Classical and quantum polarization correlations

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We compare the maximum achievable polarization correlations for classical-like separable states and quantum entangled states. For one-photon systems we find that the maximally entangled states have three times larger correlations than the maximum correlations achievable with separable states. However, for larger photon numbers we find that there are separable states with larger correlations than the maximally entangled states.

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I. INTRODUCTION

Polarization is a fundamental ingredient of light, both in the quantum and in the classical domains. In the quantum regime this variable has been crucial in order to demonstrate experimentally fundamental properties and applications of the quantum theory such as entanglement, complementarity, quantum cryptography, teleportation, and Bell inequalities [1–14]. All these examples involve sources of light with nonclassical properties and strong polarization correlations.

It might be guessed that quantum entangled states would present always larger polarization correlations than classical-like separable states. In this work we examine this issue in some detail. We compute and compare the maximum achievable correlations attainable by entangled and separable (i.e., disentangled) states. For one-photon systems we find that the maximally entangled states have three times larger correlations than the maximum correlations achievable with separable states (Sec. III). However, for larger photon numbers we find that there are separable states with larger correlations than the maximally entangled states (Sec. IV). These results are obtained by using a recently introduced measure of polarization correlations [15,16] (Sec. II). For the sake of completeness and comparison we also compute the polarization correlations for the same states by using the familiar Stokes parameters (Sec. V).

II. POLARIZATION CORRELATIONS

In a recent work [15,16] we have proposed a suitable assessment of polarization correlations in terms of the joint polarization distribution on the Poincaré sphere given by the SU(2) Q function [17]

$$Q(\Omega_1,\Omega_2)=\sum_{n_1,n_2=0}^{\infty}\frac{(n_1+1)(n_2+1)}{(4\pi)^2}\times\langle n_1,\Omega_1;n_2,\Omega_2|\rho|n_1,\Omega_1;n_2,\Omega_2\rangle, \tag{2.1}$$

where ρ is the density matrix, $|n_1,\Omega_1;n_2,\Omega_2\rangle$ is the product of SU(2) coherent states in the corresponding modes

$$|n,\Omega\rangle=\sum_{m=0}^{n}\binom{n}{m}^{1/2}\left(\sin\frac{\theta}{2}\right)^{n-m}\left(\cos\frac{\theta}{2}\right)^{m}e^{-im\phi}|m,n-m\rangle, \tag{2.2}$$

and $|n_1,n_2\rangle$ is the product of photon number states in the two field modes sustaining a single polarization mode. In these expressions Ω represents the standard parametrization of the sphere $\Omega=(\theta,\phi)$, where θ and ϕ are the polar and the azimuthal angles, respectively. The polarization correlations can be naturally measured as the distance between the joint distribution $Q(\Omega_1,\Omega_2)$ and the product of individual distributions $Q_1(\Omega_1)Q_2(\Omega_2)$, in the form

$$C=\int d\Omega_1 d\Omega_2[Q(\Omega_1,\Omega_2)-Q_1(\Omega_1)Q_2(\Omega_2)]^2, \tag{2.3}$$

where $d\Omega_j=\sin\theta_j d\theta_j d\phi_j$, $j=1,2$ is the differential of solid angle. A similar approach has been adopted to asses the visibility of multiparticle interference fringes [18]. This definition is invariant under SU(2) transformations applied to each polarization mode separately. These are linear and energy conserving transformations of the field complex amplitudes (produced by passive optical devices such as free propagation, beam splitters, phase plates, and mirrors) that modify the position and orientation of the polarization distribution on the Poincaré sphere but preserve its form.

This formalism assesses polarization correlations irrespective of its quantum or classical origin. Therefore this is a suitable tool to compare the maximum achievable correlations attainable with quantum and classical states.

As an alternative approach to this problem we can mention the use of the familiar Stokes parameters, which is further developed in the next sections. Here we just point out that the measure (2.3) is more powerful and complete than any other one based on the Stokes parameters since C involves field correlations of all orders simultaneously, while the Stokes parameters involve only the lowest orders.

