Exchange operator formalism for N-body spin models with near-neighbors interactions

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Abstract

We present a detailed analysis of the spin models with near-neighbors interactions constructed in our previous paper [1] by a suitable generalization of the exchange operator formalism. We provide a complete description of a certain flag of finite-dimensional spaces of spin functions preserved by the Hamiltonian of each model. By explicitly diagonalizing the Hamiltonian in the latter spaces, we compute several infinite families of eigenfunctions of the above models in closed form in terms of generalized Laguerre and Jacobi polynomials.

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1. INTRODUCTION

The discovery of the quantum models named after Calogero [2] and Sutherland [3, 4] is a key development in the theory of integrable systems which has exerted a far-reaching influence on many different areas of Mathematics and Physics. This is borne out by the relevance of these models in such disparate fields as group theory [5, 6], the theory of special functions and orthogonal polynomials [7, 8, 9, 10], soliton theory [11], random matrix theory [12, 13, 14], quantum field theory [15, 16, 17], etc. The Calogero and Sutherland models describe a system of N quantum particles in a line or circle, respectively, with pairwise interactions inversely proportional to the square of the distance. Over the years, many different generalizations of these models have been considered in the literature. One such significant extension was proposed in the early eighties by Olshanetsky and Perelomov [18], who showed that both the Calogero and Sutherland models are limiting cases of a more general integrable model with a two-body interaction potential of elliptic type. The integrability of the latter model was explained by expressing the Hamiltonian as one of the radial components of the Laplace–Beltrami operator in a symmetric space associated with the A_{N-1} root system. It was also shown in Ref. [18] that one can construct integrable generalizations of the Calogero–Sutherland (CS) models associated with any classical (extended) root system, like BC_N.

Another essential feature of the original Calogero and Sutherland models and their generalization to other root systems is their exact solvability, that is, the fact that the whole spectrum can be computed in closed form using algebraic techniques. In the last decade, some authors have introduced further extensions of CS models which are quasi-exactly solvable, in the sense that only part of the spectrum can be computed algebraically [19, 20, 21, 22]. In all of these quasi-exactly solvable CS models, the Hamiltonian can be expressed as a polynomial in the generators of a realization of the Lie algebra $\mathfrak{sl}_{N+1}$ in terms of first-order differential operators. Since these operators leave invariant a finite-dimensional space of functions, the Hamiltonian is guaranteed to possess a finite number of eigenfunctions belonging to this space.

A great deal of attention has also been devoted to constructing models of CS type for particles with internal degrees of freedom (typically spin), partly motivated by their intimate connection with integrable spin chains of Haldane–Shastry type [23, 24]. Two main approaches have been followed in order to incorporate spin into CS models, based either on supersymmetry [25, 26, 27, 28, 29] or the exchange (also known as Dunkl) operator formalism [8, 30, 31, 32, 33, 34]. The spin models thus obtained include the exactly solvable spin counterparts of the scalar Calogero and Sutherland models of A_N and BC_N type, as well as several quasi-exactly solvable deformations thereof, some of them with elliptic potentials [35, 36]. A common property shared by all of these models is the long-range character of the interaction potential, in the sense that all particles interact with each other.

The connection between spin CS models and spin chains of Haldane–Shastry type was first elucidated by Polychronakos through a mechanism known as the "freezing trick" [37]. The main idea is that in the large coupling constant limit the particles in a (dynamical) spin CS model freeze at the classical equilibrium of the scalar part of the potential, thus giving rise to a spin chain with long-range position-dependent interactions. In this limit the eigenfunctions of the spin CS model factorize into the product of an eigenfunction of the corresponding scalar CS model times an eigenfunction of the associated spin chain. If *all* the eigenfunctions of both the scalar and spin CS models are known, the partition function of the corresponding spin chain can be exactly computed from those of the scalar and spin CS models [38, 39, 40].

A few years ago, Jain and Khare presented a novel class of scalar CS-like models of A_N type, characterized by the fact that each particle only interacts with its nearest and next-to-nearest neighbors [41]. In a subsequent paper [42], Auberson, Jain and Khare discussed a generalization of these models to the BC_N root system and to higher dimensions. The latter papers, however, left open some important issues, such as the exact or quasi-exact solvability of these models, the derivation of general explicit formulas for their eigenfunctions and the existence of similar models for particles with spin. The last question was first addressed by Deguchi and Ghosh [43], who introduced and partially solved several spin 1/2 extensions of the scalar models of Jain and Khare using the supersymmetric approach. By a suitable generalization of the exchange operator formalism, in our previous paper [1] we constructed the three spin models of A_N type with near neighbors interactions listed in Eq. (2) below. A significant property of these models is the fact that the spin chains obtained from them by the freezing trick feature short-range position-dependent interactions, and thus occupy an intermediate position between the Heisenberg chain (with short-range position-independent interactions) and the spin chains of Haldane–Shastry type (possessing long-range position-dependent interactions). In Ref. [1] we presented without proof closed-form expressions for several infinite families of eigenfunctions of the scalar reductions of all three models, considerably generalizing the results of Ref. [42]. We were also able to derive similar expressions for a wide class of spin eigenfunctions of the models (2b) and (2c). The computation of the spin eigenfunctions for the remaining model (2a), which is probably the most interesting one due to the rich structure of its finite-dimensional invariant spaces, was not undertaken in Ref. [1].

In this paper we present a detailed analysis of the models (2), with special emphasis on the rational model (2a). In particular, we have achieved a complete description of the flag of invariant finite-dimensional spaces for the latter model presented in Ref. [1]. More importantly, we have found that this flag can be further enlarged with an additional family of spin functions. We have computed all the eigenfunctions of the model (2a) belonging to the new flag, thereby obtaining seven infinite families of spin eigenfunctions in closed form. These eigenfunctions have been expressed in all cases in a compact way in terms of generalized Laguerre and Jacobi polynomials. The resulting expressions will be used in a forthcoming paper for computing a number of eigenvalues and eigenfunctions of the spin chain obtained from the model (2a) by taking the strong coupling limit.

The paper is organized as follows. In Section 2 we define the Hamiltonians of the spin many-body models which are the subject of this work, and show that they can be expressed in terms of suitable differential operators with near-neighbors exchange terms. Section 3 is entirely devoted to the characterization of certain finite-dimensional spaces of polynomial spin functions invariant under these operators. The first part of this section deals with the construction of the latter spaces and the proof of their invariance, cf. Theorem 1 and Corollary 1. In the rest of the section we complete the description of the invariant spaces for the model (2a), by identifying the spin states which satisfy a restriction stated in Theorem 1. In Section 4 we show that the eigenvalue problems for the Hamiltonians of the models (2) restricted to their invariant spaces reduce to finding the polynomial solutions of a corresponding system of differential equations. By completely solving the latter problem, we obtain several (infinite) families of eigenfunctions for the models (2), whose explicit expressions are presented in Theorems 2–4. Finally, in Section 5 we summarize our results and outline some related open problems.

2. THE MODELS

In this section we shall introduce the three types of N-body models with near-neighbors interactions whose study is the aim of this paper. We shall also recall from our previous paper [1] the relation of each of these models with a corresponding differential-difference operator involving near-neighbors exchange operators.

Let us begin with some preliminary definitions. We shall denote by $|s_1 \dots s_N\rangle$, where $s_i = -M, -M+1, \dots, M$ and $M \in \frac{1}{2}\mathbb{N}$, the elements of the canonical basis of the space Σ of the particles' internal degrees of freedom (SU$(2M+1)$ spin). The action of the spin permutation operators S_{ij} on this basis is given by

$$S_{ij}|\dots s_i \dots s_j \dots\rangle = |\dots s_j \dots s_i \dots\rangle.$$

The operators S_{ij} can be expressed in terms of the fundamental SU$(2M+1)$ generators S_i^a, $a = 1, \dots, 4M(M+1)$, as $S_{ij} = 1/(2M+1) + \sum_a S_i^a S_j^a$. The Hamiltonians of the models we shall be concerned with are given by

$$H_\epsilon = -\sum_i \partial_{x_i}^2 + V_\epsilon \,, \qquad \epsilon = 0, 1, 2, \tag{1}$$

where

$$V_0 = \omega^2 r^2 + \sum_i \frac{2a^2}{(x_i - x_{i-1})(x_i - x_{i+1})} + \sum_i \frac{2a}{(x_i - x_{i+1})^2}\,(a - S_{i,i+1}), \tag{2a}$$

$$V_1 = \omega^2 r^2 + \sum_i \frac{b(b-1)}{x_i^2} + \sum_i \frac{8a^2 x_i^2}{(x_i^2 - x_{i-1}^2)(x_i^2 - x_{i+1}^2)} + 4a \sum_i \frac{x_i^2 + x_{i+1}^2}{(x_i^2 - x_{i+1}^2)^2}\,(a - S_{i,i+1}), \tag{2b}$$

$$V_2 = 2a^2 \sum_i \cot(x_i - x_{i-1}) \cot(x_i - x_{i+1}) + 2a \sum_i \csc^2(x_i - x_{i+1})\,(a - S_{i,i+1}), \tag{2c}$$

with $r^2 = \sum_i x_i^2$ and $a, b > 1/2$. Here and in what follows, all sums and products run from 1 to N unless otherwise stated, with the identifications $x_0 \equiv x_N$ and $x_{N+1} \equiv x_1$. A few remarks on the configuration spaces of these models are now in order. In all three models the potential diverges as $(x_i - x_{i+1})^{-2}$ on the hyperplanes $x_i = x_{i+1}$, so that the particles i and $i+1$ cannot overtake one another. Since we are interested in models with nearest and next-to-nearest neighbors interactions, we shall henceforth assume that $x_1 < \dots < x_N$. For the second potential (2b) we shall take in addition $x_1 > 0$, due to the double pole at $x_i = 0$. For a similar reason, we shall assume that $x_{i+1} - x_i < \pi$ for the potential (2c).

Remark 1. The Hamiltonians (1) admit *scalar reductions* $H_\epsilon^{\rm sc} \equiv H_\epsilon\big|_{S_{i,i+1}\to 1}$ satisfying the obvious identity

$$H_\epsilon(\psi|s\rangle) = (H_\epsilon^{\rm sc}\psi)|s\rangle\,,$$

where ψ is a scalar function of the coordinates $\mathbf{x} = (x_1, \dots, x_N)$ and $|s\rangle$ is a totally symmetric spin state. It follows that the spin Hamiltonians H_ϵ possess *factorized eigenfunctions* of the form $\Psi = \psi|s\rangle$, where ψ is an eigenfunction of the corresponding scalar Hamiltonian $H_\epsilon^{\rm sc}$ and $|s\rangle$ is again a symmetric spin state. The scalar reductions of the models (2a) and (2c) were introduced by Auberson, Jain and Khare in Ref. [42], whereas that of the model (2b) first appeared in our paper [1]. It should also be noted that for spin 1/2 the potentials V_0 and V_2 differ from those studied by Deguchi and Ghosh in Ref. [43] by a spin-dependent term.

The models (2) share a common property that is ultimately responsible for their partial solvability, namely that each Hamiltonian H_ϵ is related to a scalar differential-difference operator involving near-neighbors exchange operators. Indeed, let K_{ij} denote the operator whose action on a smooth function f of the (possibly complex) coordinates $\mathbf{z} = (z_1, \dots, z_N)$ is given by

$$(K_{ij}f)\,(z_1,\dots,z_i,\dots,z_j,\dots,z_N) = f(z_1,\dots,z_j,\dots,z_i,\dots,z_N). \qquad (3)$$

Given a scalar differential-difference operator D linear in the exchange operator K_{ij}, we shall denote by D^* the differential operator acting on $C^\infty \otimes \Sigma$ obtained from D by the replacement $K_{ij} \to S_{ij}$. One of the key ingredients in our construction is the fact that

$$D\Phi = D^*\Phi\,, \qquad \text{for all} \quad \Phi \in \Lambda\big(C^\infty \otimes \Sigma\big)\,, \qquad (4)$$

where Λ denotes the projector on states totally symmetric under simultaneous permutations of the coordinates and spins. Consider next the second-order differential-difference operators T_ϵ given by

$$T_\epsilon = \sum_i z_i^\epsilon \partial_i^2 + 2a\sum_i \frac{1}{z_i - z_{i+1}}\,(z_i^\epsilon\partial_i - z_{i+1}^\epsilon\partial_{i+1}) - 2a\sum_i \frac{\vartheta_\epsilon(z_i, z_{i+1})}{(z_i - z_{i+1})^2}\,(1 - K_{i,i+1}), \quad (5)$$

where $\partial_i = \partial_{z_i}$, $z_{N+1} \equiv z_1$, and

$$\vartheta_0(x,y) = 1\,, \qquad \vartheta_1(x,y) = \frac{1}{2}\,(x+y)\,, \qquad \vartheta_2(x,y) = xy\,.$$

Each Hamiltonian H_ϵ is related to a linear combination

$$\overline{H}_\epsilon = c\,T_\epsilon + c_- J^- + c_0 J^0 + E_0 \qquad (6)$$

of its corresponding operator T_ϵ and the first-order differential operators

$$J^- = \sum_i \partial_i\,, \qquad J^0 = \sum_i z_i\partial_i \qquad (7)$$

TABLE I: Parameters, gauge factor and change of variable in Eqs. (6) and (8).

	$\epsilon=0$	$\epsilon=1$	$\epsilon=2$
c	-1	-4	4
c_-	0	$-2(2b+1)$	0
c_0	2ω	4ω	$4(1-2a)$
E_0	$N\omega(2a+1)$	$N\omega(4a+2b+1)$	$2Na^2$
$\mu(\mathbf{x})$	$\mathrm{e}^{-\frac{\omega}{2}r^2}\prod_i \lvert x_i-x_{i+1}\rvert^a$	$\mathrm{e}^{-\frac{\omega}{2}r^2}\prod_i \lvert x_i^2-x_{i+1}^2\rvert^a\, x_i^b$	$\prod_i \sin^a \lvert x_i-x_{i+1}\rvert$
$\zeta(x)$	x	x^2	$\mathrm{e}^{\pm 2\mathrm{i}x}$

through the star mapping, a change of variables and a gauge transformation. More precisely,

$$H_\epsilon = \mu\cdot \overline{H}_\epsilon^*\big|_{z_i=\zeta(x_i)}\cdot\mu^{-1}\,,\qquad \epsilon=0,1,2, \tag{8}$$

where the constants c, c_-, c_0, E_0, the gauge factor $\mu(\mathbf{x})$, and the change of variables $\zeta(x)$ for each model are listed in Table I.

