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Equivalence between macroscopic quantum superpositions and maximally entangled states: Application to phase-shift detection

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We demonstrate that the superpositions of macroscopically distinct coherent states are maximally entangled states. Among other possible applications of this result, in this paper we show that the experimental arrangements generating superpositions of macroscopically distinct coherent states may be adapted for precision phase-shift detection reaching the maximum sensitivity allowed by quantum physics (Heisenberg limit).

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I. INTRODUCTION

Maybe the most attractive and puzzling subjects of the quantum theory can be encountered within the fuzzy boundary between the quantum and classical realms. Among the paradigms of quantum behavior revealed by the lack of classical counterpart, we may mention the quantum limits to precision measurements, coherent superpositions of macroscopically distinct states, entanglement, and extreme quantum correlations. Since these are examples of purely quantum features, it may be expected that they are tightly connected at a very fundamental level.

In this paper, we show that the quantum superposition of two macroscopically distinct coherent states is a maximally entangled state. Among other consequences, this implies that maximally entangled states may be easily produced using the same schemes devised to generate quantum superpositions of macroscopically distinct states. This is interesting since these quantum superpositions have been widely studied and have been already generated experimentally [1–5]. Other proposals to generate maximally entangled states in diverse contexts may be found in Refs. [6,7].

Furthermore, this equivalence implies that macroscopic (or mesoscopic) quantum superpositions may find applications in areas where quantum correlations are relevant such as the demonstration of quantum nonlocality, cryptography, quantum communication, and quantum computation. In particular, it is known that maximally entangled states may be applied to perform high-precision measurements [6,8]. In this paper, we will show that the superposition of macroscopically distinct coherent states allows us to perform phase-shift measurements reaching the maximum sensitivity allowed by quantum physics (i.e., the Heisenberg limit [8,9]). This is interesting because it implies that the problem of precision phase measurements can benefit from the advances achieved in the area of macroscopic quantum superpositions.

In Sec. II, we demonstrate the equivalence between maximally entangled states and the superposition of macroscopically distinct coherent states. In Sec. III, we show how these states may be used to reach the Heisenberg limit in precision phase measurements. We propose a practical scheme based on currently available experimental arrangements already used to generate these states in the field of quantum optics [2,3].

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II. SUPERPOSITIONS OF MACROSCOPICALLY DISTINCT COHERENT STATES AS MAXIMALLY ENTANGLED STATES

Throughout, we will deal with a system made of two independent degrees of freedom represented by bosonic operators a_1, a_2. This can be describing very different practical situations such as traveling electromagnetic fields, standing waves contained in cavities, the vibrational motion of a trapped ion, or even Bose-Einstein condensates with atoms in two different internal states. In other words, the associated number basis may be describing photons, phonons, or atoms.

We consider the coherent superposition of two distinguishable coherent states

$$|\psi\rangle \propto |\alpha\rangle + e^{i\phi_0}|-\alpha\rangle, \quad (1)$$

where $|\pm\alpha\rangle$ are coherent states and ϕ_0 is a constant phase. Since $|\pm\alpha\rangle$ must be distinguishable, we impose $|\alpha|\gg 1$ so that $\langle\alpha|-\alpha\rangle\simeq 0$.

On the other hand, the maximally entangled states are two-mode states of the form [6,10]

$$|\eta_n\rangle_a = \frac{1}{\sqrt{2}}(|n,0\rangle_a + e^{i\phi_0}|0,n\rangle_a), \quad (2)$$

where $|n_1,n_2\rangle_a$ are number states in modes a_1 and a_2. For convenience, the constant phase ϕ_0 is assumed to be the same as in Eq. (1). These states are maximally entangled in the sense that if we find any particle (photon, phonon, or atom) in one of the modes (a_1 or a_2 with equal probabilities), all the n particles must be found in the same mode. For $n\neq 0$, these states represent a superposition of two distinguishable states specially when n is large [10]. What we will prove is that the converse is also true: the quantum superposition (1) implies the maximal entanglement (2).

To this end, we assume that mode a_1 is in the state (1), while mode a_2 is in a coherent state of the same complex amplitude α

$$|\psi\rangle=\mathcal{N}(|\alpha\rangle_{a_1}+e^{i\phi_0}|-\alpha\rangle_{a_1})|\alpha\rangle_{a_2}, \tag{3}$$

where $\mathcal{N}$ is a normalization constant with $\mathcal{N}\simeq 1/\sqrt{2}$ when $|\alpha|\gg 1$.