III. ONE-PHOTON STATES

In a first approach to the problem let us examine exhaustively the simple but fully relevant case of a single photon in each polarization mode. In this case any joint density matrix can be expressed as a linear combination of the operators $s_{1,k}\otimes s_{2,\ell}$, for $k,\ell=0,x,y,z$, having the following expres-

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sions in the photon-number basis:

$$\begin{aligned} s_{j,0} &= |1,0\rangle_j\langle 1,0| + |0,1\rangle_j\langle 0,1|, \\ s_{j,x} &= |0,1\rangle_j\langle 1,0| + |1,0\rangle_j\langle 0,1|, \\ s_{j,y} &= i(|0,1\rangle_j\langle 1,0| - |1,0\rangle_j\langle 0,1|), \\ s_{j,z} &= |1,0\rangle_j\langle 1,0| - |0,1\rangle_j\langle 0,1|, \end{aligned} \tag{3.1}$$

with

$$\mathrm{tr}[(s_{1,k}\otimes s_{2,\ell})(s_{1,m}\otimes s_{2,r})] = 4\delta_{k,m}\delta_{\ell,r}. \tag{3.2}$$

These are the restrictions to the subspace of one photon per mode of the Stokes operators

$$\begin{aligned} S_{j,0} &= a_{j,1}^\dagger a_{j,1} + a_{j,2}^\dagger a_{j,2}, \\ S_{j,x} &= a_{j,2}^\dagger a_{j,1} + a_{j,1}^\dagger a_{j,2}, \\ S_{j,y} &= i(a_{j,2}^\dagger a_{j,1} - a_{j,1}^\dagger a_{j,2}), \\ S_{j,z} &= a_{j,1}^\dagger a_{j,1} - a_{j,2}^\dagger a_{j,2}, \end{aligned} \tag{3.3}$$

where $a_{j,k}$ are the corresponding complex amplitude operators.

Every one-photon pure state is a SU(2) coherent state $|1,\Omega\rangle$ and

$$\begin{aligned} \langle 1,\Omega|\mathbf{S}|1,\Omega\rangle &= \langle 1,\Omega|\mathbf{s}|1,\Omega\rangle \\ &= \mathbf{\Omega} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \end{aligned} \tag{3.4}$$

leading to a joint polarization distribution

$$\begin{aligned} Q(\Omega_1,\Omega_2) = \frac{1}{(4\pi)^2}[1 &+ \mathbf{\Omega}_1\cdot\langle\mathbf{s}_1\rangle + \mathbf{\Omega}_2\cdot\langle\mathbf{s}_2\rangle \\ &+ \langle(\mathbf{\Omega}_1\cdot\mathbf{s}_1)\otimes(\mathbf{\Omega}_2\cdot\mathbf{s}_2)\rangle], \end{aligned} \tag{3.5}$$

while the associated individual distributions are

$$Q_j(\Omega_j) = \frac{1}{4\pi}(1 + \mathbf{\Omega}_j\cdot\langle\mathbf{s}_j\rangle). \tag{3.6}$$

From Eqs. (2.3) and (3.5) it can be seen that the degree of polarization correlations admits a simple and natural expression in terms of the mean values of the Stokes operators

$$C_{\text{one photon}} = \frac{1}{(12\pi)^2}\sum_{k,\ell=x,y,z}(\langle s_{1,k}\otimes s_{2,\ell}\rangle - \langle s_{1,k}\rangle\langle s_{2,\ell}\rangle)^2, \tag{3.7}$$

which resembles previous approaches to this problem [19]. We stress that this equivalence occurs only for one-photon states since in this case the Stokes parameters are the only nontrivial moments of the polarization distribution.

The joint Q function for one-photon states can be easily determined experimentally by measuring the probability $P(\Omega_1,\Omega_2)$ that the two photons cross simultaneously two analyzers

$$\begin{aligned} P(\Omega_1,\Omega_2) &= \mathrm{tr}(\rho|1,\Omega_1\rangle_1\langle 1,\Omega_1|\otimes|1,\Omega_2\rangle_2\langle 1,\Omega_2|) \\ &= 4\pi^2 Q(\Omega_1,\Omega_2), \end{aligned} \tag{3.8}$$

where $|1,\Omega_j\rangle_j$ represent the polarization states crossing the corresponding analyzer with certainty. The statistics of such a measurement coincides with the definition of the two-mode polarization distribution $Q(\Omega_1,\Omega_2)$.