From Eqs. (4) and (8) it follows that if $\Phi(\mathbf{z})\in\Lambda\big(C^\infty\otimes\Sigma\big)$ is a symmetric eigenfunction of $\overline{H}_\epsilon$, then

$$\Psi(\mathbf{x})=\mu(\mathbf{x})\Phi(\mathbf{z})\big|_{z_i=\zeta(x_i)} \tag{9}$$

is a (formal) eigenfunction of H_ϵ with the same eigenvalue. In this paper we shall construct a flag $\mathcal{H}_\epsilon^0\subset\mathcal{H}_\epsilon^1\subset\cdots$ of finite-dimensional subspaces of $\Lambda\big(\mathbb{C}[\mathbf{z}]\otimes\Sigma\big)$ invariant under each $\overline{H}_\epsilon$. We will show that the problem of diagonalizing $\overline{H}_\epsilon$ in each subspace $\mathcal{H}_\epsilon^n$ is equivalent to the computation of the polynomial solutions of a system of linear differential equations. We shall completely solve this problem, thereby obtaining several infinite families of eigenfunctions of H_ϵ for each ϵ. From the expressions for the change of variable and the gauge factor in Table I, and the fact that the functions Φ in Eq. (9) are in all cases polynomials, it immediately follows that the eigenfunctions thus obtained are in fact normalizable.

Remark 2. The operators (5) can be expressed as quadratic combinations of the first-order operators

$$D_i^\epsilon = z_i^\epsilon\partial_i\,,\qquad Q_i^\epsilon=\frac{\vartheta_\epsilon(z_i,z_{i+1})}{z_i-z_{i+1}}\,(1-K_{i,i+1})+\frac{\vartheta_\epsilon(z_{i-1},z_i)}{z_i-z_{i-1}}\,(1-K_{i-1,i})\,,$$

where $\epsilon=0,1$, as follows:

$$T_0=\sum_i\Big[\big(D_i^0\big)^2+a\big\{D_i^0,Q_i^0\big\}\Big]\,,\qquad T_1=\sum_i\Big[D_i^1D_i^0+a\big\{D_i^1,Q_i^0\big\}\Big]\,,$$

$$T_2=\sum_i\Big[\big(D_i^1\big)^2+\Big\{D_i^1,aQ_i^1+a-\frac{1}{2}\Big\}\Big]\,.$$

For each nonnegative integer n, the space $\mathcal{P}^n$ of polynomials in $\mathbf{z}$ of total degree at most n is invariant under the operators Q_i^ϵ (see Ref. [35]), and hence also under both T_ϵ and $\overline{H}_\epsilon$. Note, however, that the operators $\overline{H}_\epsilon$ do not commute with the symmetrizer Λ, and thus the previous observation does not imply that they preserve the space $\Lambda(\mathcal{P}^n \otimes \Sigma)$ of symmetric spin functions of polynomial type. Consequently, $\overline{H}_\epsilon$ is not guaranteed *a priori* to admit finite-dimensional invariant subspaces of $\Lambda\big(\mathbb{C}[\mathbf{z}]\otimes\Sigma\big)$. This is in fact the main difference with the usual solvable spin CS models [8, 30, 35, 36, 44, 45, 46], for which the operators analogous to $\overline{H}_\epsilon$ preserve $\mathcal{P}_n$ and commute with Λ, and hence automatically leave invariant the space $\Lambda(\mathcal{P}^n \otimes \Sigma)$.

3. THE INVARIANT SPACES

In this section we shall prove that each operator T_ϵ leaves invariant a flag $\mathcal{T}^0_\epsilon \subset \mathcal{T}^1_\epsilon \subset \cdots$, where $\mathcal{T}^n_\epsilon$ is a finite-dimensional subspace of $\Lambda(\mathcal{P}^n \otimes \Sigma)$. This result will then be used to construct a corresponding invariant flag $\mathcal{H}^0_\epsilon \subset \mathcal{H}^1_\epsilon \subset \cdots$ for the operator $\overline{H}_\epsilon$, where $\mathcal{H}^n_\epsilon \subset \mathcal{T}^n_\epsilon$ for all n.

Let us first introduce the following two sets of elementary symmetric polynomials:

$$\sigma_k = \sum_i z_i^k\,, \qquad \tau_k = \sum_{i_1<\cdots<i_k} z_{i_1}\cdots z_{i_k}\,; \qquad k=1,\dots,N\,.$$

It is well known that any symmetric polynomial in $\mathbf{z}$ can be expressed as a polynomial in either $\boldsymbol{\sigma} \equiv (\sigma_1,\dots,\sigma_N)$ or $\boldsymbol{\tau} \equiv (\tau_1,\dots,\tau_N)$.

We shall denote by $2aX_\epsilon$ the terms of T_ϵ linear in derivatives, that is

$$X_\epsilon = \sum_i \frac{1}{z_i - z_{i+1}}\,\big(z_i^\epsilon \partial_i - z_{i+1}^\epsilon \partial_{i+1}\big)\,.$$

In the next lemma we show that each vector field X_ϵ leaves invariant a corresponding flag $\mathcal{X}^0_\epsilon \subset \mathcal{X}^1_\epsilon \subset \cdots$ of finite-dimensional subspaces of the space $\mathcal{S} \equiv \mathbb{C}[\boldsymbol{\sigma}] = \mathbb{C}[\boldsymbol{\tau}]$ of symmetric polynomials in $\mathbf{z}$.

Lemma 1. *For each $n=0,1,\dots$, the operator X_ϵ leaves invariant the linear space $\mathcal{X}^n_\epsilon$, where*

$$\mathcal{X}^n_0 = \mathbb{C}[\sigma_1,\sigma_2,\sigma_3]\cap\mathcal{P}^n, \quad \mathcal{X}^n_1 = \mathbb{C}[\sigma_1,\sigma_2,\tau_N]\cap\mathcal{P}^n, \quad \mathcal{X}^n_2 = \mathbb{C}[\sigma_1,\tau_{N-1},\tau_N]\cap\mathcal{P}^n.$$

Proof. If f is a function of the symmetric variables $\sigma_1,\sigma_2,\sigma_3,\tau_{N-1},\tau_N$, we shall use from now on the convenient notation

$$f_k = \begin{cases} \partial_{\sigma_k} f\,, & k=1,2,3,\\ \partial_{\tau_k} f\,, & k=N-1,N. \end{cases}$$

Let us first consider the vector field X_0. Since

$$X_0\sigma_k = k\sum_i z_i^{k-1} X_0 z_i = k\bigg(\sum_i \frac{z_i^{k-1}}{z_i - z_{i+1}} - \sum_i \frac{z_i^{k-1}}{z_{i-1} - z_i}\bigg) = \begin{cases} 0\,, & k=1\,,\\ 2N\,, & k=2\,,\\ 6\sigma_1\,, & k=3\,, \end{cases}$$

if $f \in \mathcal{X}_0^n$ we have

$$X_0 f = 2(N f_2 + 3\sigma_1 f_3) \in \mathcal{X}_0^n \,. \tag{10a}$$

The proof for the remaining two cases follows from the analogous formulas

$$X_1 f = N f_1 + 4\sigma_1 f_2 \,, \qquad f \in \mathcal{X}_1^n \,; \tag{10b}$$

$$X_2 f = 2\sigma_1 f_1 + N(\tau_{N-1} f_{N-1} + \tau_N f_N) \,, \qquad f \in \mathcal{X}_2^n \,. \tag{10c}$$

□

Remark 3. It should be noted that these flags cannot be trivially enlarged, since, e.g.,

$$\frac{1}{4} X_0 \sigma_4 = 2\sigma_2 + \sum_i z_i z_{i+1} \,,$$

$$\frac{1}{3} X_1 \sigma_3 = 2\sigma_2 + \sum_i z_i z_{i+1} \,, \qquad X_1 \tau_{N-1} = \tau_N \sum_i (z_i z_{i+1})^{-1} \,,$$

$$\frac{1}{2} X_2 \sigma_2 = 2\sigma_2 + \sum_i z_i z_{i+1} \,, \qquad X_2 \tau_{N-2} = N\tau_{N-2} - \tau_N \sum_i (z_i z_{i+1})^{-1}$$

are not symmetric polynomials.
We note that the restriction of T_ϵ to $\mathcal{X}_\epsilon^n \subset \mathcal{S}$ obviously satisfies

$$T_\epsilon|_{\mathcal{X}_\epsilon^n} = \sum_i z_i^\epsilon \partial_i^2 + 2a X_\epsilon \,. \tag{11}$$

The second-order terms of the operator (11), however, do not preserve the corresponding space $\mathcal{X}_\epsilon^n$, unless one imposes the additional restrictions specified in the following proposition:

Proposition 1. *For each $n = 0, 1, \dots,$ the operator T_ϵ leaves invariant the linear space $\mathcal{S}_\epsilon^n$, where*

$$\mathcal{S}_0^n = \{ f \in \mathcal{X}_0^n \mid f_{33} = 0 \} \,,$$

$$\mathcal{S}_1^n = \{ f \in \mathcal{X}_1^n \mid f_{22} = f_{NN} = 0 \} \,,$$

$$\mathcal{S}_2^n = \{ f \in \mathcal{X}_2^n \mid f_{11} = f_{N-1,N-1} = 0 \} \,.$$

Proof. Let us begin with the operator T_0. If $f \in \mathcal{X}_0^n$, an elementary computation shows that

$$\partial_i f = f_1 + 2 z_i f_2 + 3 z_i^2 f_3 \tag{12}$$

and therefore

$$\sum_i \partial_i^2 f = N(f_{11} + 2f_2) + 2(2f_{12} + 3f_3)\sigma_1 + 2(3f_{13} + 2f_{22})\sigma_2 + 12 f_{23}\sigma_3 + 9 f_{33}\sigma_4 \,. \tag{13}$$

From the previous formula and Eq. (10a) it follows that $T_0 f \in \mathcal{S}_0^n$ whenever $f \in \mathcal{S}_0^n$. Similarly, if $f \in \mathcal{X}_1^n$ we have

$$\partial_i f = f_1 + 2 z_i f_2 + z_i^{-1} \tau_N f_N \,, \tag{14}$$

so that

$$\sum_i z_i\partial_i^2 f = (f_{11}+2f_2)\sigma_1 + 4f_{12}\sigma_2 + 4f_{22}\sigma_3 \\ + 2Nf_{1N}\tau_N + 4f_{2N}\sigma_1\tau_N + f_{NN}\tau_{N-1}\tau_N\,, \quad (15)$$

which together with Eq. (10b) implies that $T_1 f\in\mathcal{S}_1^n$ for all $f\in\mathcal{S}_1^n$. Finally, if $f\in\mathcal{X}_2^n$ then

$$\partial_i f = f_1 + \big(z_i^{-1}\tau_{N-1} - z_i^{-2}\tau_N\big)f_{N-1} + z_i^{-1}\tau_N f_N \quad (16)$$

and hence

$$\sum_i z_i^2\partial_i^2 f = f_{11}\sigma_2 + 2f_{1,N-1}(\sigma_1\tau_{N-1} - N\tau_N) \\ + 2f_{1N}\sigma_1\tau_N + f_{N-1,N-1}\big[(N-1)\tau_{N-1}^2 - 2\tau_{N-2}\tau_N\big] \\ + 2(N-1)f_{N-1,N}\tau_{N-1}\tau_N + Nf_{NN}\tau_N^2\,. \quad (17)$$

The statement follows again from the previous equation and Eq. (10c). □

The last proposition implies that each operator T_ϵ preserves "trivial" symmetric spaces $\mathcal{S}_\epsilon^n\otimes\Lambda(\Sigma)$ spanned by factorized states. The main theorem of this section shows that in fact the latter operator leaves invariant a flag of nontrivial finite-dimensional subspaces of $\Lambda(\mathcal{P}^n\otimes\Sigma)$. Before stating this theorem we need to make a few preliminary definitions. Given a spin state $|s\rangle\in\Sigma$, we set

$$|s_i\rangle = \frac{1}{N!}\sum_{\substack{\pi\in S_N\\ \pi(1)=i}}\pi|s\rangle\,, \qquad |s_{ij}^{\pm}\rangle = \frac{1}{N!}\sum_{\substack{\pi\in S_N\\ \pi(1)=i,\pi(2)=j}}\pi(1\pm S_{12})|s\rangle\,, \quad (18)$$

where S_N is the symmetric group on N elements. Here and throughout the paper we identify an abstract permutation π with its realization as a permutation of the particles' spins. From Eq. (18) we have

$$\Lambda\big(f(z_1)|s\rangle\big) = \sum_i f(z_i)|s_i\rangle\,, \qquad \Lambda\big(g^{\pm}(z_1,z_2)|s\rangle\big) = \sum_{i<j} g^{\pm}(z_i,z_j)|s_{ij}^{\pm}\rangle\,, \quad (19)$$

where the last identity holds if $g^{\pm}(z_2,z_1)=\pm g^{\pm}(z_1,z_2)$. We also define the subspace

$$\Sigma' = \Big\{\,|s\rangle\in\Sigma \;\Big|\; \sum_i |s_{i,i+1}^{+}\rangle\in\Lambda(\Sigma)\Big\}\subset\Sigma\,. \quad (20)$$

Theorem 1. *Let*

$$\mathcal{T}_0^n = \big\langle f(\sigma_1,\sigma_2,\sigma_3)\Lambda|s\rangle, g(\sigma_1,\sigma_2,\sigma_3)\Lambda(z_1|s\rangle), h(\sigma_1,\sigma_2)\Lambda(z_1^2|s\rangle), \\ \tilde h(\sigma_1,\sigma_2)\Lambda(z_1z_2|s'\rangle), w(\sigma_1,\sigma_2)\Lambda(z_1z_2(z_1-z_2)|s\rangle) \mid f_{33}=g_{33}=0\big\rangle\,,$$

$$\mathcal{T}_1^n = \big\langle f(\sigma_1,\sigma_2,\tau_N)\Lambda|s\rangle, g(\sigma_1,\tau_N)\Lambda(z_1|s\rangle) \mid f_{22}=f_{NN}=g_{NN}=0\big\rangle\,,$$

$$\mathcal{T}_2^n = \big\langle f(\sigma_1,\tau_{N-1},\tau_N)\Lambda|s\rangle, g(\tau_{N-1},\tau_N)\Lambda(z_1|s\rangle), \tau_N q(\sigma_1,\tau_N)\Lambda(z_1^{-1}|s\rangle) \\ \mid f_{11}=f_{N-1,N-1}=g_{N-1,N-1}=q_{11}=0\big\rangle\,,$$

where $|s\rangle\in\Sigma$, $|s'\rangle\in\Sigma'$, $\deg f\le n$, $\deg g\le n-1$, $\deg h\le n-2$, $\deg\tilde h\le n-2$, $\deg w\le n-3$, $\deg q\le n-N+1$, *and* $\deg$ *is the total degree in* $\mathbf{z}$. *Then* $\mathcal{T}_\epsilon^n$ *is invariant under* T_ϵ *for all* $n=0,1,\dots$.

