f(\mathcal{E}) = (m + n)^{-N} \sum_{i=1}^L d_i \delta(\mathcal{E} - \mathcal{E}_i), \tag{74}$$

where $\mathcal{E}_1 < \dots < \mathcal{E}_L$ are the distinct energy levels and d_i is the degeneracy of \mathcal{E}_i , can be approximated with great accuracy by the Gaussian law

$$g(\mathcal{E}) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(\mathcal{E}-\mu)^2}{2\sigma^2}} \tag{75}$$

with parameters μ and σ given by the mean and standard deviation of the chain’s spectrum. Since the energy levels are equidistant, this means that

$$\frac{d_i}{\delta\mathcal{E}(m + n)^N} \simeq g(\mathcal{E}_i) \quad (N \gg 1). \tag{76}$$

As an illustration, in Fig. 1 we have plotted both sides of the latter equation in the case $m = n = 1$ and $\epsilon' = -\epsilon = 1$ for $N = 15$ and $N = 30$. The approximate Gaussian character of the level density is thus a property shared by all spin chains of HS type studied so far, of both types A and B [20,27,31,33,34].

Since, by Eq. (76), the level density is asymptotically given by the Gaussian law (75), it is of interest to compute the mean energy and its variance as functions of N , m , n , ϵ and ϵ' , as has been done for other spin chains of HS type (including the purely fermionic version of the chain (5)). Indeed, in Appendix B we show that μ and σ^2 are given by

$$\mu = \frac{1}{2} \left(1 - \frac{m-n}{(m+n)^2} \right) N(N-1) + \left(1 - \frac{\epsilon p_m + \epsilon' p_n}{m+n} \right) \frac{N}{2}, \quad (77)$$

$$\begin{aligned} \sigma^2 = & \left(1 - \frac{(m-n)^2}{(m+n)^4} \right) \frac{N}{36} (4N^2 + 6N - 1) + \frac{32mn}{9(m+n)^4} N(N-1)(N-2) \\ & + \frac{2}{(m+n)^3} (n\epsilon p_m - m\epsilon' p_n) N(N-1) \\ & + \frac{N}{4(m+n)^2} \left(\left(\frac{m-n}{m+n} \right)^2 - (\epsilon p_m + \epsilon' p_n)^2 \right), \end{aligned} \quad (78)$$

where $p_m, p_n \in \{0, 1\}$ are the parities of m and n , respectively. In particular, when N tends to infinity μ and σ^2 grow as N^2 and N^3 , as for the ordinary (non-supersymmetric) PF chains of types *A* and *B* (cf. Refs. [33,34]).

Another property of the chain's spectrum worth studying is the distribution of spacings between consecutive "unfolded" levels. In general [38], the unfolding of the levels \mathcal{E}_i of a spectrum is the mapping $\mathcal{E}_i \mapsto \eta_i \equiv \eta(\mathcal{E}_i)$, where $\eta(\mathcal{E})$ is the continuous part of the cumulative level density

$$F(\mathcal{E}) \equiv \int_{-\infty}^{\mathcal{E}} f(\mathcal{E}') d\mathcal{E}' = \frac{1}{(m+n)^N} \sum_{i; \mathcal{E}_i \leq \mathcal{E}} d_i.$$

It can be easily shown that the unfolded spectrum $\{\eta_i\}_{i=1}^L$ is uniformly distributed regardless of the initial level density, making it meaningful to compare different spectra. In our case, by the previous discussion we can take $\eta(\mathcal{E})$ as the cumulative Gaussian density (75), namely

$$\eta(\mathcal{E}) = \int_{-\infty}^{\mathcal{E}} g(\mathcal{E}') d\mathcal{E}' = \frac{1}{2} \left[1 + \operatorname{erf} \left(\frac{\mathcal{E} - \mu}{\sqrt{2}\sigma} \right) \right]. \quad (79)$$

One then defines the normalized spacings

$$s_i = (\eta_{i+1} - \eta_i) / \Delta, \quad i = 1, \dots, L-1,$$

where $\Delta \equiv (\eta_L - \eta_1) / (L-1)$ is the mean spacing of the unfolded energies, so that $\{s_i\}_{i=1}^{L-1}$ has unit mean.