A. Maximum correlations for separable states

With the help of the above results we can investigate the maximum polarization correlations attainable with separable states with a single photon in each mode. To this end we construct the most general separable state considering the diagonal form of the density matrix, whose eigenvectors must be orthogonal product states. Without loss of generality we can always choose one of the eigenvectors to be the product of number states $|1,0\rangle_1|1,0\rangle_2$ [this is because of the double SU(2) symmetry of C mentioned above]. The other eigenvectors must be orthogonal to this one. The procedure to construct the other most general orthogonal states is rather simple due to the reduced dimensionality. Specifically (i) every one-photon pure state is a SU(2) coherent state $|1,\Omega\rangle$, and (ii) the only state orthogonal to $|1,\Omega\rangle$ is the antipodal state $|1,-\Omega\rangle$, where $-\Omega = (\pi-\theta,\phi+\pi)$. The result for ρ_s is of the form

$$\begin{aligned} \rho_s = \alpha|1,0\rangle_1\langle 1,0|\otimes|1,0\rangle_2\langle 1,0| &+ \beta|1,0\rangle_1\langle 1,0|\otimes|0,1\rangle_2 \\ \times\langle 0,1| &+ \gamma|0,1\rangle_1\langle 0,1|\otimes|1,\Omega\rangle_2\langle 1,\Omega| + \delta|0,1\rangle_1 \\ \times\langle 0,1|\otimes|1,-\Omega\rangle_2\langle 1,-\Omega|, \end{aligned} \tag{3.9}$$

or an equivalent expression where the modes 1 and 2 are interchanged. In this expression α, β, γ, δ are real nonnegative parameters with $\alpha+\beta+\gamma+\delta=1$, $|n,m\rangle$ are number states and $|1,\pm\Omega\rangle$ are SU(2) coherent states. The individual density matrices are

$$\begin{aligned} \rho_1 &= (\alpha+\beta)|1,0\rangle\langle 1,0| + (\gamma+\delta)|0,1\rangle\langle 0,1|, \\ \rho_2 &= \alpha|1,0\rangle\langle 1,0| + \beta|0,1\rangle\langle 0,1| + \gamma|1,\Omega\rangle\langle 1,\Omega| \\ &\quad + \delta|1,-\Omega\rangle\langle 1,-\Omega|, \end{aligned} \tag{3.10}$$

and then

$$\begin{aligned} \langle\mathbf{s}_1\rangle &= (\alpha+\beta-\delta-\gamma)\mathbf{u}_z, \\ \langle\mathbf{s}_2\rangle &= (\alpha-\beta)\mathbf{u}_z + (\gamma-\delta)\mathbf{\Omega}, \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} \langle s_{1,k\neq z}\otimes\mathbf{s}_2\rangle &= 0, \\ \langle s_{1,z}\otimes\mathbf{s}_2\rangle &= (\alpha-\beta)\mathbf{u}_z - (\gamma-\delta)\mathbf{\Omega}, \end{aligned} \tag{3.12}$$

where $\mathbf{u}_z$ is a unit real vector in the z direction. Therefore

$$\langle s_{1,z}\otimes \mathbf{s}_2\rangle - \langle s_{1,z}\rangle\langle \mathbf{s}_2\rangle = 2(\alpha-\beta)(\delta+\gamma)\mathbf{u}_z + 2(\alpha+\beta)(\delta-\gamma)\mathbf{\Omega}. \quad (3.13)$$

It can be easily seen that the maximum of C occurs when $\mathbf{u}_z=\mathbf{\Omega}$ with, either $\alpha=\delta=1/2$, $\beta=\gamma=0$, or $\alpha=\delta=0$, $\beta=\gamma=1/2$. We have also the possibility of $\mathbf{u}_z=-\mathbf{\Omega}$ interchanging $\alpha\leftrightarrow\beta$. The maximum of the polarization correlations for one-photon separable states is

$$C_{\mathrm{sm}}=\frac{1}{(12\pi)^2}. \quad (3.14)$$

One of the states reaching the maximum is, for example,

$$\rho_{\mathrm{sm}}=\tfrac{1}{2}(|1,0\rangle_1\langle 1,0|\otimes|1,0\rangle_2\langle 1,0|+|0,1\rangle_1\langle 0,1|\otimes|0,1\rangle_2\langle 0,1|) = \tfrac{1}{4}(I+s_{1,z}\otimes s_{2,z}), \quad (3.15)$$

where I represents the identity in the subspace of one-photon states, i.e., $I=s_{1,0}\otimes s_{2,0}$. The corresponding joint Q function is

$$Q_{\mathrm{sm}}(\Omega_1,\Omega_2)=\frac{1}{(4\pi)^2}(1+\cos\theta_1\cos\theta_2), \quad (3.16)$$

with uniform individual distributions $Q_1(\Omega_1)=Q_2(\Omega_2)=1/(4\pi)$.