Proof. By Proposition 1, it suffices to show that T_ϵ maps $\mathcal{T}_\epsilon^n/\big(\mathcal{S}_\epsilon^n\otimes\Lambda(\Sigma)\big)$ into $\mathcal{T}_\epsilon^n$. We shall first deal with the operator T_0. Consider the states of the form $g\Lambda(z_1|s\rangle)$, with $g\in\mathcal{S}_0^{n-1}$. Since

$$(\partial_l-\partial_{l+1})z_i=\frac{1}{z_l-z_{l+1}}\,(1-K_{l,l+1})z_i\,,\qquad\forall\,i,l,$$

we have

$$T_0(gz_i)=(T_0g)z_i+2\partial_ig\,.$$

Calling

$$\Phi^{(k)}\equiv\Lambda(z_1^k|s\rangle),\qquad k\in\mathbb{Z}\,,\tag{21}$$

from Eqs. (12) and (19) we obtain

$$T_0\big(g\Phi^{(1)}\big)=\sum_iT_0(gz_i)|s_i\rangle=(T_0g)\Phi^{(1)}+2\sum_{k=1}^3kg_k\Phi^{(k-1)}\in\mathcal{T}_0^{n-2}\,.\tag{22}$$

Similarly, if $h(\sigma_1,\sigma_2)\in\mathcal{S}_0^{n-2}$, the identity

$$(\partial_l-\partial_{l+1})z_i^2=\frac{1}{z_l-z_{l+1}}\,(1-K_{l,l+1})z_i^2+(z_l-z_{l+1})(\delta_{li}+\delta_{l,i-1})\,,\qquad\forall\,i,l$$

implies that

$$T_0(hz_i^2)=(T_0h)z_i^2+4z_i\partial_ih+2(2a+1)h\,,$$

and therefore

$$\begin{aligned}T_0\big(h\Phi^{(2)}\big)&=\sum_iT_0(hz_i^2)|s_i\rangle\\&=(T_0h+8h_2)\Phi^{(2)}+4h_1\Phi^{(1)}+2(2a+1)h\Phi^{(0)}\end{aligned}\tag{23}$$

belongs to $\mathcal{T}_0^{n-2}$ on account of Eqs. (10a) and (13). On the other hand, from the equality

$$(\partial_l-\partial_{l+1})z_iz_j=\frac{1}{z_l-z_{l+1}}(1-K_{l,l+1})z_iz_j-(z_l-z_{l+1})\delta_{j,i+1}\delta_{l,i},\quad\forall\,i<j,\forall\,l$$

it follows that

$$T_0(\tilde hz_iz_j)=(T_0\tilde h)z_iz_j+2(z_i\partial_j\tilde h+z_j\partial_i\tilde h)-2a\tilde h\delta_{j,i+1}\,.$$

Setting

$$\widetilde\Phi^{(2)}\equiv\Lambda(z_1z_2|s\rangle)\tag{24}$$

and using again Eqs. (12) and (19) we then have

$$\begin{aligned}T_0\big(\tilde h\widetilde\Phi^{(2)}\big)&=\sum_{i<j}T_0(\tilde hz_iz_j)|s_{ij}^+\rangle\\&=(T_0\tilde h+8\tilde h_2)\widetilde\Phi^{(2)}+2\tilde h_1\Lambda\big[(z_1+z_2)|s\rangle\big]-2a\tilde h\sum_i|s_{i,i+1}^+\rangle\,.\end{aligned}\tag{25}$$

Since $\Lambda\big[(z_1+z_2)|s\rangle\big] = \Lambda\big[z_1(1+S_{12})|s\rangle\big]$, the RHS of Eq. (25) belongs to $\mathcal{T}_0^{n-2}$ if and only if $|s\rangle \in \Sigma'$. The last type of states generating the module $\mathcal{T}_0^n$ are of the form $w(\sigma_1,\sigma_2)\widehat{\Phi}^{(3)}$, where

$$\widehat{\Phi}^{(3)} \equiv \Lambda\big(z_1 z_2(z_1-z_2)|s\rangle\big)\,. \tag{26}$$

From the equality

$$\frac{1}{z_l-z_{l+1}}\,(\partial_l-\partial_{l+1})\big[z_i z_j(z_i-z_j)\big] = \frac{1}{(z_l-z_{l+1})^2}\,(1-K_{l,l+1})\big[z_i z_j(z_i-z_j)\big] \\ +(\delta_{l,i-1}+\delta_{li})z_j-(\delta_{l,j-1}+\delta_{lj})z_i, \qquad \forall\, i<j, \forall\, l$$

it follows that

$$T_0\big[w z_i z_j(z_i-z_j)\big] = (T_0 w) z_i z_j(z_i-z_j) + 2z_j(2z_i-z_j)\partial_i w \\ -2z_i(2z_j-z_i)\partial_j w - 2(2a+1)(z_i-z_j)w\,. \tag{27}$$

Using again Eqs. (12) and (19) we obtain

$$T_0\big(w\widehat{\Phi}^{(3)}\big) = \sum_{i<j} T_0\big(w z_i z_j(z_i-z_j)\big)|s_{ij}^-\rangle = (T_0 w + 12 w_2)\widehat{\Phi}^{(3)} \\ +2w_1\Lambda\big[(z_1^2-z_2^2)|s\rangle\big] - 2(2a+1)w\Lambda\big[(z_1-z_2)|s\rangle\big]\,. \tag{28}$$

Since $\Lambda\big[(z_1^k-z_2^k)|s\rangle\big] = \Lambda\big[z_1^k(1-S_{12})|s\rangle\big]$, the RHS of the latter equation clearly belongs to $\mathcal{T}_0^{n-2}$. This shows that $T_0(\mathcal{T}_0^n)\subset\mathcal{T}_0^{n-2}\subset\mathcal{T}_0^n$.

Consider next the action of the operator T_1 on a state of the form $g(\sigma_1,\tau_N)\Phi^{(1)}$, with $g\in\mathcal{S}_1^{n-1}$. From the identity

$$(z_l\partial_l - z_{l+1}\partial_{l+1})z_i = \frac{1}{2}\,\frac{z_l+z_{l+1}}{z_l-z_{l+1}}\,(1-K_{l,l+1})z_i + \frac{1}{2}\,(z_l-z_{l+1})(\delta_{l,i}+\delta_{l,i-1})\,, \quad \forall\, i,l$$

we easily obtain

$$T_1(g z_i) = (T_1 g)z_i + 2z_i\partial_i g + 2ag\,,$$

and therefore, by Eqs. (10b), (14) and (15),

$$T_1\big(g\Phi^{(1)}\big) = \sum_i T_1(g z_i)|s_i\rangle = (T_1 g + 2g_1)\Phi^{(1)} + 2(ag+\tau_N g_N)\Phi^{(0)} \in \mathcal{T}_1^{n-1}\,. \tag{29}$$

Thus $T_1(\mathcal{T}_1^n)\subset\mathcal{T}_1^{n-1}\subset\mathcal{T}_1^n$, as claimed.

Consider, finally, the operator T_2. If $g(\tau_{N-1},\tau_N)\in\mathcal{S}_2^{n-1}$, the identity

$$(z_l^2\partial_l - z_{l+1}^2\partial_{l+1})z_i = \frac{z_l z_{l+1}}{z_l-z_{l+1}}(1-K_{l,l+1})z_i + z_i(z_l-z_{l+1})(\delta_{l,i}+\delta_{l,i-1}), \quad \forall\, i,l$$

yields

$$T_2(g z_i) = (T_2 g)z_i + 2z_i^2\partial_i g + 4a z_i g\,,$$

and hence, by Eq. (16),

$$T_2(g\Phi^{(1)}) = \sum_i T_2(g z_i)|s_i\rangle \\ = \big[T_2 g + 2(\tau_{N-1}g_{N-1}+\tau_N g_N + 2ag)\big]\Phi^{(1)} - 2\tau_N g_{N-1}\Phi^{(0)} \tag{30}$$

clearly belongs to $\mathcal{T}_2^n$ on account of Eqs. (10c) and (17). The last type of spin states we need to study are of the form $\hat{q}\Phi^{(-1)}$, where $\hat{q} \equiv \tau_N q(\sigma_1, \tau_N)$ with $q_{11} = 0$. Since

$$(z_l^2\partial_l - z_{l+1}^2\partial_{l+1})z_i^{-1} = \frac{z_l z_{l+1}}{z_l - z_{l+1}}\,(1 - K_{l,l+1})z_i^{-1}\,, \qquad \forall\, i, l,$$

we obtain

$$T_2\big(\hat{q}z_i^{-1}\big) = (T_2\hat{q})z_i^{-1} - 2\partial_i\hat{q} + 2\hat{q}z_i^{-1}\,,$$

and thus, by Eqs. (16) and (19),

$$T_2\big(\hat{q}\Phi^{(-1)}\big) = \sum_i T_2(\hat{q}z_i^{-1})|s_i\rangle = (T_2\hat{q} - 2\tau_N\hat{q}_N + 2\hat{q})\Phi^{(-1)} - 2\hat{q}_1\Phi^{(0)}\,. \tag{31}$$

From Eqs. (10c) and (17), it follows that the RHS of the previous equation belongs to $\mathcal{T}_2^n$. Hence $T_2(\mathcal{T}_2^n) \subset \mathcal{T}_2^n$, which concludes the proof. □

Remark 4. We have chosen to allow a certain overlap between the different types of states spanning the spaces $\mathcal{T}_\epsilon^n$. For instance, if $|s\rangle$ is symmetric the state $g(\sigma_1, \sigma_2, \sigma_3)\Lambda(z_1|s\rangle) \in \mathcal{T}_0^n$ is also of the form $f(\sigma_1, \sigma_2, \sigma_3)\Lambda|s\rangle$. Less trivially, if $|s\rangle$ involves only two distinct spin components and is antisymmetric under S_{12}, then we have

$$\widehat{\Phi}^{(3)} = \frac{2}{N}\big(\sigma_1\Phi^{(2)} - \sigma_2\Phi^{(1)}\big)\,,$$

where $\Phi^{(k)}$ and $\widehat{\Phi}^{(3)}$ are respectively defined in Eqs. (21) and (26). Hence, for spin 1/2 the states of the form $w(\sigma_1, \sigma_2)\widehat{\Phi}^{(3)}$ in the space $\mathcal{T}_0^n$ can be expressed in terms of the other generators of this space.

The main result of this section follows easily from the previous theorem:

Corollary 1. *For each $\epsilon = 0, 1, 2$, the gauge Hamiltonian $\overline{H}_\epsilon$ leaves invariant the space $\mathcal{H}_\epsilon^n$ defined by*

$$\mathcal{H}_0^n = \mathcal{T}_0^n\,, \qquad \mathcal{H}_1^n = \mathcal{T}_1^n\big|_{f_N = g_N = 0}\,, \qquad \mathcal{H}_2^n = \mathcal{T}_2^n\,. \tag{32}$$

Proof. We shall begin by showing that each space $\mathcal{T}_\epsilon^n$ is invariant under the operator J^0. Note first that

$$J^0\Phi^{(j)} = j\,\Phi^{(j)}\,, \qquad J^0\widetilde{\Phi}^{(2)} = 2\widetilde{\Phi}^{(2)}\,, \qquad J^0\widehat{\Phi}^{(3)} = 3\widehat{\Phi}^{(3)}\,; \qquad j \in \mathbb{Z}\,, \tag{33}$$

where the states $\Phi^{(j)}$, $\widetilde{\Phi}^{(2)}$ and $\widehat{\Phi}^{(3)}$ are defined in Eqs. (21), (24) and (26), respectively. Using Eqs. (12), (14) and (16) one can immediately establish the identities

$$J^0 f = \sigma_1 f_1 + 2\sigma_2 f_2 + 3\sigma_3 f_3\,, \qquad \forall f(\sigma_1, \sigma_2, \sigma_3)\,, \tag{34a}$$

$$J^0 f = \sigma_1 f_1 + 2\sigma_2 f_2 + N\tau_N f_N\,, \qquad \forall f(\sigma_1, \sigma_2, \tau_N)\,, \tag{34b}$$

$$J^0 f = \sigma_1 f_1 + (N-1)\tau_{N-1} f_{N-1} + N\tau_N f_N\,, \qquad \forall f(\sigma_1, \tau_{N-1}, \tau_N)\,. \tag{34c}$$

From Eqs. (33)-(34) and the fact that J^0 is a derivation it follows that J^0 leaves invariant the spaces $\mathcal{T}_\epsilon^n$ for all $\epsilon = 0, 1, 2$. This implies that $\overline{H}_\epsilon$ preserves $\mathcal{T}_\epsilon^n$ for

$\epsilon = 0, 2$, since the coefficient c_- vanishes in these cases (cf. Table I). On the other hand, for $\epsilon = 1$ the coefficient c_- is nonzero, and thus we have to consider the action of the operator J^- on the space $\mathcal{T}_1^n$. We now have

$$J^- \Phi^{(j)} = j\, \Phi^{(j-1)} , \qquad j \in \mathbb{Z} , \tag{35}$$

and, from Eq. (14),

$$J^- f = N f_1 + 2\sigma_1 f_2 + \tau_{N-1} f_N , \qquad \forall f(\sigma_1, \sigma_2, \tau_N) . \tag{36}$$

Hence J^- leaves invariant the subspace $\mathcal{H}_1^n$ of $\mathcal{T}_1^n$ defined by the restrictions $f_N = g_N = 0$. From the obvious identity $T_1\big(f\Phi^{(0)}\big) = \big(T_1 f\big)\Phi^{(0)}$ and Eq. (29), together with (10b), (11) and (15), it follows that the operator T_1 also preserves $\mathcal{H}_1^n$. Likewise, Eqs. (33) and (34b) imply that $\mathcal{H}_1^n$ is invariant under J^0, and hence under the gauge Hamiltonian $\overline{H}_1$. □

Theorem 1 characterizes the invariant space $\mathcal{T}_0^n$ in terms of the subspace $\Sigma' \subset \Sigma$ in Eq. (20) that we shall now study in detail. In fact, from the definition of the invariant space $\mathcal{T}_0^n$ it follows that we can consider without loss of generality the quotient space $\Sigma'/\!\sim$, where $|s\rangle \sim |\tilde{s}\rangle$ if $\Lambda(z_1 z_2 |s\rangle) = \Lambda(z_1 z_2 |\tilde{s}\rangle)$. For instance, from Eq. (18) it immediately follows that if $|s\rangle \in \Sigma'$ and $\pi \in S_N$ is a permutation such that $\pi(i) \in \{1, 2\}$ for $i = 1, 2$, then $\pi|s\rangle$ belongs to Σ' and is equivalent to $|s\rangle$.