Let us consider the pair of modes b_1, b_2 defined by

$$b_1=\frac{1}{\sqrt{2}}(a_1+a_2), \quad b_2=\frac{1}{\sqrt{2}}(a_2-a_1). \tag{4}$$

It may be easily checked that the following equalities hold:

$$\begin{aligned}|\psi\rangle&=\mathcal{N}(|\alpha,\alpha\rangle_a+e^{i\phi_0}|-\alpha,\alpha\rangle_a)\\&=\mathcal{N}(|\sqrt{2}\alpha,0\rangle_b+e^{i\phi_0}|0,\sqrt{2}\alpha\rangle_b)\\&=e^{-|\alpha|^2}\sum_{n=0}^{\infty}\frac{(\sqrt{2}\alpha)^n}{\sqrt{n!}}|\eta_n\rangle_b,\end{aligned} \tag{5}$$

where $|\alpha_1,\alpha_2\rangle_{a,b}$ are the product of coherent states in the corresponding modes. These equalities prove that the superpositions of macroscopically distinct coherent states may be also regarded as maximally entangled states. These two view points are connected simply by the definition of the proper mode basis.

We have followed a passive picture of the relation between modes a and b. It is also possible to adopt an active point of view in which we deal always with the same pair of modes, say a_1 and a_2. In such a case, the mode relation (4) is an input-output transformation performed by the unitary operator

$$R=e^{(\pi/4)(a_1^\dagger a_2-a_2^\dagger a_1)}, \tag{6}$$

such that $R|\alpha,\alpha\rangle_a=|\sqrt{2}\alpha,0\rangle_a$, and $R|-\alpha,\alpha\rangle_a=|0,\sqrt{2}\alpha\rangle_a$. The maximally entangled state is obtained by applying the transformation R to the input state (3). This mode-coupling R may be easily implemented for traveling waves by using beam splitters or phase plates. In the case of standing waves in the same cavity, R may be achieved by means of phase plates controlled via electrooptical effects or via dispersive interactions with atoms crossing the cavity. For field modes in different cavities, R may be performed via cavity couplings such as the one already carried out in Ref. [3]. Concerning the vibrational motion of trapped ions, the operator (6) may be realized as proposed in Ref. [11] while for Bose-Einstein condensates, it may be carried out as proposed in Ref. [12].

Incidentally, let us note that the states (5) are examples of the so-called entangled coherent states [13]. Moreover, states of the forms (2) and (3) satisfy the eigenvalue equation $b_1b_2|\eta_n\rangle_b=b_1b_2|\psi\rangle=0$, so they are closely related to the so-called pair coherent states [14].

Finally, we may note that transformations of the form (4) have been used previously to transform product states (single-mode squeezed vacuum states) into strongly correlated quantum states (two-mode squeezed vacuum), and vice versa [15].

III. PRECISION PHASE-SHIFT DETECTION USING SUPERPOSITIONS OF MACROSCOPICALLY DISTINCT COHERENT STATES

It is known that quantum physics imposes a limit to the precision of phase-shift measurements. By a variety of different arguments, it has been shown that the minimum detectable phase shift scales as the inverse of the total number of particles used in the measurement (Heisenberg limit) [8,9]. Among the different strategies proposed to reach the Heisenberg limit, we may single out the possibility of using maximally entangled states [6].

Following the equivalence demonstrated above, in this section, we show how the superpositions of macroscopically distinct coherent states may be used to approach the Heisenberg limit. We propose a phase-shift measurement that is directly based on experimental arrangements that have already demonstrated their usefulness in the generation of these quantum superpositions for the electromagnetic field in cavities [2,3]. Nevertheless, this same scheme may be easily translated to other contexts such as traveling electromagnetic fields or the vibrational motion of trapped ions, for example [4,11,16–19].

The system of interest is made of two cavity field modes a_1, a_2. In such a case, the initial state (3) may be generated via a dispersive interaction of mode a_1 with a two-level atom crossing the cavity [2,3]. The modes a_1, a_2 and b_1, b_2 are defined so that b_2 is the mode where the phase-shift ϕ occurs

$$\begin{aligned}|\psi_\phi\rangle&=e^{i\phi b_2^\dagger b_2}|\psi\rangle\\&=\mathcal{N}(|\sqrt{2}\alpha,0\rangle_b+e^{i\phi_0}|0,\sqrt{2}\alpha e^{i\phi}\rangle_b)\\&=\mathcal{N}(|\alpha,\alpha\rangle_a+e^{i\phi_0}|-\alpha e^{i\phi},\alpha e^{i\phi}\rangle_a),\end{aligned} \tag{7}$$

where ϕ is an unknown nonrandom classical parameter to be disclosed by a suitable measurement. To this end, the determination of the mean value of any number operator does not serve since all the mean values $\langle\psi_\phi|a_j^\dagger a_k|\psi_\phi\rangle$ for $j,k=1,2$ are independent of ϕ when $|\alpha|\gg 1$. A suitable phase-dependent observable is the parity of one of the modes, say a_1. This measurement may be carried out in practice by detecting the internal state of atoms after they cross the cavity interacting with the field mode a_1 via a purely dispersive coupling [2,3,20]. The parity measurement has only two possible outcomes, + and −. For the state $|\psi_\phi\rangle$ in Eq. (7) with $|\alpha|\gg 1$, the corresponding probabilities are

$$p_\pm\simeq\frac{1}{2}[1\pm e^{-2\bar{n}\sin^2(\phi/2)}\cos(\bar{n}\sin\phi+\phi_0)], \tag{8}$$

where $\bar{n}=2|\alpha|^2$ is the total mean number of particles.