B. Maximally entangled state

This maximum for separable states can be compared with the correlations for the maximally entangled state

$$|\psi\rangle_{\mathrm{me}}=\frac{1}{\sqrt{2}}(|1,0\rangle_1|1,0\rangle_2+|0,1\rangle_1|0,1\rangle_2), \quad (3.17)$$

or, equivalently,

$$\rho_{\mathrm{me}}=\tfrac{1}{4}(I+s_{1,x}\otimes s_{2,x}-s_{1,y}\otimes s_{2,y}+s_{1,z}\otimes s_{2,z}). \quad (3.18)$$

As has been shown in Ref. [16], this is the pure state with maximum polarization correlations. The joint Q function is

$$Q_{\mathrm{me}}(\Omega_1,\Omega_2)=\frac{1}{(4\pi)^2}[1+\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos(\phi_1+\phi_2)], \quad (3.19)$$

while the individual distributions are uniform $Q_1(\Omega_1)=Q_2(\Omega_2)=1/(4\pi)$. This leads to

$$C_{\mathrm{me}}=\frac{3}{(12\pi)^2}=3C_{\mathrm{sm}}. \quad (3.20)$$

The correlations arising from maximal entanglement are three times larger than the maximum achievable with separable states.

C. Separable counterpart of the maximally entangled state

A separable state that has been often considered as the separable counterpart of the maximally entangled state is [20]

$$\rho_{\mathrm{sc}}=\frac{1}{4\pi}\int d\Omega|1,\Omega\rangle_1\langle 1,\Omega|\otimes|1,\Omega\rangle_2\langle 1,\Omega| = \frac{1}{4}\left[I+\frac{1}{3}(s_{1,x}\otimes s_{2,x}+s_{1,y}\otimes s_{2,y}+s_{1,z}\otimes s_{2,z})\right], \quad (3.21)$$

leading to

$$Q_{\mathrm{sc}}(\Omega_1,\Omega_2)=\frac{1}{(4\pi)^2}\left\{1+\frac{1}{3}[1+\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos(\phi_1-\phi_2)]\right\}, \quad (3.22)$$

and again uniform individual distributions $Q_1(\Omega_1)=Q_2(\Omega_2)=1/(4\pi)$. In this case

$$C_{\mathrm{sc}}=\frac{1}{3(12\pi)^2}=\frac{1}{3}C_{\mathrm{sm}}=\frac{1}{9}C_{\mathrm{me}}, \quad (3.23)$$

and this separable counterpart of maximal entanglement does not provide maximum classical polarization correlations.

Despite the very different nature of the states (3.21) and (3.17), we have that the corresponding polarization distributions Q_{sc} and Q_{me} have the same structure (up to a trivial reflection $\phi_2\rightarrow-\phi_2$). The only relevant difference is the relative height of the distribution above the uniform constant background, so that, leaving aside the above-mentioned reflection,

$$Q_{\mathrm{sc}}=\frac{2}{3}\frac{1}{(4\pi)^2}+\frac{1}{3}Q_{\mathrm{me}}. \quad (3.24)$$

This means that the state (3.21) is a mixture of maximal entanglement and fully unpolarized light.

IV. MULTIPHOTON STATES

In what follows we address the generalization of these conclusions to the case of an arbitrary number of photons n in each polarization mode. For arbitrary dimension we have not been able to obtain the maximum C for separable states. Nevertheless, we can still obtain meaningful conclusions if we focus on suitable generalizations of the classical and quantum states considered above.