In the rest of this section, we shall denote $|s_{ij}^+\rangle$ simply as $|s_{ij}\rangle$ for the sake of conciseness. From Eq. (18) it easily follows that any symmetric state belongs to Σ', since

$$\sum_i |s_{i,i+1}\rangle = \frac{2}{N-1}\,|s\rangle , \quad \text{for all } |s\rangle \in \Lambda(\Sigma) . \tag{37}$$

On the other hand, if $|s\rangle \in \Lambda(\Sigma)$ the corresponding state $h(\sigma_1, \sigma_2)\Lambda(z_1 z_2 |s\rangle)$ is a trivial (factorized) state. We shall next show that the reciprocal of this statement is also true, up to equivalence.

Lemma 2. *For every $|s\rangle \in \Sigma$, $\Lambda(z_1 z_2 |s\rangle)$ is a factorized state if and only if $|s\rangle \sim \Lambda|s\rangle$.*

Proof. Suppose that

$$\Lambda(z_1 z_2 |s\rangle) = |\hat{s}\rangle \sum_{i<j} c_{ij} z_i z_j$$

is a factorized state. Since the LHS of the previous formula is symmetric, $c_{ij} = c$ for all i, j and $|\hat{s}\rangle \in \Lambda(\Sigma)$. By absorbing the constant c into $|\hat{s}\rangle$ we can take $c = 1$ without loss of generality, and therefore

$$\Lambda(z_1 z_2 |s\rangle) = \sum_{i<j} z_i z_j |s_{ij}\rangle = \tau_2 |\hat{s}\rangle \quad \Longrightarrow \quad |s_{ij}\rangle = |\hat{s}\rangle , \qquad i, j = 1, \dots, N.$$

From Eq. (19) with $f(z_1, z_2) = 1$ it then follows that

$$\Lambda|s\rangle = \sum_{i<j} |s_{ij}\rangle = \frac{1}{2}\, N(N-1)|\hat{s}\rangle .$$

Setting $|s_0\rangle = |s\rangle - \Lambda|s\rangle$ and using the previous identity we obtain

$$\Lambda(z_1 z_2|s_0\rangle) = \Lambda(z_1 z_2|s\rangle) - \frac{2\tau_2}{N(N-1)}\,\Lambda|s\rangle = |\hat{s}\rangle\tau_2 - \frac{2\tau_2}{N(N-1)}\,\Lambda|s\rangle = 0\,.$$

Hence $|s\rangle \sim \Lambda|s\rangle$, as claimed. □

By the previous observations, it suffices to characterize the nonsymmetric states in Σ'. To this end, let us introduce the linear operator $A : \Sigma \to \Sigma$ by

$$A|s\rangle = \sum_i |s_{i,i+1}\rangle. \tag{38}$$

Given an element $|\mathbf{s}\rangle \equiv |s_1 \dots s_N\rangle$ of the canonical basis of Σ, we shall also denote by $\{s^1, \dots, s^n\}$ the set of distinct components of $\mathbf{s} \equiv (s_1, \dots, s_N)$, and by ν_i the number of times that s^i appears among the components of $\mathbf{s}$. For instance, if $|\mathbf{s}\rangle = |-2, 0, 1, -2, 1\rangle$, then we can take $s^1 = -2$, $s^2 = 0$, $s^3 = 1$, so that $\nu_1 = \nu_3 = 2$, $\nu_2 = 1$. Consider the spin states $|\chi_i(\mathbf{s})\rangle \equiv |\chi_i\rangle$, $i = 1, \dots, n$, given by

$$|\chi_i\rangle = \nu_i(\nu_i - 1)|s^i s^i \dots\rangle - \sum_{\substack{1\le j,k\le n\\ j,k\neq i}} \nu_j(\nu_k - \delta_{jk})|s^j s^k \dots\rangle\,, \qquad \nu_i > 1\,, \tag{39a}$$

$$|\chi_i\rangle = \sum_{\substack{1\le j\le n\\ j\neq i}} \nu_j\left(|s^i s^j \dots\rangle + |s^j s^i \dots\rangle\right), \qquad \nu_i = 1\,. \tag{39b}$$

Here we have adopted the following convention: an ellipsis inside a ket stands for an arbitrary ordering of the components in $\mathbf{s}$ not indicated explicitly. Note that the states (39) are defined only up to equivalence, and that $|\chi_i(\mathbf{s})\rangle = |\chi_i(\pi\mathbf{s})\rangle$ for any permutation $\pi \in S_N$.

Proposition 2. *Given a basic spin state $|\mathbf{s}\rangle$, the associated spin states $|\chi_i(\mathbf{s})\rangle$ are all in $\Sigma'/\sim$.*

Proof. Consider first a state $|\chi_i\rangle$ of the type (39a). Using the definition of the operator A in Eq. (38) we obtain

$$\begin{aligned} N!A|\chi_i\rangle = {} & 2\nu_i(\nu_i-1)\sum_l \sum_{\pi\in S_{N-2}} \pi|\dots \underset{\substack{\downarrow\\ l}}{s^i} s^i \dots\rangle \\ & - 2\sum_l \sum_{\substack{1\le j,k\le n\\ j,k\neq i}} \sum_{\pi\in S_{N-2}} \nu_j(\nu_k-\delta_{jk})\pi|\dots \underset{\substack{\downarrow\\ l}}{s^j} s^k \dots\rangle\,, \end{aligned} \tag{40}$$

where the permutations π act only on the $N-2$ spin components specified by the ellipses. On the other hand, we have

$$\begin{aligned} N\cdot N!\Lambda|\mathbf{s}\rangle = {} & \sum_l \sum_{\pi\in S_{N-2}} \nu_i(\nu_i-1)\pi|\dots \underset{\substack{\downarrow\\ l}}{s^i} s^i \dots\rangle \\ & + 2\sum_l \sum_{\substack{1\le j\le n\\ j\neq i}} \sum_{\pi\in S_{N-1}} \nu_j\pi|\dots \underset{\substack{\downarrow\\ l}}{s^j} \dots\rangle \\ & - \sum_l \sum_{\substack{1\le j,k\le n\\ j,k\neq i}} \sum_{\pi\in S_{N-2}} \nu_j(\nu_k-\delta_{jk})\pi|\dots \underset{\substack{\downarrow\\ l}}{s^j} s^k \dots\rangle\,. \end{aligned} \tag{41}$$

Comparing Eqs. (40) and (41) we obtain

$$A|\chi_i\rangle = 2\Big(N\Lambda|\mathbf{s}\rangle - \frac{2}{N!}\sum_{\substack{1\le j\le n\\ j\ne i}}\nu_j\sum_l\sum_{\pi\in S_{N-1}}\pi|\dots\underset{l}{\overset{\downarrow}{s^j}}\dots\rangle\Big)$$

$$= 2\Big(N - 2\sum_{\substack{1\le j\le n\\ j\ne i}}\nu_j\Big)\Lambda|\mathbf{s}\rangle = 2(2\nu_i - N)\Lambda|\mathbf{s}\rangle\,. \tag{42}$$

This shows that any state of the form (39a) belongs to $\Sigma'/\!\sim$. Suppose next that $\nu_i = 1$, so that $|\chi_i\rangle$ is given by Eq. (39b). Since

$$A|\chi_i\rangle = \frac{2}{N!}\sum_l\sum_{\substack{1\le j\le n\\ j\ne i}}\sum_{\pi\in S_{N-2}}\nu_j\pi\big(|\dots\underset{l}{\overset{\downarrow}{s^i}}s^j\dots\rangle + |\dots\underset{l}{\overset{\downarrow}{s^j}}s^i\dots\rangle\big) = 4\Lambda|\mathbf{s}\rangle\,, \tag{43}$$

it follows that in this case $|\chi_i\rangle$ is also in $\Sigma'/\!\sim$. □

Remark 5. Just as symmetric spin states, cf. Eq. (37), the states $|\chi_i\rangle$ satisfy the relation

$$A|\chi_i\rangle = \frac{2}{N-1}\,\Lambda|\chi_i\rangle\,. \tag{44}$$

Indeed, if $\nu_i > 1$, from Eqs. (39a) and (42) we have

$$\Lambda|\chi_i\rangle = \Big[\nu_i(\nu_i - 1) + \sum_{\substack{1\le j\le n\\ j\ne i}}\nu_j - \sum_{\substack{1\le j,k\le n\\ j,k\ne i}}\nu_j\nu_k\Big]\Lambda|\mathbf{s}\rangle$$

$$= \big[\nu_i(\nu_i-1) + N - \nu_i - (N-\nu_i)^2\big]\Lambda|\mathbf{s}\rangle$$

$$= (N-1)(2\nu_i - N)\Lambda|\mathbf{s}\rangle = \frac{N-1}{2}\,A|\chi_i\rangle\,.$$

On the other hand, if $\nu_i = 1$ Eqs. (39b) and (43) imply that

$$\Lambda|\chi_i\rangle = 2\Big(\sum_{\substack{1\le j\le n\\ j\ne i}}\nu_j\Big)\Lambda|\mathbf{s}\rangle = 2(N-1)\Lambda|\mathbf{s}\rangle = \frac{N-1}{2}\,A|\chi_i\rangle\,.$$

Example 1. We shall now present all the states of the form (39) for spin 1/2. In this case, up to a permutation the basic state $|\mathbf{s}\rangle$ is given by

$$|\mathbf{s}\rangle = |\overbrace{+\cdots+}^{\nu}\overbrace{-\cdots-}^{N-\nu}\rangle\,. \tag{45}$$

If ν is either 0 or N, then $n = 1$ and thus $|\chi_1\rangle$ is of the type (39a) and proportional to $|\mathbf{s}\rangle$. If $\nu = 1$, then $n = 2$ and we can take (dropping inessential factors)

$$|\chi_1\rangle = \frac{1}{2}\big(|{+}{-}\cdots\rangle + |{-}{+}\cdots\rangle\big) \sim |{+}{-}\cdots\rangle\,, \qquad |\chi_2\rangle = |{-}{-}\cdots\rangle\,.$$

Although the states $|\chi_1\rangle$ and $|\chi_2\rangle$ are linearly independent, the combination $2|\chi_1\rangle + (N-2)|\chi_2\rangle$ is equivalent to a symmetric state. In the case $\nu = N-1$ the states $|\chi_i\rangle$

are obtained from the previous ones by flipping the spins. Finally, if $2 \le \nu \le N-2$ then $n = 2$ and the states $|\chi_i\rangle$ are now given by

$$|\chi_1\rangle = -|\chi_2\rangle = \nu(\nu-1)|{+}{+}\cdots\rangle - (N-\nu)(N-\nu-1)|{-}{-}\cdots\rangle\,. \tag{46}$$

According to the previous example, for spin 1/2 there are exactly $n-1$ independent states of the form (39) associated to each basic state $|\mathbf{s}\rangle$, up to symmetric states. We shall see next that this fact actually holds for arbitrary spin:

Proposition 3. *Let $|\mathbf{s}\rangle$ be a basic spin state. If n is the number of distinct components of $\mathbf{s}$, there are exactly $n-1$ independent states of the form* (39) *modulo symmetric states.*

Proof. Let p be the number of distinct components s^i of $\mathbf{s}$ such that $\nu_i > 1$. A straightforward computation shows that the combination

$$\sum_{i=1}^{n}|\chi_i\rangle \sim (2-p)\sum_{i,j=1}^{n}\nu_i(\nu_j-\delta_{ij})|s^i s^j\cdots\rangle \sim (2-p)N(N-1)\Lambda|\mathbf{s}\rangle \tag{47}$$

is equivalent to a symmetric state. Suppose first that $p \neq 2$. It is immediate to check that in this case the set $\{|\chi_i\rangle \mid i = 1,\ldots,n\}$ is linearly independent. If a linear combination $\sum_{i=1}^{n} c_i|\chi_i\rangle$ is equivalent to a symmetric state $|\hat{s}\rangle$, then $|\hat{s}\rangle$ must be proportional to $\Lambda|\mathbf{s}\rangle$, so that we can write

$$\sum_{i=1}^{n} c_i|\chi_i\rangle \sim \lambda(2-p)N(N-1)\Lambda|\mathbf{s}\rangle\,.$$

Hence $\sum_{i=1}^{n}(c_i-\lambda)|\chi_i\rangle \sim 0$, and the linear independence of the states $|\chi_i\rangle$ implies that $c_i = \lambda$ for all i. On the other hand, if $p = 2$ the set $\{|\chi_i\rangle \mid i = 1,\ldots,n\}$ is linearly dependent on account of Eq. (47), but removing one of the two states with $\nu_i > 1$ clearly yields a linearly independent set. It is also obvious from the coefficients of the states $|s^i s^i\ldots\rangle$ that no linear combination $\sum_{i=1}^{n} c_i|\chi_i\rangle$ can be equivalent to a nonzero symmetric state. ☐

The next natural question to be addressed is whether the states of the form (39) span the space $\Sigma'/{\sim}$ up to symmetric states:

Proposition 4. *The space $\big(\Sigma'/\Lambda(\Sigma)\big)/{\sim}$ is spanned by states of the form* (39).