This process may be regarded as an interferometric arrangement. The modes a_1, a_2 play the role of the input and output modes of the interferometer, while modes b_1, b_2 are the internal paths where the phase shift occur. We can appreciate that for small ϕ, the interference term depends on $\bar{n}\phi$ rather than the dependence on ϕ of classical interferometry. Therefore, we may say that in this arrangement, the phase-shift ϕ is amplified by a factor $\bar{n}$ [6,8,21]. (Note that this differs from the phase amplification concept studied in Ref. [22].) The interference is modulated by an exponential factor that determines the visibility defining an effective range of coherence via the condition $\bar{n}\phi^2 \ll 1$.

In order to study the sensitivity of this phase-shift measurement, we assume that ϕ is within the coherence interval $\phi \ll 1/\sqrt{\bar{n}}$ and we take $\phi_0 = -\pi/2$ for convenience. In these conditions, the probabilities (8) may be approximated as

$$p_{\pm} \simeq \frac{1}{2}[1 \pm \sin(\bar{n}\phi)]. \tag{9}$$

In order to obtain a meaningful estimation of ϕ, the measurement must be repeated several times. After N runs, the probability that the outcome $+$ is obtained m times is given by the binomial distribution

$$P_N(m) = \binom{N}{m} p_+^m p_-^{N-m}. \tag{10}$$

In the limit of large N, the quotient m/N may be regarded as effectively continuous and the binomial distribution tends to be Gaussian

$$P_N(\tilde{x}) \simeq \sqrt{\frac{N}{2\pi}} e^{-(N/2)(\tilde{x}-x)^2}, \tag{11}$$

where $\tilde{x} = 2m/N - 1$, $x = \sin(\bar{n}\phi)$, and we have assumed that $x \ll 1$. We can see that $\tilde{x}$ is a suitable estimator of the true but unknown x with uncertainty

$$\Delta\tilde{x} = \frac{1}{\sqrt{N}}. \tag{12}$$

Since $x \ll 1$, we may consider the linearization $x \simeq \bar{n}\phi$ and $\tilde{x} \simeq \bar{n}\tilde{\phi}$, where $\tilde{\phi}$ is the estimate of ϕ. Then, Eq. (12) leads to

$$\Delta\tilde{\phi} = \frac{1}{\bar{n}\sqrt{N}}. \tag{13}$$

The phase resolution scales as the inverse of the mean number of particles so this measuring strategy approaches the Heisenberg limit.

Strictly speaking, the phase resolution (13) only applies provided that we have a prior knowledge of ϕ with accuracy of the order of $1/\bar{n}$. This is because of the combination of periodicity and amplification in Eqs. (8) and (9): two phase shifts differing by $2\pi/\bar{n}$ within the coherence range are indistinguishable since they lead to the same probabilities $p_{\pm}$. In the preceding calculations, we have removed this ambiguity assuming that ϕ was close enough to zero. This situation parallels the free spectral range in spectroscopic measurements using Fabry-Perot interferometers or diffraction gratings [23].

This demonstrates the feasibility of the use of macroscopic quantum superpositions for precision measurements reaching the Heisenberg limit. As a matter of fact, probabilities of the form (8) have been already obtained experimentally in the process of generation and detection of these quantum superpositions. This may be checked in Eq. (2) of Ref. [3] and Eq. (5) of Ref. [4] (see also Eq. (16) of Ref. [18]). Closely related measuring schemes are the so-called de Broglie interferometers [24].

We have examined just a particular example of the general framework developed in the preceding section. We could follow the same procedure for any other proposal for the generation of macroscopic quantum superpositions. For example, the preparation of these states and the parity measurement for the motion of a trapped ion may be accomplished following steps equivalent to the ones analyzed above [11,17,18]. For traveling fields, we may single out a practical arrangement based on the optical Kerr nonlinearity and made of three Mach-Zehnder interferometers in series [13,19]. It is formally very close to the example analyzed above and may be regarded as an example of de Broglie interferometer.

Finally, we may quote another proposal for the use of macroscopic quantum superpositions for precision interferometric measurements [25] where these states are employed as states with squeezed quadrature fluctuations instead of using the quantum entanglement exploited here.

IV. CONCLUSIONS

In this paper, we have demonstrated the equivalence between the superpositions of macroscopically distinct coherent states and maximally entangled states. In particular, this implies that most of the methods proposed so far for the generation of macroscopic quantum superpositions may be suitably adapted for obtaining and applying maximal quantum entanglement.

In this context, we have proposed phase-shift measurements reaching the Heisenberg limit based on experimental processes that have been already used to generate superpositions of macroscopically distinct coherent states in electromagnetic fields in cavities. The possible areas of application of these results go beyond cavity fields and quantum optics and may be used for precision spectroscopy and metrology using trapped and cooled ions for example.

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