As mentioned in the Introduction, the main motivation for studying the spacings density $p(s)$ lies in the Berry–Tabor conjecture, which states that the distribution of spacings of a "generic" quantum integrable system should obey Poisson's law $p(s) = e^{-s}$. By contrast, for a chaotic system the spacings distribution is expected to follow Wigner's surmise $p(s) = (\pi s/2) \exp(-\pi s^2/4)$, as is approximately the case for the Gaussian ensembles in random matrix theory [38]. A detailed study of the spacings distribution has been performed for many spin chains of HS type, namely the PF chains of types *A* and *B* [33,34], the original (type *A*) HS chain [20,34], as well as the supersymmetric version of the latter chain [27]. The

spacings distributions of all these chains, which turn out to be qualitatively very similar, differ essentially from both Wigner’s and Poisson’s distributions. More precisely, for the ordinary (non-supersymmetric) chains mentioned above we showed in our recent papers [33,34] that the cumulative spacings distribution $P(s) \equiv \int_0^s p(s') ds'$ is approximately given by

$$P(s) \simeq 1 - \frac{2}{\sqrt{\pi s_{\max}}} \sqrt{\log\left(\frac{s_{\max}}{s}\right)}, \tag{80}$$

where s_{\max} is the maximum spacing. As a matter of fact, in Ref. [33] we proved that the previous approximation holds for any spectrum $\mathcal{E}_{\min} \equiv \mathcal{E}_1 < \dots < \mathcal{E}_L \equiv \mathcal{E}_{\max}$, provided that the following conditions are satisfied:

- (i) The energies are equispaced, i.e., $\mathcal{E}_{i+1} - \mathcal{E}_i = \delta\mathcal{E}$ for $i = 1, \dots, L - 1$.
- (ii) The level density (normalized to unity) is approximately given by the Gaussian law (75).
- (iii) $\mathcal{E}_{\max} - \mu, \mu - \mathcal{E}_{\min} \gg \sigma$.
- (iv) \mathcal{E}_{\min} and \mathcal{E}_{\max} are approximately symmetric with respect to μ , namely $|\mathcal{E}_{\min} + \mathcal{E}_{\max} - 2\mu| \ll \mathcal{E}_{\max} - \mathcal{E}_{\min}$.

Moreover, when these conditions are satisfied the maximum spacing can be estimated with great accuracy as

$$s_{\max} = \frac{\mathcal{E}_{\max} - \mathcal{E}_{\min}}{\sqrt{2\pi}\sigma}. \tag{81}$$

It should also be noted that Eq. (80) is valid only for spacings $s \geq s_0$, where s_0 is the unique zero of the RHS of this equation, namely

$$s_0 = s_{\max} e^{-\frac{\pi}{4} s_{\max}^2}. \tag{82}$$

Since, by Eq. (81) and condition (iii), we have (for instance)

$$\frac{\pi}{4} s_{\max}^2 > \frac{(\mathcal{E}_{\max} - \mu)^2}{8\sigma^2} \gg 1,$$

it follows that $s_0 \ll s_{\max}$. The validity of the above conditions for the chain (5) when $N \gg 1$ was checked in Ref. [33] in the purely fermionic case, which automatically implies their fulfillment in the purely bosonic case on account of the duality relation (73). For this reason, we shall assume in the rest of this section that $m, n \geq 1$. We shall see below that in this case conditions (i)–(iii) are also satisfied when $N \gg 1$, while condition (iv) only holds for $m = n$. It turns out, however, that when one drops condition (iv) the approximate formula (80) still holds, albeit in a slightly smaller range.

Indeed, it was proved in Ref. [33] that conditions (i)–(iii) imply that the spacings s_i are approximately related to the energies \mathcal{E}_i by

$$s_i \simeq s_{\max} e^{-\frac{(\mathcal{E}_i - \mu)^2}{2\sigma^2}}. \tag{83}$$

Hence

$$P(s) \equiv \frac{\#(s_i \leq s)}{L - 1} = \frac{1}{L - 1} [\#(\mathcal{E}_{\min} \leq \mathcal{E}_i \leq \mathcal{E}_-) + \#(\mathcal{E}_+ \leq \mathcal{E}_i < \mathcal{E}_{\max})], \tag{84}$$

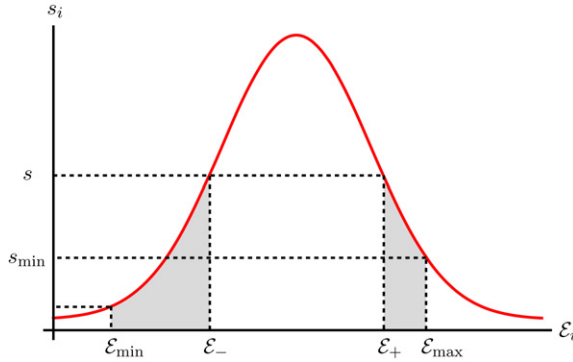


Fig. 2. Plot of the RHS of Eq. (83) showing the points \mathcal{E}_{\min} , \mathcal{E}_{\max} , and the roots \mathcal{E}_{\pm} corresponding to a spacing $s \geq s_{\min}$.