Proof. For conciseness, we present the proof of this result only for the case $M = 1/2$. Let Σ_ν denote the subspace of Σ spanned by basic spin states with ν "+" spins, and set $\Sigma'_\nu = \Sigma' \cap \Sigma_\nu$. Since the operators A and Λ involved in the definition (20) of Σ' clearly preserve Σ_ν, it suffices to show that the states $|\chi_i(\mathbf{s})\rangle$ with $\mathbf{s}$ given by (45) span the space $\Sigma'_\nu/{\sim}$ up to symmetric states. Note first that the statement is trivial for $\nu = 0, 1, N-1, N$, since in this case the states of the form (39) obviously generate the whole space $\Sigma_\nu/{\sim}$. Suppose, therefore, that $2 \le \nu \le N-2$, so that

$$\Sigma_\nu/{\sim} = \big\langle\, |{+}{+}\cdots\rangle,\, |{+}{-}\cdots\rangle,\, |{-}{-}\cdots\rangle \,\big\rangle\,.$$

Since the state (46) and the symmetric state (up to equivalence)

$$\nu(\nu-1)|{+}{+}\cdots\rangle + 2\nu(N-\nu)|{+}{-}\cdots\rangle + (N-\nu)(N-\nu-1)|{-}{-}\cdots\rangle$$

both belong to $\Sigma'_\nu/\!\sim$, we need only show that (for instance) $|{+}{-}\cdots\rangle$ is not in $\Sigma'_\nu/\!\sim$, i.e., that $A|{+}{-}\cdots\rangle$ is not symmetric. But this is certainly the case, since a state of the form

$$|\overbrace{+-\cdots+-}^{2k}\overbrace{-\cdots-}^{N-\nu-k}\overbrace{+\cdots+}^{\nu-k}\rangle\,,\qquad k=1,2,\dots,\min(\nu,N-\nu)\,,$$

appears in $A|{+}{-}\cdots\rangle$ with coefficient $2k(\nu-1)!(N-\nu-1)!$ depending on k. □

4. THE ALGEBRAIC EIGENFUNCTIONS

In the previous section we have provided a detailed description of the spaces $\mathcal{H}^n_\epsilon\subset\Lambda\big(\mathbb{C}[\mathbf{z}]\otimes\Sigma\big)$ invariant under the corresponding gauge Hamiltonian $\overline{H}_\epsilon$. In this section we shall explicitly compute all the eigenfunctions of the restrictions of the operators $\overline{H}_\epsilon$ to their invariant spaces $\mathcal{H}^n_\epsilon$. This yields several infinite[1] families of eigenfunctions for each of the models (2), which is the main result of this paper. We shall use the term *algebraic* to refer to these eigenfunctions and their corresponding energies. It is important to observe that the eigenfunctions of the gauge Hamiltonian $\overline{H}_\epsilon$ that can be constructed in this way are necessarily invariant under the whole symmetric group, in spite of the fact that $\overline{H}_\epsilon$ is symmetric only under cyclic permutations. In fact, the explicit solutions of all known CS models with near-neighbors interactions (both in the scalar and spin cases) can be factorized as the product of a simple gauge factor analogous to μ times a completely symmetric function [41, 42, 43, 48]. This, however, does not rule out the existence of other eigenfunctions of the gauge Hamiltonian $\overline{H}_\epsilon$ invariant only under the subgroup of cyclic permutations, which is indeed an interesting open problem.

Case a

We shall begin with the model (2a), which is probably the most interesting one due to the rich structure of its associated invariant flag. In order to find the algebraic energies of the model, note first that one can clearly construct a basis $\mathcal{B}^n_0$ of $\mathcal{H}^n_0$ whose elements are homogeneous polynomials in $\mathbf{z}$ with coefficients in Σ. If $F\in\mathcal{B}^n_0$ has degree k, then $J^0F=kF$ and T_0F has degree at most $k-2$. If $\mathcal{B}^n_0$ is ordered according to the degree, the operator $\overline{H}_0$ is represented in this basis by a triangular matrix with diagonal elements E_0+kc_0, where $k=0,\dots,n$ is the degree. Thus the algebraic energies are the numbers

$$E_k=E_0+2k\omega,\qquad k=0,1,\dots.$$

We shall next show that the algebraic eigenfunctions of $\overline{H}_0$ can be expressed in closed form in terms of generalized Laguerre and Jacobi polynomials. The computation is basically a two-step procedure. In the first place, one encodes the eigenvalue

[1] For the model (2c) we shall see below that the number of eigenfunctions with a given total momentum is finite.

problem in the invariant space $\mathcal{H}_0^n$ as a system of linear partial differential equations. The second step then consists in finding the polynomial solutions of this system.

Regarding the first step, we shall need the following preliminary lemma:

Lemma 3. *The operator $\overline{H}_0$ preserves the following subspaces of $\mathcal{H}_0^n$:*

$$\overline{\mathcal{H}}^n_{0,|s\rangle} = \langle f\Phi^{(0)}, g\Phi^{(1)}, h\Phi^{(2)}\rangle\,, \qquad |s\rangle\in\Sigma\,, \tag{48}$$

$$\widetilde{\mathcal{H}}^n_{0,|s\rangle} = \overline{\mathcal{H}}^n_{0,|s\rangle} + \langle \tilde{h}\widetilde{\Phi}^{(2)}\rangle\,, \qquad |s\rangle\in\Sigma'\,,\quad S_{12}|s\rangle = |s\rangle\,, \tag{49}$$

$$\widehat{\mathcal{H}}^n_{0,|s\rangle} = \overline{\mathcal{H}}^n_{0,|s\rangle} + \langle w\widehat{\Phi}^{(3)}\rangle\,, \qquad |s\rangle\in\Sigma\,,\quad S_{12}|s\rangle = -|s\rangle\,, \tag{50}$$

where f, g, h, $\tilde{h}$, w are as in the definition of $\mathcal{T}_0^n$ in Theorem 1, and $\Phi^{(k)}$, $\widetilde{\Phi}^{(2)}$, $\widehat{\Phi}^{(3)}$ are respectively given by (21), (24) *and* (26).

Proof. The identity $T_0\big(f\Phi^{(0)}\big) = (T_0 f)\Phi^{(0)}$ and Eqs. (22), (23) and (33) clearly imply that the subspace $\overline{\mathcal{H}}^n_{0,|s\rangle}$ is invariant under $\overline{H}_0$. Consider next the action of $\overline{H}_0$ on a function of the form $\tilde{h}\widetilde{\Phi}^{(2)}$. Since $|s\rangle$ is symmetric under S_{12}, we can replace $\Lambda\big[(z_1+z_2)|s\rangle\big]$ by $2\Phi^{(1)}$ in Eq. (25). Secondly, any state $|s\rangle\in\Sigma'$ satisfies the identity

$$\sum_i |s^+_{i,i+1}\rangle = \frac{2}{N-1}\,\Lambda|s\rangle\,. \tag{51}$$

Indeed, we already know that the previous identity holds for symmetric states (cf. Eq. (37)) and for states of the form (39) (cf. Eqs. (38) and (44)). On the other hand, by Proposition 4 every state in Σ' is a linear combination of a symmetric state, states of the form (39), and a state $|s\rangle$ such that $\Lambda\big(z_1z_2|s\rangle\big)=0$. But for the latter "null" state $|s_{ij}\rangle = 0$ for all $i<j$, and hence $A|s\rangle = \Lambda|s\rangle = 0$. Therefore, Eq. (25) can be written as

$$T_0\big(\tilde{h}\widetilde{\Phi}^{(2)}\big) = (T_0\tilde{h} + 8\tilde{h}_2)\widetilde{\Phi}^{(2)} + 4\tilde{h}_1\Phi^{(1)} - \frac{4a}{N-1}\,\tilde{h}\Phi^{(0)}\,. \tag{52}$$

From the previous equation and Eq. (33) it follows that $\overline{H}_0\big(\tilde{h}\widetilde{\Phi}^{(2)}\big)\in\widetilde{\mathcal{H}}^n_{0,|s\rangle}$. Finally, if $S_{12}|s\rangle = -|s\rangle$, Eq. (28) reduces to

$$T_0\big(w\widehat{\Phi}^{(3)}\big) = (T_0 w + 12w_2)\widehat{\Phi}^{(3)} + 4w_1\Phi^{(2)} - 4(2a+1)w\Phi^{(1)}\,, \tag{53}$$

which, together with Eq. (33), implies that $\overline{H}_0\big(w\widehat{\Phi}^{(3)}\big)\in\widehat{\mathcal{H}}^n_{0,|s\rangle}$. □

Remark 6. The requirement that $|s\rangle$ be symmetric (respectively antisymmetric) under S_{12} in the definition of the space $\widetilde{\mathcal{H}}^n_{0,|s\rangle}$ (respectively $\widehat{\mathcal{H}}^n_{0,|s\rangle}$) is no real restriction, since the antisymmetric (respectively symmetric) part of $|s\rangle$ does not contribute to the state $\widetilde{\Phi}^{(2)}$ (respectively $\widehat{\Phi}^{(3)}$).

By the previous lemma, we can consider without loss of generality eigenfunctions of $\overline{H}_0$ of the form

$$\Phi = f\Phi^{(0)} + g\Phi^{(1)} + h\Phi^{(2)} + \tilde{h}\widetilde{\Phi}^{(2)} + w\widehat{\Phi}^{(3)}\,, \qquad \deg\Phi = k\,, \tag{54}$$

where the spin functions $\Phi^{(k)}$, $\widetilde{\Phi}^{(2)}$ and $\widehat{\Phi}^{(3)}$ are all built from the same spin state $|s\rangle$. Note that we can assume that $\tilde{h}w=0$, and that the spin state $|s\rangle$ is symmetric under S_{12} and belongs to Σ' if $\tilde{h}\neq 0$, whereas it is antisymmetric under S_{12} if $w\neq 0$.

Using Eqs. (22), (23), (33), (52) and (53), it is straightforward to show that the eigenvalue equation $\overline{H}_0\Phi=(E_0+2k\omega)\Phi$ is equivalent to the system

$$\big[-T_0+2\omega(J^0+3-k)\big]w-12w_2=0\,, \tag{55a}$$

$$\big[-T_0+2\omega(J^0+2-k)\big]\tilde{h}-8\tilde{h}_2=0\,, \tag{55b}$$

$$\big[-T_0+2\omega(J^0+2-k)\big]h-8h_2=6g_3+4w_1\,, \tag{55c}$$

$$\big[-T_0+2\omega(J^0+1-k)\big]g-4g_2=4(h_1+\tilde{h}_1)-4(2a+1)w\,, \tag{55d}$$

$$\big[-T_0+2\omega(J^0-k)\big]f=2\Big(g_1+(2a+1)h-\frac{2a}{N-1}\,\tilde{h}\Big)\,. \tag{55e}$$

Since f and g are linear in σ_3, we can write

$$f=p+\sigma_3 q\,,\qquad g=u+\sigma_3 v\,, \tag{56}$$

where p, q, u and v are polynomials in σ_1 and σ_2. Taking into account that the action of T_0 on scalar symmetric functions is given by the RHS of Eq. (11) with $\epsilon=0$, and using Eqs. (10a), (13) and (34a), we finally obtain the following linear system of PDEs:

$$\big[L_0-2\omega(k-3)\big]w-12w_2=0\,, \tag{57a}$$

$$\big[L_0-2\omega(k-2)\big]\tilde{h}-8\tilde{h}_2=0\,, \tag{57b}$$

$$\big[L_0-2\omega(k-2)\big]h-8h_2=6v+4w_1\,, \tag{57c}$$

$$\begin{aligned}\big[L_0-2\omega(k-1)\big]u-4u_2=4h_1+4\tilde{h}_1+6\sigma_2 v_1\\ +6(2a+1)\sigma_1 v-4(2a+1)w\,,\end{aligned} \tag{57d}$$

$$\big[L_0-2\omega(k-4)\big]v-16v_2=0\,, \tag{57e}$$

$$\big(L_0-2\omega k\big)p=2u_1+2(2a+1)h-\frac{4a}{N-1}\,\tilde{h}+6\sigma_2 q_1+6(2a+1)\sigma_1 q\,, \tag{57f}$$

$$\big[L_0-2\omega(k-3)\big]q-12q_2=2v_1\,, \tag{57g}$$

where

$$\begin{aligned}L_0=-\big(N\partial^2_{\sigma_1}+4\sigma_1\partial_{\sigma_1}\partial_{\sigma_2}+4\sigma_2\partial^2_{\sigma_2}+2(2a+1)N\partial_{\sigma_2}\big)\\ +2\omega(\sigma_1\partial_{\sigma_1}+2\sigma_2\partial_{\sigma_2})\,.\end{aligned} \tag{58}$$

As a consequence of the general discussion of the previous Section, the latter system is guaranteed to possess polynomials solutions. In fact, these polynomial solutions can be expressed in closed form in terms of generalized Laguerre polynomials L_ν^λ and Jacobi polynomials $P_\nu^{(\gamma,\delta)}$.