A. Separable states

We can begin with the generalization of Eq. (3.21),

$$\rho_{\mathrm{sc}}=\frac{1}{4\pi}\int d\Omega|n,\Omega\rangle_1\langle n,\Omega|\otimes|n,\Omega\rangle_2\langle n,\Omega|. \quad (4.1)$$

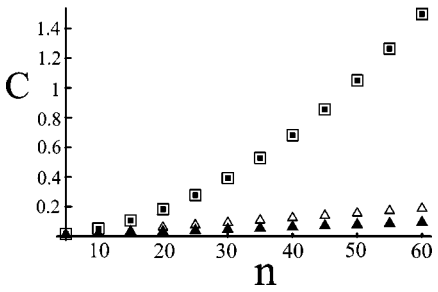

FIG. 1. Plot of the polarization correlations C as a function of the number of photons n for several field states: filled triangles for ρ_{sc} in Eq. (4.1); empty squares for ρ_{sm} in Eq. (4.7); empty triangles for the maximally entangled state $|\psi\rangle_{\mathrm{me}}$ in Eq. (4.12); filled squares for the entangled state $|\psi\rangle_{\mathrm{qe}}$ in Eq. (4.16). It can be appreciated that C_{sm} and C_{qe} coincide, giving more polarization correlations than ρ_{sc} and $|\psi\rangle_{\mathrm{me}}$. It can be also noticed that C_{sm} and C_{qe} depend quadratically on n, while for C_{sc} and C_{me} the dependence is linear.

In order to compute Q we will approximate the scalar product between SU(2) coherent states

$$|\langle n,\Omega|n,\Omega'\rangle|^2=\frac{1}{2^n}(1+\mathbf{\Omega}\cdot\mathbf{\Omega}')^n=\left[1-\frac{1}{4}(\mathbf{\Omega}-\mathbf{\Omega}')^2\right]^n. \tag{4.2}$$

Since this scalar product differs from zero significantly only when $\mathbf{\Omega}\simeq\mathbf{\Omega}'$ we can approximate the term in the square brackets by a Gaussian, so that

$$|\langle n,\Omega|n,\Omega'\rangle|^2\simeq e^{-n(\mathbf{\Omega}-\mathbf{\Omega}')^2/4}=e^{-n(1-\mathbf{\Omega}\cdot\mathbf{\Omega}')/2}, \tag{4.3}$$

so that Q can be easily computed,

$$Q_{\mathrm{sc}}(\Omega_1,\Omega_2)=\left(\frac{n+1}{4\pi}\right)^2 e^{-n}\frac{\sinh\left(\frac{n}{\sqrt{2}}\sqrt{1+\mathbf{\Omega}_1\cdot\mathbf{\Omega}_2}\right)}{\frac{n}{\sqrt{2}}\sqrt{1+\mathbf{\Omega}_1\cdot\mathbf{\Omega}_2}}. \tag{4.4}$$

The reduced individual distributions are uniform $Q_1(\Omega_1)=Q_2(\Omega_2)=1/(4\pi)$, so that C_{sc} can be expressed as

$$C_{\mathrm{sc}}=\frac{1}{(4\pi)^2}\left[2\frac{(n+1)^4}{n^2}e^{-2n}\int_0^1 dz\frac{\sinh^2(nz)}{z}-1\right], \tag{4.5}$$

which we have computed numerically obtaining the results represented in Figs. 1 (filled triangles) and 2 as a function of n. For $n\gg 1$ the rough approximation $[\sinh^2(nz)]/z\simeq\sinh^2(nz)$ is valid for the integration interval in Eq. (4.5). This allow us to compute the above integral leading to

$$C_{\mathrm{sc}}\simeq\frac{n}{2^6\pi^2}. \tag{4.6}$$

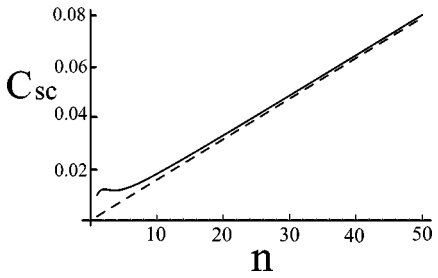

FIG. 2. Plot of the polarization correlations C_{sc} as a function of the number of photons n. In solid line we represent Eq. (4.5) while the dashed line corresponds to the approximation (4.6).

The usefulness of this approximation can be checked in Fig. 2, where we have represented the numerical evaluation of Eq. (4.5) along with the approximation (4.6).