where

$$\mathcal{E}_{\pm} = \mu \pm \sqrt{2\sigma} \sqrt{\log\left(\frac{s_{\max}}{s}\right)} \tag{85}$$

are the roots of the equation $s = s_{\max} e^{-\frac{(\mathcal{E}-\mu)^2}{2\sigma^2}}$, cf. Fig. 2. Using condition (i) to estimate the RHS of the latter equation we easily obtain

$$P(s) \simeq \frac{1}{(L-1)\delta\mathcal{E}} [\max(\mathcal{E}_- - \mathcal{E}_{\min}, 0) + \max(\mathcal{E}_{\max} - \mathcal{E}_+, 0)]. \tag{86}$$

If

$$\begin{aligned} s_{\min} &= \max\left(s_{\max} e^{-\frac{(\mathcal{E}_{\min}-\mu)^2}{2\sigma^2}}, s_{\max} e^{-\frac{(\mathcal{E}_{\max}-\mu)^2}{2\sigma^2}}\right) \\ &\equiv s_{\max} e^{-\frac{1}{2\sigma^2} [\min(\mu - \mathcal{E}_{\min}, \mathcal{E}_{\max} - \mu)]^2}, \end{aligned} \tag{87}$$

then for $s \geq s_{\min}$ we have $\mathcal{E}_- \geq \mathcal{E}_{\min}$ and $\mathcal{E}_+ \leq \mathcal{E}_{\max}$; see again Fig. 2. Hence, for $s \geq s_{\min}$ Eq. (86) becomes

$$P(s) \simeq \frac{1}{(L-1)\delta\mathcal{E}} (\mathcal{E}_- - \mathcal{E}_{\min} + \mathcal{E}_{\max} - \mathcal{E}_+). \tag{88}$$

Since (again by condition (i)) $\mathcal{E}_{\max} - \mathcal{E}_{\min} = (L-1)\delta\mathcal{E}$, substituting Eq. (85) into (88) and using (81) we immediately arrive at the approximation (80) for $P(s)$. We have thus shown that conditions (i)–(iii) imply that Eq. (80) holds for $s \geq s_{\min}$. Note, finally, that the minimum spacing s_{\min} satisfies the inequalities

$$s_0 \leq s_{\min} \ll s_{\max}.$$

Indeed, the second inequality is an immediate consequence of Eq. (87) and condition (iii). As to the first one, by Eqs. (81), (82) and (87) we have

$$\begin{aligned} \log\left(\frac{s_0}{s_{\max}}\right) &= -\frac{1}{8\sigma^2} (\mathcal{E}_{\max} - \mathcal{E}_{\min})^2 \\ &\leq -\frac{1}{8\sigma^2} [2 \min(\mu - \mathcal{E}_{\min}, \mathcal{E}_{\max} - \mu)]^2 = \log\left(\frac{s_{\min}}{s_{\max}}\right), \end{aligned}$$

where the equality holds if and only if \mathcal{E}_{\min} and \mathcal{E}_{\max} are symmetric about μ .

Let us next verify that the spectrum of the chain (5) satisfies conditions (i)–(iii) when $N \gg 1$ and $m, n \geq 1$. We have already seen at the beginning of this section that conditions (i) and (ii) hold for sufficiently large N . In order to check the validity of condition (iii), we need to compute in closed form the chain’s maximum and minimum energies. By Eq. (11), the minimum energy is given by

$$\mathcal{E}_{\min} = \lim_{a \rightarrow \infty} \frac{1}{a} (E_{\min} - E_{\min}^{\text{sc}}),$$

where E_{\min}^{sc} and E_{\min} are the minimum energies of the scalar and spin dynamical models (9) and (8), respectively. From the discussion in Section 3 it follows that $E_{\min}^{\text{sc}} = E_0$, so that (cf. Eq. (21)) \mathcal{E}_{\min} is the minimum value of $|\mathbf{k}|$, where \mathbf{k} is any multi-index compatible with conditions (i)–(iii) in Section 3. If $m > 1$ we can obviously take $\mathbf{k} = 0$ (and, for instance, $\mathbf{s} = ((s, 0), \dots, (s, 0))$, where s is any positive spin component), so that $\mathcal{E}_{\min} = 0$ in this case. Similarly, when $m = \epsilon = 1$ the minimum energy is again zero, since the pair $\mathbf{k} = 0$ and $\mathbf{s} = ((0, 0), \dots, (0, 0))$ satisfies the required conditions. However, if $m = 1$ and $\epsilon = -1$, condition (iii) implies that k_i must be odd when $s_i^{\downarrow} = 0$. Hence, the multi-index \mathbf{k} yielding the minimum energy is of the form