Theorem 2. *Let*

$$\alpha=N(a+\frac{1}{2})-\frac{3}{2}\,,\qquad \beta\equiv\beta(m)=1-m-N\Big(a+\frac{1}{2}\Big)\,,\qquad t=\frac{2r^2}{N\bar{x}^2}-1\,,$$

where $\bar{x} = \frac{1}{N}\sum_i x_i$ is the center of mass coordinate. The Hamiltonian H_0 possesses the following families of spin eigenfunctions with eigenvalue $E_{lm} = E_0 + 2\omega(2l+m)$, with $l \geq 0$ and m as indicated in each case:

$$
\begin{aligned}
\Psi^{(0)}_{lm} &= \mu\bar{x}^m L_l^{-\beta}(\omega r^2) P^{(\alpha,\beta)}_{[\frac{m}{2}]}(t)\,\Phi^{(0)}\,, \qquad m \geq 0\,,\\
\Psi^{(1)}_{lm} &= \mu\bar{x}^{m-1} L_l^{-\beta}(\omega r^2) P^{(\alpha+1,\beta)}_{[\frac{m-1}{2}]}(t)\big(\Phi^{(1)} - \bar{x}\,\Phi^{(0)}\big)\,, \qquad m \geq 1\,,\\
\Psi^{(2)}_{lm} &= \mu\bar{x}^{m-2} L_l^{-\beta}(\omega r^2)\bigg[P^{(\alpha+2,\beta)}_{[\frac{m}{2}]-1}(t)\big(\Phi^{(2)} - 2\bar{x}\,\Phi^{(1)}\big)\\
&\qquad + \bar{x}^2\bigg(P^{(\alpha+2,\beta)}_{[\frac{m}{2}]-1}(t) - \frac{2(\alpha+1)}{2[\frac{m-1}{2}]+1}\,P^{(\alpha+1,\beta)}_{[\frac{m}{2}]-1}(t)\bigg)\Phi^{(0)}\bigg]\,, \quad m \geq 2\,,\\
\widetilde{\Psi}^{(2)}_{lm} &= \mu\bar{x}^{m-2} L_l^{-\beta}(\omega r^2)\bigg[P^{(\alpha+2,\beta)}_{[\frac{m}{2}]-1}(t)\big(\widetilde{\Phi}^{(2)} - 2\bar{x}\,\Phi^{(1)}\big)\\
&\qquad + \bar{x}^2\bigg(P^{(\alpha+2,\beta)}_{[\frac{m}{2}]-1}(t) + \frac{2(\alpha+1)}{\big(2[\frac{m-1}{2}]+1\big)(N-1)}\,P^{(\alpha+1,\beta)}_{[\frac{m}{2}]-1}(t)\bigg)\Phi^{(0)}\bigg]\,, \quad m \geq 2\,,\\
\Psi^{(3)}_{lm} &= \mu\bar{x}^{m-3} L_l^{-\beta}(\omega r^2)\bigg[\frac{2}{3N}\,P^{(\alpha+3,\beta)}_{[\frac{m-3}{2}]}(t)\sum_i x_i^3 + \bar{x}^3\varphi_m(t)\bigg]\,\Phi^{(0)}\,, \qquad m \geq 3\,,\\
\widehat{\Psi}^{(3)}_{lm} &= \mu\bar{x}^{m-3} L_l^{-\beta}(\omega r^2)\bigg[P^{(\alpha+3,\beta)}_{[\frac{m-3}{2}]}(t)\big(\widehat{\Phi}^{(3)} - 2\bar{x}\,\Phi^{(2)}\big)\\
&\qquad + 2\bar{x}^2\bigg(P^{(\alpha+3,\beta)}_{[\frac{m-3}{2}]}(t) + \frac{2(\alpha+2)}{2[\frac{m}{2}]-1}\,P^{(\alpha+2,\beta)}_{[\frac{m-3}{2}]}(t)\bigg)\Phi^{(1)}\\
&\qquad - 2\bar{x}^3\bigg(\frac{1}{3}\,P^{(\alpha+3,\beta)}_{[\frac{m-3}{2}]}(t) + \frac{1}{2[\frac{m}{2}]-1}\,P^{(\alpha+2,\beta)}_{[\frac{m-3}{2}]}(t)\\
&\qquad\qquad + \frac{2\alpha+3}{m(m-2)}\,\varepsilon(m)P^{(\alpha+1,\beta)}_{\frac{m-3}{2}}(t)\bigg)\Phi^{(0)}\bigg]\,, \quad m \geq 3\,,\\
\Psi^{(4)}_{lm} &= \mu\bar{x}^{m-4} L_l^{-\beta}(\omega r^2)\bigg[\frac{3}{2([\frac{m-3}{2}]+\frac{1}{2})}\,\bar{x}^2 P^{(\alpha+3,\beta)}_{[\frac{m}{2}]-2}(t)\,\Phi^{(2)}\\
&\qquad + \Big(\frac{3}{2}\,\bar{x}^3\phi_m(t) - \frac{1}{N}\,P^{(\alpha+4,\beta)}_{[\frac{m}{2}]-2}(t)\sum_i x_i^3\Big)\Phi^{(1)}\\
&\qquad + \Big(\frac{1}{N}\,\bar{x}P^{(\alpha+4,\beta)}_{[\frac{m}{2}]-2}(t)\sum_i x_i^3 + \frac{3}{2}\,\bar{x}^4\chi_m(t)\Big)\Phi^{(0)}\bigg]\,, \quad m \geq 4\,.
\end{aligned}
$$

Here $[\cdot]$ denotes the integer part, $\varepsilon(m) = \big(1-(-1)^m\big)/2$, and

$$
\Phi^{(k)} = \Lambda(x_1^k|s\rangle), \qquad \widetilde{\Phi}^{(2)} = \Lambda(x_1x_2|s\rangle), \qquad \widehat{\Phi}^{(3)} = \Lambda(x_1x_2(x_1-x_2)|s\rangle),
$$

where the spin state $|s\rangle$ is symmetric under S_{12} and belongs to Σ' for the eigenfunction $\widetilde{\Psi}^{(2)}_{lm}$, and is antisymmetric under S_{12} for the eigenfunction $\widehat{\Psi}^{(3)}_{lm}$. The functions φ_m, ϕ_m and χ_m are polynomials given explicitly by

$$
\varphi_m = \frac{m+2\alpha+2}{m-1}\,P^{(\alpha+2,\beta-2)}_{\frac{m}{2}} - P^{(\alpha+3,\beta-1)}_{\frac{m}{2}-1} - \frac{4\alpha+7}{m-1}\,P^{(\alpha+2,\beta-1)}_{\frac{m}{2}-1} + \frac{1}{3}\,P^{(\alpha+3,\beta)}_{\frac{m}{2}-2}\,,
$$

$$
\begin{aligned}
\phi_m = {}& P^{(\alpha+4,\beta-1)}_{\frac{m}{2}-1} - 2P^{(\alpha+3,\beta-1)}_{\frac{m}{2}-1} - \frac{m+2\alpha+3}{(m-1)(m-3)}\, P^{(\alpha+2,\beta-1)}_{\frac{m}{2}-1} \\
& - \frac{1}{3}\, P^{(\alpha+4,\beta)}_{\frac{m}{2}-2} + \frac{m+2\alpha-1}{m-3}\, P^{(\alpha+3,\beta)}_{\frac{m}{2}-2}\,, \\
\chi_m = {}& \frac{3m+2\alpha}{(m-1)(m-3)}\, P^{(\alpha+2,\beta-1)}_{\frac{m}{2}-1} + \frac{2m-7}{m-3}\, P^{(\alpha+3,\beta-1)}_{\frac{m}{2}-1} - P^{(\alpha+4,\beta-1)}_{\frac{m}{2}-1} \\
& - \frac{m+2\alpha+2}{(m-1)(m-3)}\, P^{(\alpha+2,\beta)}_{\frac{m}{2}-2} - \frac{m+2\alpha}{m-3}\, P^{(\alpha+3,\beta)}_{\frac{m}{2}-2} + \frac{1}{3}\, P^{(\alpha+4,\beta)}_{\frac{m}{2}-2}\,,
\end{aligned}
$$

for even m, and

$$
\begin{aligned}
\varphi_m = {}& 2P^{(\alpha+2,\beta-1)}_{\frac{m-1}{2}} - P^{(\alpha+3,\beta-1)}_{\frac{m-1}{2}} + \frac{1}{3}\, P^{(\alpha+3,\beta)}_{\frac{m-3}{2}} + \frac{m+2\alpha+2}{m(m-2)}\, P^{(\alpha+1,\beta)}_{\frac{m-3}{2}} \\
& - \frac{m+2\alpha+2}{m-2}\, P^{(\alpha+2,\beta)}_{\frac{m-3}{2}}\,, \\
\phi_m = {}& P^{(\alpha+4,\beta-1)}_{\frac{m-3}{2}} - \frac{2m-5}{m-2}\, P^{(\alpha+3,\beta)}_{\frac{m-3}{2}} - \frac{1}{3}\, P^{(\alpha+4,\beta)}_{\frac{m-5}{2}} + \frac{m+2\alpha-1}{m-2}\, P^{(\alpha+3,\beta)}_{\frac{m-5}{2}}\,, \\
\chi_m = {}& \frac{2m-3}{m(m-2)}\, P^{(\alpha+2,\beta-1)}_{\frac{m-3}{2}} + \frac{2(m-3)}{m-2}\, P^{(\alpha+3,\beta-1)}_{\frac{m-3}{2}} - P^{(\alpha+4,\beta-1)}_{\frac{m-3}{2}} \\
& - \frac{m+2\alpha+1}{m(m-2)}\, P^{(\alpha+2,\beta)}_{\frac{m-5}{2}} - \frac{m+2\alpha}{m-2}\, P^{(\alpha+3,\beta)}_{\frac{m-5}{2}} + \frac{1}{3}\, P^{(\alpha+4,\beta)}_{\frac{m-3}{2}}\,,
\end{aligned}
$$

for odd m. For every $n = 0, 1, \dots,$ the above eigenfunctions with $2l + m \le n$ span the whole H_0-invariant space $\mu\mathcal{H}_0^n$.

Proof. Recall, to begin with, that the algebraic eigenfunctions of H_0 are of the form $\Psi = \mu\Phi$, with μ given in Table I and Φ an eigenfunction of $\overline{H}_0$ of the form (54)-(56). In order to determine Φ, we must find the most general polynomial solution of the linear system (57). From the structure of this system it follows that there are seven types of independent solutions, characterized by the vanishing of certain subsets of the unknown functions $p, q, u, v, h, \tilde{h}, w$. These types are listed in Table II, where in the last column we have indicated the eigenfunction of H_0 obtained from each case. We shall present here in detail the solution of the system (57) for the case $q = v = h = \tilde{h} = w = 0$ and $u \neq 0$, which yields the eigenfunctions of the form $\Psi^{(1)}_{lm}$ (the procedure for the other cases is essentially the same). In this case the system (57) reduces to

$$
\big[L_0 - 2\omega(k-1)\big]u - 4u_2 = 0\,, \qquad \big(L_0 - 2\omega k\big)p = 2u_1\,. \tag{59}
$$

Let us begin with the homogeneous equation for u. We shall look for polynomial solutions of this equation of the form $u = Q(\sigma_1, \sigma_2)R(\sigma_2)$, where Q is a homogeneous polynomial of degree $m-1$ in $\mathbf{z}$ and R is a polynomial of degree l in σ_2, so that $k = \deg\Phi = 2l + m$ by Eq. (54). From Eq. (58) and the homogeneity of Q we have

$$
\begin{aligned}
L_0(QR) &= (L_0 Q)R + Q(L_0 R) - 4\sigma_1 Q_1 R_2 - 8\sigma_2 Q_2 R_2 \\
&= (L_0 Q)R + Q(L_0 - 4(m-1)\partial_{\sigma_2})R\,.
\end{aligned}
$$

Hence the equation for u can be written as

$$
R\big(\widehat{L}_0 - 4\partial_{\sigma_2}\big)Q = Q\big(-L_0 + 4m\partial_{\sigma_2} + 4l\omega\big)R\,,
$$

TABLE II: The seven types of solutions of the system (57) and their corresponding eigenfunctions.

Conditions	Eigenfunction
$q=u=v=h=\tilde h=w=0, \quad p\neq 0$	$\Psi^{(0)}_{lm}$
$u=v=h=\tilde h=w=0, \quad q\neq 0$	$\Psi^{(3)}_{lm}$
$q=v=h=\tilde h=w=0, \quad u\neq 0$	$\Psi^{(1)}_{lm}$
$q=v=\tilde h=w=0, \quad h\neq 0$	$\Psi^{(2)}_{lm}$
$q=v=h=w=0, \quad \tilde h\neq 0$	$\widetilde\Psi^{(2)}_{lm}$
$q=v=\tilde h=0, \quad w\neq 0$	$\widehat\Psi^{(3)}_{lm}$
$\tilde h=w=0, \quad v\neq 0$	$\Psi^{(4)}_{lm}$

where $\widehat L_0 = L_0|_{\omega=0}$. Since $\big(\widehat L_0 - 4\partial_{\sigma_2}\big)Q$ is a homogeneous polynomial of degree $m-3$ in $\mathbf{z}$, both sides of the latter equation must vanish separately. We are thus led to the following decoupled equations for Q and R:

$$\big(\widehat L_0 - 4\partial_{\sigma_2}\big)Q = 0\,, \tag{60}$$

$$\big(-L_0 + 4m\partial_{\sigma_2} + 4l\omega\big)R = 0\,. \tag{61}$$

In terms of the variable $\rho=\omega\sigma_2$, Eq. (61) can be written as

$$4\omega\mathcal{L}_l^{-\beta}(R) = 0\,,$$

where

$$\mathcal{L}_\nu^\lambda = \rho\partial_\rho^2 + (\lambda+1-\rho)\partial_\rho + \nu \tag{62}$$

is the generalized Laguerre operator. Hence R is proportional to the generalized Laguerre polynomial $L_l^{-\beta}(\omega\sigma_2)$. On the other hand, we can write $Q=\sigma_1^{m-1}P(t)$ where P is a polynomial in the homogeneous variable $t=\frac{2N\sigma_2}{\sigma_1^2}-1$. With this substitution, Eq. (60) becomes

$$4N\sigma_1^{m-3}\mathcal{J}^{(\alpha+1,\beta)}_{[\frac{m-1}{2}]}(P) = 0\,,$$

where the Jacobi operator $\mathcal{J}^{(\gamma,\delta)}_\nu$ is given by

$$\mathcal{J}^{(\gamma,\delta)}_\nu = (1-t^2)\partial_t^2 + \big[\delta-\gamma-(\gamma+\delta+2)t\big]\,\partial_t + \nu(\nu+\gamma+\delta+1)\,.$$

Thus $P(t)$ is proportional to the Jacobi polynomial $P^{(\alpha+1,\beta)}_{[\frac{m-1}{2}]}(t)$, so that we can take

$$u = \sigma_1^{m-1}L_l^{-\beta}(\omega\sigma_2)P^{(\alpha+1,\beta)}_{[\frac{m-1}{2}]}(t)\,. \tag{63}$$

We must next find a particular solution of the inhomogeneous equation for p in (59), since the general solution of the corresponding homogeneous equation yields an eigenfunction of the simpler type $\Psi^{(0)}_{lm}$. Since

$$u_1 = \sigma_1^{m-2}L_l^{-\beta}(\omega\sigma_2)\Big[(m-1)P^{(\alpha+1,\beta)}_{[\frac{m-1}{2}]}(t) - 2(t+1)\dot P^{(\alpha+1,\beta)}_{[\frac{m-1}{2}]}(t)\Big]$$

(where the dot denotes derivative with respect to t), we make the ansatz $p = \overline{Q}(\sigma_1,\sigma_2)\overline{R}(\sigma_2)$, where $\overline{Q}$ is a homogeneous polynomial of degree m in $\mathbf{z}$ and $\overline{R}$ is a polynomial of degree l in σ_2. Substituting this ansatz into the second equation in (59) and proceeding as before we immediately obtain

$$\overline{R}\big(\widehat{L}_0\overline{Q}\big) + \overline{Q}\big(L_0 - 4m\partial_{\sigma_2} - 4l\omega\big)\overline{R} = 2u_1\,.$$