On the other hand, a suitable generalization of the separable state ρ_{sm} (3.15) is provided by the state

$$\rho_{\mathrm{sm}}=\tfrac{1}{2}(|n,0\rangle_1\langle n,0|\otimes|n,0\rangle_2\langle n,0| +|0,n\rangle_1\langle 0,n|\otimes|0,n\rangle_2\langle 0,n|), \tag{4.7}$$

leading to a joint Q function

$$Q_{\mathrm{sm}}(\Omega_1,\Omega_2)=\frac{1}{2}\left(\frac{n+1}{4\pi}\right)^2\left[\left(\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\right)^{2n} +\left(\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\right)^{2n}\right], \tag{4.8}$$

and individual distributions

$$Q_j(\Omega_j)=\frac{n+1}{8\pi}\left(\cos^{2n}\frac{\theta_j}{2}+\sin^{2n}\frac{\theta_j}{2}\right), \tag{4.9}$$

so that

$$C_{\mathrm{sm}}=\frac{1}{2^6\pi^2}\frac{(n+1)^4}{(2n+1)^2}\left(1-\frac{n!^2}{(2n)!}\right)^2, \tag{4.10}$$

which is represented in Fig. 1 (empty squares) as a function of n. In the limit of $n\gg 1$ the following approximation is valid:

$$C_{\mathrm{sm}}\simeq\frac{n^2}{2^8\pi^2}. \tag{4.11}$$

It can be appreciated that this state carries larger polarization correlations than the state (4.1). On the other hand the individual distributions (4.9) are clearly not uniform, in spite of the fact that $\langle\mathbf{S}_j\rangle=0$.

B. Entangled states

We can consider at least two suitable multiphoton generalizations for the one-photon maximally entangled state (3.17). For example, we have the maximally entangled state

$$|\psi\rangle_{\rm me}=\frac{1}{\sqrt{n+1}}\sum_{m=0}^{n}|m,n-m\rangle_1|m,n-m\rangle_2, \quad (4.12)$$

recently studied in Ref. [21], and whose polarization properties have been already examined in Ref. [16]. In this case

$$Q_{\rm me}(\Omega_1,\Omega_2)=\frac{n+1}{(4\pi)^2}\left|\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}+e^{i(\phi_1+\phi_2)}\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\right|^{2n}, \quad (4.13)$$

and $Q_1(\Omega_1)=Q_2(\Omega_2)=1/(4\pi)$, so that

$$C_{\rm me}=\frac{1}{2^4\pi^2}\frac{n^2}{2n+1}, \quad (4.14)$$

which is represented in Fig. 1 (empty triangles). For $n\gg 1$ this scales linearly on n,

$$C_{\rm me}\simeq\frac{n}{2^5\pi^2}. \quad (4.15)$$

It is worth noting that for increasing n this state carries lesser polarization correlations than the separable state $\rho_{\rm sm}$ in Eq. (4.7). This seemingly paradoxical result might be ascribed to the fact that for the separable state $\rho_{\rm sm}$ in Eq. (4.7) the polarization distribution is concentrated around just two extreme polarization states, while the structure of the maximally entangled state (4.12) implies a more uniform distribution.

Finally we may consider a second generalization of the quantum entangled state (3.17) of the form

$$|\psi\rangle_{\rm qe}=\frac{1}{\sqrt{2}}(|n,0\rangle_1|n,0\rangle_2+|0,n\rangle_1|0,n\rangle_2), \quad (4.16)$$

which is a pure state counterpart of the separable state $\rho_{\rm sm}$ in Eq. (4.7). This is sometimes referred to either as a Schrödinger cat state or even as a maximally entangled state [22–24]. In this case, the joint polarization distribution is

$$Q_{\rm qe}(\Omega_1,\Omega_2)=\frac{(n+1)^2}{2^5\pi^2}\left\{\left(\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\right)^{2n}+\left(\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\right)^{2n}+\frac{1}{2^{2n-1}}(\sin\theta_1\sin\theta_2)^n\cos[n(\phi_1+\phi_2)]\right\}, \quad (4.17)$$

while the individual distributions are the same as in Eq. (4.9). This leads to

$$C_{\rm qe}=C_{\rm sm}+\frac{1}{2^5\pi^2}\frac{(n+1)!^4}{(2n+1)!^2}, \quad (4.18)$$