$$\mathbf{k} = \begin{cases} (\overbrace{1, \dots, 1}^{N-\bar{n}(0)}, \overbrace{0, \dots, 0}^{\bar{n}(0)}), & \bar{n}(0) < N, \\ (0, \dots, 0), & \bar{n}(0) \geq N, \end{cases}$$

with $\mathcal{E}_{\min} = \max(N - \bar{n}(0), 0)$, where the notation $\bar{n}(k_j)$ is defined by the fermionic analogue of Eq. (20). (A compatible vector \mathbf{s} is obtained by filling the rightmost $\bar{n}(0)$ components with all possible nonnegative fermionic spin values, and the rest (if any) with the unique bosonic spin $s_i^{\downarrow} = 0$.) In summary, if $m, n \geq 1$ the minimum energy is given by

$$\mathcal{E}_{\min} = \begin{cases} 0, & m > 1, \text{ or } m = \epsilon = 1, \\ \max(N - \bar{n}(0), 0), & m = -\epsilon = 1. \end{cases} \tag{89}$$

As to the maximum energy, the duality relation (73) and the previous equation imply that

$$\mathcal{E}_{\max} = \begin{cases} N^2, & n > 1, \text{ or } n = -\epsilon' = 1, \\ N^2 - \max(N - \bar{m}(0), 0), & n = \epsilon' = 1. \end{cases} \tag{90}$$

From Eqs. (77), (78) and the last two equations, it easily follows that both $(\mathcal{E}_{\max} - \mu)/\sigma$ and $(\mu - \mathcal{E}_{\min})/\sigma$ are $O(N^{\frac{1}{2}})$ as $N \rightarrow \infty$, so that condition (iii) above is indeed satisfied. On the other hand, by Eqs. (77), (89) and (90) we have

$$\frac{|\mathcal{E}_{\min} + \mathcal{E}_{\max} - 2\mu|}{\mathcal{E}_{\max} - \mathcal{E}_{\min}} = \frac{|m - n|}{(m + n)^2} + O(N^{-1}) \quad (m, n \geq 1),$$

so that condition (iv) is not satisfied unless $m = n$, as claimed above.

We have verified that the approximation (80)–(81) is in excellent agreement with the numerical data obtained using our exact formulas for the partition function for many different values of $m, n \geq 1$ and $N \geq 15$. For instance, in Fig. 3 we plot the cumulative spacings distribution $P(s)$ versus its analytic approximation (80) in the cases $m = n = -\epsilon = \epsilon' = 1$ and $m = 2, n = \epsilon = \epsilon' = 1$ for $N = 15$ and $N = 30$ spins. The respective root mean square errors (normalized to the mean) drop from 9.4×10^{-3} and 4.6×10^{-3} for $N = 15$, to 2.4×10^{-3} and 1.1×10^{-3} for $N = 30$. It is worth noting that the approximation (80)–(81) for the cumulative