If we set $\overline{R} = L_l^{-\beta}(\omega\sigma_2)$ the second term of the LHS vanishes, and cancelling the common factor $L_l^{-\beta}(\omega\sigma_2)$ we are left with the following equation for $\overline{Q}$:

$$\widehat{L}_0\overline{Q} = 2\sigma_1^{m-2}\Big[(m-1)P^{(\alpha+1,\beta)}_{[\frac{m-1}{2}]}(t) - 2(t+1)\dot{P}^{(\alpha+1,\beta)}_{[\frac{m-1}{2}]}(t)\Big]\,.$$

The form of the RHS of this equation suggests the ansatz $\overline{Q} = \sigma_1^m\overline{P}(t)$, with $\overline{P}$ a polynomial in the variable t. The previous equation then yields

$$\mathcal{J}^{(\alpha,\beta)}_{[\frac{m}{2}]}(\overline{P}) = \frac{1}{2N}\Big[(m-1)P^{(\alpha+1,\beta)}_{[\frac{m-1}{2}]}(t) - 2(t+1)\dot{P}^{(\alpha+1,\beta)}_{[\frac{m-1}{2}]}(t)\Big]\,. \tag{64}$$

From the definition of the Jacobi operator we easily obtain

$$\mathcal{J}^{(\alpha,\beta)}_{[\frac{m}{2}]} = \mathcal{J}^{(\alpha+1,\beta)}_{[\frac{m-1}{2}]} + (1+t)\,\partial_t - \frac{1}{2}\,(m-1)\,,$$

which implies that $\overline{P} = -\frac{1}{N}\,P^{(\alpha+1,\beta)}_{[\frac{m-1}{2}]}$ is a particular solution of Eq. (64). Hence

$$p = -\frac{1}{N}\,\sigma_1^m L_l^{-\beta}(\omega\sigma_2)P^{(\alpha+1,\beta)}_{[\frac{m-1}{2}]}(t) \tag{65}$$

is a particular solution of the inhomogeneous equation in (59). We have thus shown that $\Phi = p\,\Phi^{(0)} + u\,\Phi^{(1)}$, with u and p respectively given by Eqs. (63) and (65), is an eigenfunction of $\overline{H}_0$ with eigenvalue $E_0 + 2\omega(2l+m)$. Multiplying Φ by the gauge factor μ we obtain the eigenfunction $\Psi^{(1)}_{lm}$ of H_0 in the statement.

It remains to show that the states $\Psi^{(k)}_{lm}$ $(k = 0,\dots,4)$, $\widetilde{\Psi}^{(2)}_{lm}$ and $\widehat{\Psi}^{(3)}_{lm}$ with $2l+m \le n$ generate the spaces (48)–(50). Consider first the "monomials" of the form $\mu\sigma_1^m\sigma_2^l\Phi^{(0)}$, which belong to $\mu\overline{\mathcal{H}}^n_{0,|s\rangle}$ if $2l+m\le n$. We can order such monomials as follows: we say that $\mu\sigma_1^{m'}\sigma_2^{l'}\Phi^{(0)} \prec \mu\sigma_1^m\sigma_2^l\Phi^{(0)}$ if $2l'+m' < 2l+m$, or $2l'+m' = 2l+m$ and $m'<m$. From the expansion [47, Eq. 8.962.1]

$$P^{(\gamma,\delta)}_\nu(t) = \frac{1}{\nu!}\sum_{k=0}^{\nu}\frac{1}{2^k k!}\,(-\nu)_k(\gamma+\delta+\nu+1)_k(\gamma+k+1)_{\nu-k}\,(1-t)^k\,,$$

where $(x)_k$ is the Pochhammer symbol

$$(x)_k = x(x+1)\cdots(x+k-1)\,,$$

it follows that $P^{(\gamma,\delta)}_\nu(0) > 0$ provided that $\gamma+1>0$ and $\gamma+\delta+2\nu<0$. In particular, $P^{(\alpha,\beta)}_{[\frac{m}{2}]}(0) > 0$ since

$$\alpha+1 = N\Big(a+\frac{1}{2}\Big) - \frac{1}{2} > N - \frac{1}{2} > 0\,,\quad \alpha+\beta+2\Big[\frac{m}{2}\Big] = 2\Big[\frac{m}{2}\Big] - m - \frac{1}{2} \le -\frac{1}{2} < 0\,.$$

Hence we can write

$$\Psi^{(0)}_{lm} = \mu\Phi^{(0)}\big(c_{lm}\sigma_1^m\sigma_2^l + \text{l.o.t.}\big)\,,$$

where $c_{lm} \neq 0$, so that

$$\big\langle \Psi^{(0)}_{lm} \mid 2l+m \leq n\big\rangle = \big\langle \mu\sigma_1^m\sigma_2^l\Phi^{(0)} \mid 2l+m \leq n\big\rangle\,.$$

Likewise, a similar argument shows that for $m \geq 1$

$$\big\langle \mu\bar{x}^{m-1}L_l^{-\beta}(\omega r^2)P^{(\alpha+1,\beta)}_{[\frac{m-1}{2}]}(t)\,\Phi^{(1)} \mid 2l+m \leq n\big\rangle = \big\langle \mu\sigma_1^{m-1}\sigma_2^l\Phi^{(1)} \mid 2l+m \leq n\big\rangle\,,$$

and therefore

$$\big\langle \Psi^{(0)}_{lm},\, \Psi^{(1)}_{lm} \mid 2l+m \leq n\big\rangle = \big\langle \mu\sigma_1^m\sigma_2^l\Phi^{(0)},\, \mu\sigma_1^{m-1}\sigma_2^l\Phi^{(1)} \mid 2l+m \leq n\big\rangle\,.$$

Proceeding in the same way with the remaining spin eigenfunctions we can finally show that

$$\big\langle \Psi^{(k)}_{lm} \mid k = 0,\dots,4\,,\; 2l+m \leq n\big\rangle = \mu\overline{\mathcal{H}}^n_{0,|s\rangle}\,,$$

and that

$$\mu\overline{\mathcal{H}}^n_{0,|s\rangle} + \big\langle \widetilde{\Psi}^{(2)}_{lm} \mid 2l+m \leq n\big\rangle = \mu\widetilde{\mathcal{H}}^n_{0,|s\rangle}\,, \qquad |s\rangle \in \Sigma', \quad S_{12}|s\rangle = |s\rangle,$$
$$\mu\overline{\mathcal{H}}^n_{0,|s\rangle} + \big\langle \widehat{\Psi}^{(3)}_{lm} \mid 2l+m \leq n\big\rangle = \mu\widehat{\mathcal{H}}^n_{0,|s\rangle}\,, \qquad S_{12}|s\rangle = -|s\rangle,$$

as claimed. □

Remark 7. By Remark 1, the coefficients of $\Phi^{(0)}$ in the spin eigenfunctions $\Psi^{(0)}_{lm}$ and $\Psi^{(3)}_{lm}$ yield the two families of eigenfunctions of the scalar reduction H_0^{sc} of the model (2a) presented without proof in our previous paper [1]. Earlier work on the scalar model H_0^{sc} had established the existence of two families of eigenfunctions of the form $\mu L_l^{-\beta}(\omega r^2)p_\nu(\mathbf{x})$, with p_ν a homogeneous polynomial of degree $\nu \geq 3$, only for $\nu \leq 6$ and $N \geq \nu$ [42]. More recently, Ezung et al. [48] have rederived a very small subset of these scalar eigenfunctions by mapping H_0^{sc} to N decoupled oscillators.

Case b

Since $c_0 = 4\omega$ in this case, reasoning as before we conclude that the algebraic energies are the numbers

$$E_k = E_0 + 4k\omega, \qquad k = 0,1,\dots,$$

where k is the degree in $\mathbf{z}$ of the corresponding eigenfunctions of $\overline{H}_1$. We shall see below that these eigenfunctions can be written in terms of generalized Laguerre polynomials. To this end, we begin by identifying certain subspaces of $\mathcal{H}_1^n$ invariant under $\overline{H}_1$.

Lemma 4. *For any given spin state $|s\rangle \in \Sigma$, the operator $\overline{H}_1$ preserves the subspace*

$$\mathcal{H}^n_{1,|s\rangle} = \big\langle f(\sigma_1,\sigma_2)\,\Phi^{(0)},\, g(\sigma_1)\,\Phi^{(1)} \mid f_{22} = 0\big\rangle \subset \mathcal{H}_1^n\,, \tag{66}$$

where f and g are polynomials of degrees at most n and $n-1$ in $\mathbf{z}$, respectively, and $\Phi^{(k)}$ is given by (21).

Proof. The statement follows from the obvious identity $T_1\big(f\Phi^{(0)}\big) = (T_1 f)\Phi^{(0)}$ and Eqs. (10b), (11), (15), (29), (33) and (35). $\square$

By the previous lemma we can assume that the eigenfunctions of $\overline{H}_1$ in $\mathcal{H}^n_{1,|s\rangle}$ are of the form

$$\Phi = f\Phi^{(0)} + g\Phi^{(1)}\,, \qquad \deg\Phi = k \le n\,. \tag{67}$$

From Eqs. (29), (33) and (35) it easily follows that the eigenvalue equation $\overline{H}_1\Phi = (E_0 + 4k\omega)\Phi$ can be cast into the system

$$\Big[-T_1 + \omega(J^0 + 1 - k) - \Big(b + \frac{1}{2}\Big)J^-\Big]g - 2g_1 = 0\,, \tag{68a}$$

$$\Big[-T_1 + \omega(J^0 - k) - \Big(b + \frac{1}{2}\Big)J^-\Big]f = \Big(2a + b + \frac{1}{2}\Big)g\,. \tag{68b}$$

Since f is linear in σ_2 (cf. Eq. (66)), we can write

$$f = p + \sigma_2 q, \tag{69}$$

where p and q are polynomials in σ_1. Using Eqs. (10b), (11), (15), (34b) and (36) we can easily rewrite the system (68) as follows:

$$\big[L_1 - \omega(k-1)\big]g - 2g_1 = 0\,, \tag{70a}$$

$$\big[L_1 - \omega(k-2)\big]q - 4q_1 = 0\,, \tag{70b}$$

$$\big(L_1 - \omega k\big)p = \Big(2a + b + \frac{1}{2}\Big)g + 2\Big(4a + b + \frac{3}{2}\Big)\sigma_1 q\,, \tag{70c}$$

where

$$L_1 = -\sigma_1\partial^2_{\sigma_1} + \Big[\omega\sigma_1 - \Big(2a + b + \frac{1}{2}\Big)N\Big]\partial_{\sigma_1}\,. \tag{71}$$

The last step is to construct the polynomials solutions of the system (70), which can be expressed in terms of generalized Laguerre polynomials, according to the following theorem.

Theorem 3. *The Hamiltonian H_1 possesses the following families of spin eigenfunctions with eigenvalue $E_k = E_0 + 4k\omega$:*

$$\Psi^{(0)}_k = \mu L^{\alpha-1}_k(\omega r^2)\,\Phi^{(0)}\,, \qquad k \ge 0\,,$$

$$\Psi^{(1)}_k = \mu L^{\alpha+1}_{k-1}(\omega r^2)\big[N\Phi^{(1)} - r^2\Phi^{(0)}\big]\,, \qquad k \ge 1\,,$$

$$\Psi^{(2)}_k = \mu L^{\alpha+3}_{k-2}(\omega r^2)\Big[N(\alpha+1)\sum_i x_i^4 - \beta r^4\Big]\,\Phi^{(0)}\,, \qquad k \ge 2\,,$$

where $\alpha = N(2a+b+\frac{1}{2})$, $\beta = N(4a+b+\frac{3}{2})$, and $\Phi^{(j)} = \Lambda\big(x_1^{2j}|s\rangle\big)$, with $j = 0, 1$ and $|s\rangle \in \Sigma$. For each $|s\rangle \in \Sigma$ and $n = 0, 1, \dots,$ the above eigenfunctions with $k \le n$ span the whole H_1-invariant space $\mu\mathcal{H}^n_{1,|s\rangle}$.

Proof. As in the previous case, the algebraic eigenfunctions of H_1 are of the form $\Psi = \mu\Phi$, where μ is given in Table I and Φ is an eigenfunction of $\overline{H}_1$ of the form (67)-(69). The functions p, q and g are polynomials in σ_1 determined by the system (70), which in terms of the variable $t = \omega\sigma_1 \equiv \omega r^2$ can be written as

$$\mathcal{L}^{\alpha+1}_{k-1}g = \mathcal{L}^{\alpha+3}_{k-2}q = 0\,, \qquad \mathcal{L}^{\alpha-1}_k p = -\frac{\alpha}{N\omega}\,g - \frac{2\beta}{N\omega^2}\,tq\,, \tag{72}$$

where $\mathcal{L}_\nu^\lambda$ is the generalized Laguerre operator (cf. Eq. (62)). The general polynomial solutions of the first two equations in (72) are respectively given by

$$g = c_1 L_{k-1}^{\alpha+1}(t), \qquad q = c_2 L_{k-2}^{\alpha+3}(t). \tag{73}$$

On the other hand, from the elementary identity

$$\mathcal{L}_\nu^\lambda\Big(t^l L_{\nu-l}^{\lambda+2l}(t)\Big) = l(l+\lambda)t^{l-1} L_{\nu-l}^{\lambda+2l}(t)\,,$$

it follows that the general polynomial solution of the third equation in (72) is given by

$$p = c_0 L_k^{\alpha-1}(t) - \frac{c_1}{N\omega}\, t L_{k-1}^{\alpha+1}(t) - \frac{c_2\beta}{N\omega^2(\alpha+1)}\, t^2 L_{k-2}^{\alpha+3}(t)\,. \tag{74}$$

Equations (73) and (74) immediately yield the formulas of the eigenfunctions in this case. The last assertion in the statement of the theorem follows from the fact that the functions p, q and g in Eqs. (73) and (74) are the most general polynomial solution of the system (72). □

Remark 8. The spin eigenfunctions $\Psi_k^{(j)}$, $j = 0, 1, 2$, listed in the previous theorem are essentially the same as those presented in Ref. [1] (note that in the latter reference there is a typo in the formula for the scalar eigenfunction $\psi_n^{(1)}$, namely the coefficient α multiplying r^4 should be replaced by the parameter β defined in Theorem 3).