where $C_{\rm sm}$ is given by Eq. (4.10). This has been also represented in Fig. 1 (filled squares) as a function of n. We can appreciate that the polarization properties of this state are very similar to the corresponding ones for the separable state $\rho_{\rm sm}$ in Eq. (4.7). The only difference is the presence of an extra term in Eq. (4.17), which can be considered as a quantum interference effect. Its contribution to $C_{\rm qe}$ is positive so that the polarization correlations are always larger for the entangled state. Nevertheless, we have that for large number of photons $n\gg 1$ the extra terms tend to vanish so that the separable and the entangled state carry the same polarization correlations $C_{\rm qe}\simeq C_{\rm sm}$. As a matter of fact, for $n=3$ the difference between $C_{\rm qe}$ and $C_{\rm sm}$ is only 0.5%.

V. CORRELATIONS VIA STOKES PARAMETERS

For the sake of completeness and comparison we can compute the polarization correlations for multiphoton states by using a generalization of Eq. (3.7) involving the Stokes parameters

$$\tilde{C}=\frac{1}{(12\pi)^2}\sum_{k,\ell=x,y,z}(\langle S_{1,k}\otimes S_{2,\ell}\rangle-\langle S_{1,k}\rangle\langle S_{2,\ell}\rangle)^2. \quad (5.1)$$

We stress that Eqs. (2.3) and (5.1) are intrinsically different and they only coincide for one-photon states, while for multiphoton case they lead to different results as we see below.

For all the multiphoton states studied in the preceding section we have

$$\langle S_{j,k}\rangle=0. \quad (5.2)$$

For the separate state $\rho_{\rm sm}$ in Eq. (4.7) we have

$$\langle S_{1,k}\otimes S_{2,k'}\rangle=n^2\delta_{k,z}\delta_{k',z}. \quad (5.3)$$

Exactly the same result is obtained for the entangled state $\rho_{\rm qe}$ in Eq. (4.16), since for $n>1$ the contributions of crossed nondiagonal terms vanish. Therefore,

$$\tilde{C}_{\rm sm}=\tilde{C}_{\rm qe}=\frac{n^4}{(12\pi)^2}. \quad (5.4)$$

On the other hand, for the separable state $\rho_{\rm sc}$ in Eq. (4.1) we get

$$\langle S_{1,k}\otimes S_{2,k'}\rangle=\frac{n^2}{3}\delta_{k,k'}, \quad (5.5)$$

so that

$$\tilde{C}_{\rm sc}=\frac{n^4}{3(12\pi)^2}. \quad (5.6)$$

Finally, for the maximally entangled state ρ_{me} in Eq. (4.12) we get that the only nonvanishing terms are

$$\langle S_{1,z}\otimes S_{2,z}\rangle=\langle S_{1,x}\otimes S_{2,x}\rangle=-\langle S_{1,y}\otimes S_{2,y}\rangle=\tfrac{1}{3}n(n+2), \tag{5.7}$$

so that

$$\widetilde{C}_{\mathrm{me}}=\frac{n^2(n+2)^2}{3(12\pi)^2}. \tag{5.8}$$

We can appreciate that for all of them $\widetilde{C}$ scales as n^4. This is the main difference with the results of the preceding section. As we have argued before, the reason for this difference is that Eq. (2.3) involves all the moments of the Stokes operators, not only the first ones. Therefore, Eq. (2.3) provides a more complete assessment of polarization correlation properties. In other words, the examples analyzed above demonstrate the relevance of higher-order moments in order to properly asseses polarization properties.

VI. CONCLUSIONS

Summarizing, we have analyzed the polarization correlations of entangled and separable states. We have found the maximum polarization correlations for separable states with a single photon in each polarization mode. We have found that this maximum is three times smaller than the value achieved by the maximally entangled states.

When examining larger photon numbers we have found that there are separable states with larger polarization correlations than the maximally entangled states. Moreover, for increasing photon numbers entangled and separable states tend to have the same degree of polarization correlations. It is worth noting that this suggests that, in general, for multiphoton states, entanglement does not necessarily imply larger polarization correlations.

Finally, the examples analyzed in this work demonstrate the relevance in quantum optics of higher-order moments of polarization variables beyond the Stokes parameters.

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