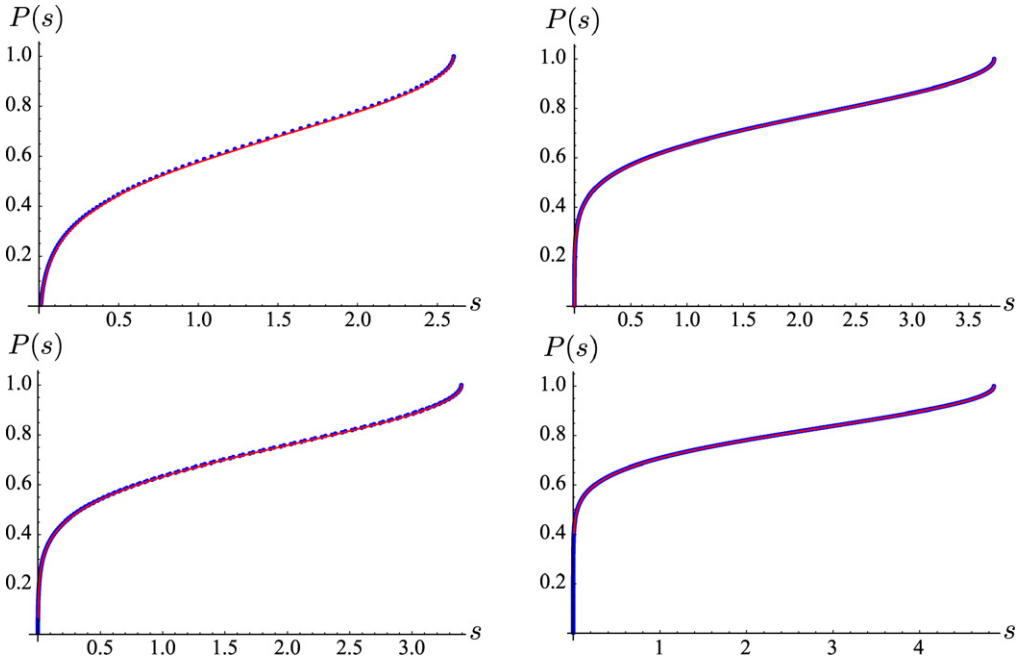


Fig. 3. Cumulative spacings distribution $P(s)$ (blue dots) and its analytic approximation (80)–(81) (continuous red line) in the cases $m = n = -\epsilon = \epsilon' = 1$ (top) and $m = 2, n = \epsilon = \epsilon' = 1$ (bottom) for $N = 15$ (left) and $N = 30$ (right). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this paper.)

spacings distribution contains no adjustable parameters, since the maximum spacing s_{\max} is completely determined by Eqs. (78), (89) and (90). In fact, from the previous equations one easily obtains the following asymptotic formula for the maximum spacing in the genuinely supersymmetric case $m, n \geq 1$:

$$s_{\max} = \frac{3}{\sqrt{2\pi}} \left(1 + \frac{34mn - m^2 - n^2}{(m+n)^4} \right)^{-\frac{1}{2}} N^{\frac{1}{2}} + O(N^{-\frac{1}{2}}).$$

Thus the behavior of s_{\max} for large N is qualitatively the same as in the non-supersymmetric cases studied in Ref. [33].

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Appendix A. Some useful q -number identities

In this appendix we shall collect several identities involving q -numbers that are used in the simplification of the partition function $Z_{\epsilon\epsilon'}^{(m|n)}$ of the chain (5) when either m or n are odd. Given

a real number $q \in (0, 1)$ and a nonnegative integer k , we define the symbols $(q)_k$ and $[k]_q$ by

$$(q)_k = \prod_{i=1}^k (1 - q^i), \quad [k]_q = \frac{1 - q^k}{1 - q} = \sum_{i=0}^{k-1} q^i, \tag{A.1}$$

with $(q)_0 \equiv 1$. The q -factorial of k is then defined as

$$[k]_q! = \prod_{i=1}^k [i]_q.$$

Finally, if k, l are nonnegative integers with $k \geq l$, we define the q -binomial coefficient

$$\begin{bmatrix} k \\ l \end{bmatrix}_q = \frac{[k]_q!}{[l]_q! [k-l]_q!}.$$

From the above definitions it immediately follows the useful equality

$$\begin{bmatrix} k \\ l \end{bmatrix}_q = \frac{(q)_k}{(q)_l (q)_{k-l}}. \tag{A.2}$$

An important identity satisfied by the q -binomial coefficients is the following instance of Newton’s q -binomial formula [39, Eq. (8)]:

$$\sum_{i=0}^k \begin{bmatrix} k \\ i \end{bmatrix}_q q^{\binom{i}{2}} x^i = \prod_{i=0}^{k-1} (1 + xq^i). \tag{A.3}$$

From the previous equation with q replaced by q^2 and x by q we obtain the formula

$$\sum_{i=0}^k \begin{bmatrix} k \\ i \end{bmatrix}_{q^2} q^{i^2} = \prod_{i=0}^{k-1} (1 + q^{2i+1}). \tag{A.4}$$

Replacing q by q^2 and setting $x = q^{-1}$ in Eq. (A.3) we obtain the analogous relation

$$\sum_{i=0}^k \begin{bmatrix} k \\ i \end{bmatrix}_{q^2} q^{i(i-2)} = \prod_{i=0}^{k-1} (1 + q^{2i-1}). \tag{A.5}$$

The last three identities are used in Section 5 to compute the partition function $\mathcal{Z}_{\epsilon\epsilon'}^{(1|1)}$.