Remark 9. It should be noted that for $\omega = 0$ the potentials (2a) and (2b) scale as r^{-2} under dilations of the coordinates (as is the case for the original Calogero model). The standard argument used in the solution of the Calogero model shows that there is a basis of eigenfunctions of these models of the form $\mu(\mathbf{x})L_\nu^\lambda(\omega r^2)F(\mathbf{x})$, where F is a homogeneous spin-valued function. The eigenfunctions presented in Theorems 2 and 3 are indeed of this form.

Case c

The model (2c) is of less interest than the previous ones, since we shall see that in this case the number of independent algebraic eigenfunctions is essentially finite. We shall take, for definiteness, the plus sign in the change of variable listed in Table I (it will be apparent from the discussion that follows that the minus sign does not yield additional solutions).

Let us first note that the potential for this model is translationally invariant, so that the total momentum operator $P = -\mathrm{i}\sum_k \partial_{x_k}$ commutes with the Hamiltonian H_2. Hence the eigenfunctions of H_2 can be assumed to have well-defined total momentum. Equivalently, since

$$\mu^{-1}\cdot P\cdot \mu\Big|_{x_k = -\frac{\mathrm{i}}{2}\log z_k} = 2J^0, \tag{75}$$

the eigenfunctions of $\overline{H}_2$ can be assumed to be homogeneous in $\mathbf{z}$. Let Φ be a homogeneous eigenfunction of $\overline{H}_2$ of degree k and eigenvalue E, so that $\Psi = \mu\Phi$ is an eigenfunction of H_2 with total momentum $2k$ (cf. Eq. (75)) and energy E. By Eq. (75), the function $\tau_N^\lambda\Psi$ clearly has total momentum $2(k+N\lambda)$. In fact, the following lemma implies that $\tau_N^\lambda\Psi$ is also an eigenfunction of H_2 with a suitably boosted energy:

Lemma 5. *Let Φ be a homogeneous eigenfunction of $\overline{H}_2$ of degree k and eigenvalue E. Then $\tau_N^\lambda\Phi$ is an eigenfunction of $\overline{H}_2$ with eigenvalue $E+8k\lambda+4N\lambda^2$.*

Proof. From the identity

$$\sum_i \frac{1}{z_i-z_{i+1}}\left(z_i^2\partial_i - z_{i+1}^2\partial_{i+1}\right) = J^0 + \frac{1}{2}\sum_i \frac{z_i+z_{i+1}}{z_i-z_{i+1}}\left(D_i - D_{i+1}\right),$$

where $D_i = z_i\partial_i$, we immediately obtain the following expression for the gauge Hamiltonian $\overline{H}_2$:

$$\frac{1}{4}\left(\overline{H}_2 - E_0\right) = \sum_i D_i^2 + a\sum_i \frac{z_i+z_{i+1}}{z_i-z_{i+1}}\left(D_i - D_{i+1}\right) - 2a\sum_i \frac{z_i z_{i+1}}{(z_i-z_{i+1})^2}\left(1-K_{i,i+1}\right). \quad (76)$$

Since $\tau_N^{-\lambda}D_i\tau_N^\lambda = D_i+\lambda$ for any real λ, it follows that

$$\tau_N^{-\lambda}\overline{H}_2\tau_N^\lambda = \overline{H}_2 + 8\lambda J^0 + 4N\lambda^2\,.$$

Taking into account that $J^0\Phi = k\Phi$, we conclude that

$$\overline{H}_2\big(\tau_N^\lambda\Phi\big) = \big(E+8k\lambda+4N\lambda^2\big)\big(\tau_N^\lambda\Phi\big)\,,$$

as claimed. □

By the previous discussion, in what follows any two eigenfunctions of $\overline{H}_2$ that differ by a power of τ_N shall be considered equivalent. From Theorem 1 and Corollary 1 it easily follows that in this case the number of independent algebraic eigenfunctions is finite, up to equivalence. More precisely:

Lemma 6. *Up to equivalence, the algebraic eigenfunctions of $\overline{H}_2$ can be assumed to belong to a space of the form*

$$\mathcal{H}_{2,|s\rangle} = \langle \sigma_1,\, \tau_{N-1},\, \sigma_1\tau_{N-1},\, \tau_N\rangle\, \Phi^{(0)} + \langle 1,\, \tau_{N-1}\rangle\,\Phi^{(1)} + \langle 1,\, \sigma_1\rangle\,\tau_N\Phi^{(-1)} \quad (77)$$

for some spin state $|s\rangle$, where $\Phi^{(k)}$ is given by (21).

Proof. Given a spin state $|s\rangle$, the obvious identity

$$T_2\big(f\Phi^{(0)}\big) = (T_2 f)\Phi^{(0)} \quad (78)$$

and Eqs. (6), (30), (31), and (33) imply that the gauge Hamiltonian $\overline{H}_2$ preserves the space

$$\mathcal{H}^n_{2,|s\rangle} = \big\langle f\Phi^{(0)}, g\Phi^{(1)}, \tau_N q\Phi^{(-1)}\big\rangle\,, \quad (79)$$

where f, g and q are as in the definition of $\mathcal{T}_2^n$ in Theorem 1. Let $\Phi\in\mathcal{H}^n_{2,|s\rangle}$ be an eigenfunction of $\overline{H}_2$, which as explained above can be taken as a homogeneous function of $\mathbf{z}$. From the conditions satisfied by the functions f, g and q in (79) and the homogeneity of Φ, it readily follows that $\Phi\in\tau_N^l\mathcal{H}_{2,|s\rangle}$ for some l, as claimed. □

Theorem 4. *The Hamiltonian H_2 possesses the following spin eigenfunctions with zero momentum*

$$\Psi_0 = \mu\,\Phi^{(0)}, \qquad \Psi_{1,2} = \mu\sum_i \begin{Bmatrix}\cos\\ \sin\end{Bmatrix}\big(2(x_i-\bar{x})\big)|s_i\rangle,$$

$$\Psi_3 = \mu\bigg[\frac{2a}{2a+1}\,\Phi^{(0)} + \sum_{i\neq j}\cos\big(2(x_i-x_j)\big)|s_j\rangle\bigg], \quad \Psi_4 = \mu\sum_{i\neq j}\sin\big(2(x_i-x_j)\big)|s_j\rangle,$$

where $|s_i\rangle$ is defined in (18) *and $\bar{x}$ is the center of mass coordinate. Their energies are respectively given by*

$$E_0\,, \qquad E_{1,2} = E_0 + 4\Big(2a+1-\frac{1}{N}\Big)\,, \qquad E_{3,4} = E_0 + 8(2a+1)\,.$$

Any algebraic eigenfunction with well-defined total momentum is equivalent to a linear combination of the above eigenfunctions.

Proof. By Lemma 6, in order to compute the algebraic eigenfunctions of $\overline{H}_2$ it suffices to diagonalize $\overline{H}_2$ in the spaces $\mathcal{H}_{2,|s\rangle}$ given by (77). Equations (30), (31), (33) and (78), and the fact that $\overline{H}_2$ preserves the degree of homogeneity, imply that the following subspaces of $\mathcal{H}_{2,|s\rangle}$ are invariant under $\overline{H}_2$:

$$\langle\sigma_1\tau_{N-1}\,,\tau_N\rangle\,\Phi^{(0)}\,, \tag{80a}$$

$$\langle\sigma_1\Phi^{(0)}\rangle\,, \qquad \langle\sigma_1\Phi^{(0)},\Phi^{(1)}\rangle\,, \qquad \langle\sigma_1\tau_{N-1}\Phi^{(0)},\tau_N\Phi^{(0)},\tau_{N-1}\Phi^{(1)}\rangle\,, \tag{80b}$$

$$\langle\tau_{N-1}\Phi^{(0)}\rangle\,, \quad \langle\tau_{N-1}\Phi^{(0)},\tau_N\Phi^{(-1)}\rangle\,, \quad \langle\sigma_1\tau_{N-1}\Phi^{(0)},\tau_N\Phi^{(0)},\sigma_1\tau_N\Phi^{(-1)}\rangle\,. \tag{80c}$$

From Eqs. (80) it follows that the alternative change of variables $z_k = \mathrm{e}^{-2\mathrm{i}x_k}$ does not yield additional eigenfunctions of H_2. Indeed, the latter change corresponds to the mapping $z_k \mapsto 1/z_k$, which up to equivalence leaves the subspace (80a) invariant and exchanges each subspace in (80b) with the corresponding one in (80c). For this reason, we can safely ignore the subspaces (80c) in the computation that follows, provided that we add to the eigenfunctions of $\overline{H}_2$ obtained from the subspaces (80b) their images under the mapping $z_k \mapsto 1/z_k$.

For the subspaces (80a)-(80b), using Eqs. (6), (11), (17), (30), (33), and (34c) we easily obtain the following eigenfunctions of $\overline{H}_2$:

$$\tau_N\Phi^{(0)}, \qquad E = E_0 + 4N, \tag{81a}$$

$$\Phi^{(1)}, \qquad E = E_0 + 4(2a+1), \tag{81b}$$

$$\tau_{N-1}\Phi^{(1)} - \frac{\tau_N}{2a+1}\,\Phi^{(0)}, \qquad E = E_0 + 4(N+4a+2). \tag{81c}$$

We have omitted the two additional eigenfunctions

$$\sigma_1\Phi^{(0)}\,, \qquad \Big(\sigma_1\tau_{N-1} - \frac{N\tau_N}{2a+1}\Big)\Phi^{(0)}$$

from the above list, since they are respectively obtained from (81b) and (81c) when the spin state $|s\rangle$ is symmetric. The eigenfunctions (81) are equivalent to the following "zero momentum" eigenfunctions:

$$\Phi^{(0)}, \qquad E = E_0, \tag{82a}$$

$$\tau_N^{-1/N}\,\Phi^{(1)}, \qquad E = E_0 + 4\Big(2a+1-\frac{1}{N}\Big), \tag{82b}$$

$$\frac{\tau_{N-1}}{\tau_N}\,\Phi^{(1)} - \frac{1}{2a+1}\,\Phi^{(0)}, \qquad E = E_0 + 8(2a+1), \tag{82c}$$

where the energies have been computed from those in Eqs. (81) using Lemma 5. The eigenfunctions of H_2 listed in the statement are readily obtained from these spin functions together with the transforms of (82b) and (82c) under the mapping $z_k \mapsto 1/z_k$. □

Remark 10. If the spin state $|s\rangle$ is symmetric, then $|s_i\rangle = \Phi^{(0)}/N$ for all i, and one easily obtains from Theorem 4 the following eigenfunctions of the scalar Hamiltonian H_2^{sc}:

$$\psi_0 = \mu, \quad \psi_{1,2} = \mu \sum_i \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} \big(2(x_i - \bar{x})\big), \quad \psi_3 = \mu \left[\frac{aN}{2a+1} + \sum_{i<j} \cos\big(2(x_i - x_j)\big) \right].$$

These formulas agree with those in Refs. [1] and [48] (the expression of ψ_3 in the former reference contains an obvious erratum, while this eigenfunction is missing altogether in the latter reference).

5. SUMMARY AND OUTLOOK

In this paper we have computed in closed form several infinite families of eigenfunctions of the spin models with near-neighbors interactions (2) introduced in our previous paper [1]. Our method is based on the fact that each spin Hamiltonian H_ϵ is related to a scalar operator $\overline{H}_\epsilon$ involving difference operators which exchange pairs of neighboring particles, cf. Eqs. (5)–(8). We have explicitly constructed a flag of finite-dimensional polynomial subspaces $\mathcal{H}_\epsilon^0 \subset \mathcal{H}_\epsilon^1 \subset \cdots$ invariant under $\overline{H}_\epsilon$ (see Corollary 1). For all three models (2), we have been able to fully diagonalize the gauge Hamiltonian $\overline{H}_\epsilon$ in its invariant spaces $\mathcal{H}_\epsilon^n$. Multiplying each eigenfunction of $\overline{H}_\epsilon$ in $\mathcal{H}_\epsilon^n$ by the appropriate gauge factor μ and performing a suitable change of variables (cf. Table I) we obtain the families of eigenfunctions of H_ϵ mentioned above.

The results obtained in this paper suggest several open problems that we shall now briefly discuss. In the first place, it would be natural to study the BC_N counterparts of the models (2), for which the interaction potential also depends on the sums $x_i + x_{i+1}$. In fact, in the scalar case this question has already been addressed in Ref. [42]. It would also be of interest to construct solvable models with near-neighbors interactions of elliptic type, both in the scalar and spin cases; see Refs. [35, 36] for a list of models of this type with long-range interactions. An important problem closely related with the subject of this paper is the analysis of the spin chains obtained from the models (2) by applying the freezing trick. These chains are characterized by the fact that the interactions are restricted to nearest neighbors (as in the Heisenberg chain), but their strength depends on the distance between the sites (as in chains of Haldane–Shastry type). For this reason, we believe that the study of these new chains could prove of considerable interest.

Consider, for instance, the chain associated with the model (2a), whose Hamiltonian is given by

$$\mathsf{H}_0 = \sum_i (\xi_i - \xi_{i+1})^{-2} S_{i,i+1}, \tag{83}$$

where $(\xi_1, \dots, \xi_N)$ is the unique equilibrium of the scalar potential

$$U_0 = \frac{1}{2} r^2 + \sum_i \frac{1}{(x_i - x_{i-1})(x_i - x_{i+1})} + \sum_i \frac{1}{(x_i - x_{i+1})^2}$$

in the domain $x_1 < \cdots < x_N$. It can be shown that the chain sites ξ_i are symmetrically distributed around the origin; for instance, for $N = 4$ we have $\xi_4 = -\xi_1 = (\sqrt{3}+1)/2$, $\xi_3 = -\xi_2 = (\sqrt{3}-1)/2$. Note, in particular, that the chain sites are not equally spaced, as is the case in most spin chains of Haldane–Shastry type. In principle, it is not possible to apply the method of Refs. [38, 39, 40] to evaluate the partition function of the chain (83) in closed form, since the algebraic eigenfunctions of the model (2a) computed in Section 4 do not form a complete set. On the other hand, the explicit nature of the algebraic eigenfunctions presented in Theorem 2 makes it feasible to compute a number of eigenvalues and eigenfunctions of the spin chain (83) by taking the strong coupling limit $a \to \infty$. We emphasize that the results thus obtained would be valid for an arbitrary number of spins, and thus could be helpful in uncovering general properties of the spectrum. By combining this approach with numerical computations for fixed values of N, we expect to achieve a reasonable understanding of the properties of this novel type of chains.

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