Likewise, if δ_q is the q -dilation operator, whose action on a smooth function $f : \mathbb{R} \rightarrow \mathbb{R}$ is given by $(\delta_q f)(x) = f(qx)$, we have [39, Eq. (11)]

$$\sum_{i=0}^k \begin{bmatrix} k \\ i \end{bmatrix}_q x^i = (x + \delta_q)^k \cdot 1,$$

where 1 denotes the constant function identically equal to 1. Equating the coefficient of x^i in both sides of the previous equation we obtain the remarkable identity

$$\begin{bmatrix} k \\ i \end{bmatrix}_q = q^{-\binom{i}{2}} \tau_i(1, q, \dots, q^{k-1}), \tag{A.6}$$

where $\tau_i(x_1, \dots, x_k)$ denotes the elementary symmetric polynomial of degree i in k variables, defined by

$$\tau_i(x_1, \dots, x_k) = \sum_{1 \leq j_1 < \dots < j_i \leq k} x_{j_1} \cdots x_{j_i}. \tag{A.7}$$

In particular, from Eq. (A.6) it easily follows that the q -binomial coefficient $\begin{bmatrix} k \\ i \end{bmatrix}_q$ is a polynomial of degree $i(k - i)$ in q .

Appendix B. Computation of the mean and variance of the energy

In this appendix we shall compute the mean and variance of the energy of the chain (5) for arbitrary values of N, m, n, ϵ and ϵ' . In the purely fermionic case, these quantities were computed in Ref. [33] using the formulas for the traces (of products) of the operators S_{ij} and S_i given in Ref. [31]. In the genuinely supersymmetric case, one should evaluate the traces of products involving the more general operators $S_{ij}^{(m|n)}$ and $S_i^{\epsilon\epsilon'}$. Indeed, the traces of $S_{ij}^{(m|n)}$ and $S_{ij}^{(m|n)} S_{kl}^{(m|n)}$, which are the only ones appearing in the chain of type A, were computed in detail in Ref. [27]. The calculation of the traces involving the spin reversal operator $S_i^{\epsilon\epsilon'}$ is totally analogous, so that we shall limit ourselves to listing the final results in Table B.1. In this table $p_m, p_n \in \{0, 1\}$ are the parities of m and n , respectively, and we have omitted the traces of products which reduce to the identity matrix or to one of the entries.

In the first place, calling

$$h_{ij} = (\xi_i - \xi_j)^{-2}, \quad \tilde{h}_{ij} = (\xi_i + \xi_j)^{-2}, \quad h_i = \beta \xi_i^{-2},$$

and using the formulas in Table B.1 for the traces of the spin operators we easily obtain

$$\begin{aligned} \mu &= \frac{\text{tr} \mathcal{H}_{\epsilon\epsilon'}^{(m|n)}}{(m+n)^N} \\ &= \left(1 - \frac{m-n}{(m+n)^2}\right) \sum_{i \neq j} (h_{ij} + \tilde{h}_{ij}) + \left(1 - \frac{\epsilon p_m + \epsilon' p_n}{m+n}\right) \sum_i h_i. \end{aligned} \tag{B.1}$$

Table B.1
Traces of products of the spin operators $S_{ij}^{(m|n)}, \tilde{S}_{ij}^{(m|n)}$ and $S_i^{\epsilon\epsilon'}$

Operator	Trace
$S_i^{\epsilon\epsilon'}$	$(m+n)^{N-1}(\epsilon p_m + \epsilon' p_n)$
$S_{ij}^{(m n)}, \tilde{S}_{ij}^{(m n)}$	$(m-n)(m+n)^{N-2}$
$S_i^{\epsilon\epsilon'} S_j^{\epsilon\epsilon'}$	$(m+n)^{N-2}(\epsilon p_m + \epsilon' p_n)^2$
$S_{ij}^{(m n)} S_k^{\epsilon\epsilon'}, \tilde{S}_{ij}^{(m n)} S_k^{\epsilon\epsilon'}$	$(m-n)(m+n)^{N-3}(\epsilon p_m + \epsilon' p_n)$
$S_{ij}^{(m n)} S_i^{\epsilon\epsilon'}, \tilde{S}_{ij}^{(m n)} S_i^{\epsilon\epsilon'}$	$(m+n)^{N-2}(\epsilon p_m - \epsilon' p_n)$
$S_{ij}^{(m n)} S_{kl}^{(m n)}, S_{ij}^{(m n)} \tilde{S}_{kl}^{(m n)}, \tilde{S}_{ij}^{(m n)} \tilde{S}_{kl}^{(m n)}$	$(m-n)^2(m+n)^{N-4}$
$S_{ij}^{(m n)} S_{il}^{(m n)}, S_{ij}^{(m n)} \tilde{S}_{il}^{(m n)}, \tilde{S}_{ij}^{(m n)} \tilde{S}_{il}^{(m n)}$	$(m+n)^{N-2}$

The sums in the previous expression were evaluated in Ref. [33], with the result

$$\sum_{i \neq j} (h_{ij} + \tilde{h}_{ij}) = \frac{1}{2}N(N - 1), \quad \sum_i h_i = \frac{N}{2}.$$

Inserting these values into (B.1) we immediately arrive at Eq. (77) for the mean energy μ .

The evaluation of the standard deviation of the energy

$$\sigma^2 = \frac{\text{tr}[(\mathcal{H}_{\epsilon\epsilon'}^{(m|n)})^2]}{(m + n)^N} - \mu^2, \tag{B.2}$$

which is considerably more involved, is also performed in two steps. We begin by expanding Eq. (B.2) and simplifying the resulting expression using Eq. (B.1) and the trace formulas in Table B.1. After a long but straightforward calculation we obtain

$$\begin{aligned} \sigma^2 = & \left(1 - \frac{(m - n)^2}{(m + n)^4}\right) \left(2 \sum_{i \neq j} (h_{ij}^2 + \tilde{h}_{ij}^2) + \sum_i h_i^2\right) \\ & + \frac{4}{(m + n)^2} \left(\left(\frac{m - n}{m + n}\right)^2 - (\epsilon p_m + \epsilon' p_n)^2\right) \left(\frac{1}{4} \sum_i h_i^2 - \sum_{i \neq j} h_{ij} \tilde{h}_{ij}\right) \\ & + \frac{8}{(m + n)^3} (n \epsilon p_m - m \epsilon' p_n) \sum_{i \neq j} (h_{ij} + \tilde{h}_{ij}) h_i \\ & + \frac{16mn}{(m + n)^4} \sum'_{i,j,k} (h_{ij} + \tilde{h}_{ij})(h_{ik} + \tilde{h}_{ik}), \end{aligned} \tag{B.3}$$

where the symbol $\sum'_{i,j,k}$ denotes summation over $i \neq j \neq k \neq i$. The sums in the first two lines of Eq. (B.3), which were evaluated in Ref. [33], are given by

$$2 \sum_{i \neq j} (h_{ij}^2 + \tilde{h}_{ij}^2) + \sum_i h_i^2 = \frac{N}{36} (4N^2 + 6N - 1), \tag{B.4}$$

$$\frac{1}{4} \sum_i h_i^2 - \sum_{i \neq j} h_{ij} \tilde{h}_{ij} = \frac{N}{16}. \tag{B.5}$$

In order to evaluate the last two sums in (B.3), we recall from Appendix A of Ref. [33] the following identities involving the zeros $y_i = \xi_i^2/2$ of the Laguerre polynomial $L_N^{\beta-1}$ and their differences $y_{ij} \equiv y_i - y_j$:

$$\sum_i \frac{1}{y_i^2} = \frac{N(N + \beta)}{\beta^2(\beta + 1)}, \tag{B.6a}$$

$$\sum_i \frac{1}{y_i} = \frac{N}{\beta}, \tag{B.6b}$$

$$\sum_i y_i = N(N + \beta - 1), \tag{B.6c}$$

$$\sum_i y_i^2 = N(N + \beta - 1)(2N + \beta - 2), \tag{B.6d}$$

$$\sum_{j,j \neq i} \frac{y_i}{y_{ij}} = \frac{1}{2}(y_i - \beta), \tag{B.6e}$$

$$\sum_{j,j \neq i} \frac{y_i^2}{y_{ij}^2} = \frac{1}{12}(-y_i^2 + 2(2N + \beta)y_i - \beta(\beta + 4)), \tag{B.6f}$$

$$\sum_{i \neq j} \frac{1}{y_{ij}^2} = \frac{N(N - 1)}{4(\beta + 1)}, \tag{B.6g}$$

$$\sum_{i \neq j} \frac{y_i}{y_{ij}^2} = \frac{1}{4}N(N - 1), \tag{B.6h}$$

$$\sum_{i \neq j} \frac{y_i^2}{y_{ij}^4} = \frac{N(N - 1)((2N + 5)\beta + 2N + 14)}{144(\beta + 1)}. \tag{B.6i}$$

Multiplying Eq. (B.6e) by y_i^{-2} and y_i , summing over i the resulting identities, and using (B.6a)–(B.6d), one easily obtains

$$\sum_{i \neq j} \frac{y_i^{-1}}{y_{ij}} = -\frac{N(N - 1)}{2\beta(\beta + 1)}, \quad \sum_{i \neq j} \frac{y_i^2}{y_{ij}} = N(N - 1)(N + \beta - 1). \tag{B.7}$$

Similarly, summing Eq. (B.6f) over i and using (B.6c)–(B.6d) we get

$$\sum_{i \neq j} \frac{y_i^2}{y_{ij}^2} = \frac{1}{12}N(N - 1)(2N + 3\beta + 2). \tag{B.8}$$

The sum in the third line of Eq. (B.3) can now be easily computed. Indeed,

$$\begin{aligned} &\sum_{i \neq j} (h_{ij} + \tilde{h}_{ij})h_i \\ &= \frac{\beta}{2} \sum_{i \neq j} \frac{y_i + y_j}{y_i y_{ij}^2} = \beta \sum_{i \neq j} \frac{1}{y_{ij}^2} - \frac{\beta}{2} \sum_{i \neq j} \frac{y_i^{-1}}{y_{ij}} = \frac{1}{4}N(N - 1), \end{aligned} \tag{B.9}$$

where we have used Eqs. (B.6g) and (B.7). Turning next to the last sum in Eq. (B.3), we have

$$\begin{aligned} &\sum'_{i,j,k} (h_{ij} + \tilde{h}_{ij})(h_{ik} + \tilde{h}_{ik}) \\ &= \sum'_{i,j,k} \frac{(y_i + y_j)(y_i + y_k)}{y_{ij}^2 y_{ik}^2} = \sum'_{i,j,k} \frac{4y_i^2}{y_{ij}^2 y_{ik}^2} - \sum'_{i,j,k} \frac{4y_i}{y_{ij}^2 y_{ik}} + \sum'_{i,j,k} \frac{1}{y_{ij} y_{ik}}. \end{aligned} \tag{B.10}$$

The last sum in the previous equation clearly vanishes on account of the identity

$$\frac{1}{(z_1 - z_2)(z_1 - z_3)} + \frac{1}{(z_2 - z_3)(z_2 - z_1)} + \frac{1}{(z_3 - z_1)(z_3 - z_2)} = 0,$$

valid for arbitrary $z_1 \neq z_2 \neq z_3 \neq z_1$. On the other hand, the first two sums in the RHS of Eq. (B.10) can be computed using Eqs. (B.6)–(B.8). Indeed,

$$\begin{aligned} \sum'_{i,j,k} \frac{4y_i^2}{y_{ij}^2 y_{ik}^2} &= \sum_{i \neq j} \frac{4}{y_{ij}^2} \left(\sum_{k,k \neq i} \frac{y_i^2}{y_{ik}^2} - \frac{y_i^2}{y_{ij}^2} \right) \\ &= \frac{1}{3} \sum_{i \neq j} \frac{1}{y_{ij}^2} (-y_i^2 + 2(2N + \beta)y_i - \beta(\beta + 4)) - 4 \sum_{i \neq j} \frac{y_i^2}{y_{ij}^4} \\ &= \frac{2}{9} N(N - 1)(N - 2), \end{aligned}$$

where we have used Eqs. (B.6f)–(B.6h) and (B.8). Likewise,

$$\begin{aligned} \sum'_{i,j,k} \frac{y_i}{y_{ij}^2 y_{ik}^2} &= \sum_{i \neq j} \frac{1}{y_{ij}^2} \left(\sum_{k,k \neq i} \frac{y_i}{y_{ik}^2} - \frac{y_i}{y_{ij}^2} \right) = \frac{1}{2} \sum_{i \neq j} \frac{y_i - \beta}{y_{ij}^2} - \sum_{i \neq j} \frac{y_i}{y_{ij}^3} \\ &= \frac{1}{2} \sum_{i \neq j} \frac{y_i - \beta}{y_{ij}^2} - \frac{1}{2} \sum_{i \neq j} \frac{1}{y_{ij}^2} = \frac{1}{2} \sum_{i \neq j} \frac{y_i}{y_{ij}^2} - \frac{\beta + 1}{2} \sum_{i \neq j} \frac{1}{y_{ij}^2} = 0, \end{aligned}$$

on account of Eqs. (B.6g)–(B.6h). The LHS of Eq. (B.10) thus reduces to

$$\sum'_{i,j,k} (h_{ij} + \tilde{h}_{ij})(h_{ik} + \tilde{h}_{ik}) = \frac{2}{9} N(N - 1)(N - 2). \tag{B.11}$$

Substituting Eqs. (B.4), (B.5), (B.9) and (B.10) into (B.3) we finally obtain the expression (78) for σ^2 in Section 7.

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