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String equations in Whitham hierarchies: τ-functions and Virasoro constraints

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A scheme for solving Whitham hierarchies satisfying a special class of string equations is presented. The τ-function of the corresponding solutions is obtained and the differential expressions of the underlying Virasoro constraints are characterized. Illustrative examples of exact solutions of Whitham hierarchies are derived and applications to conformal maps dynamics are indicated. © *2006 American Institute of Physics.* [DOI: 10.1063/1.2218982]

I. INTRODUCTION

Nonlinear integrable models of dispersionless type[1–4] arise in several branches of physics and applied mathematics. They have gained prominence after the discovery of their relevance in the formalism of quantum topological fields,[5,6] and of their role in the theory of deformations of conformal and quasiconformal maps on the complex plane.[7–9] Recently, new applications have been found[10–16] which include dynamics of conformal maps, growth processes of Laplacian type and large N limits of random matrix partition functions.

From the point of view of the theory of integrable systems, these models turn to be furnished by members of the so called *universal Whitham hierarchies* introduced by Krichever in Refs. 4 and 5. A particularly important example of these hierarchies is the dispersionless Toda (dToda) hierarchy.[10–13,17,18] The solutions of dispersionless integrable models underlying many of their applications satisfy special systems of constraints called *string equations*, which posses attractive mathematical properties and interesting physical meaning. Takasaki and Takebe[19–22] showed the relevance of string equations for studying the dispersionless Kadomtsev-Petviashvilii (KP) and Toda hierarchies and, in particular, for characterizing their associated symmetry groups. Nevertheless, although some schemes for solving string equations in the dispersionless KP and Toda hierarchies were provided in Refs. 23–25, general efficient methods of solution for string equations are still lacking.

In a recent work[26] a general formalism of Whitham hierarchies based on a factorization problem on a Lie group of canonical transformations has been proposed. It leads to a natural formulation of string equations in terms of dressing transformations. The present article is concerned with the analysis of these string equations and, in particular, their applications for characterizing exact solutions of Whitham hierarchies. Thus, we provide a solution scheme for a special class of string equations which determines not only the solutions of the algebraic orbits of the Whitham hierarchy,[5] but also the solutions arising in the above-mentioned applications of dispersionless integrable models.[24,25] We characterize the τ-function corresponding to these solutions and, by taking advantage of the string equations, we also derive the differential expressions of the underlying Virasoro constraints.

The elements of the phase space for a zero genus Whitham hierarchy are characterized by a finite set

$$(q_\alpha, z_\alpha^{-1}(p)), \quad \alpha = 0, \ldots, M,$$

of *punctures* q_α, where $q_0 := \infty$, of the complex p-plane and an associated set of local coordinates of the form

$$z_\alpha = \begin{cases} p + \sum_{n=1}^{\infty} \frac{d_{0n}}{p^n}, & \alpha = 0, \\ \frac{d_i}{p - q_i} + \sum_{n=0}^{\infty} d_{in}(p - q_i)^n, & \alpha = i = 1, \ldots, M. \end{cases} \tag{1}$$

The set of flows of the Whitham hierarchy can be formulated as the following infinite system of quasiclassical Lax equations:

$$\frac{\partial z_\alpha}{\partial t_{\mu n}} = \{\Omega_{\mu n}, z_\alpha\}, \tag{2}$$

where the Poisson bracket is defined as

$$\{F, G\} := \frac{\partial F}{\partial p}\frac{\partial G}{\partial x} - \frac{\partial F}{\partial x}\frac{\partial G}{\partial p}$$

and the Hamiltonian functions are

$$\Omega_{\mu n} := \begin{cases} (z_\mu^n)_{(\mu,+)}, & n \geqslant 1, \\ -\log_i(p - q_i), & n = 0, \quad \mu = i = 1, \ldots, M. \end{cases} \tag{3}$$

Here $(\cdot)_{(i,+)}$ and $(\cdot)_{(0,+)}$ stand for the projectors on the subspaces generated by $\{(p-q_i)^{-n}\}_{n=1}^{\infty}$ and $\{p^n\}_{n=0}^{\infty}$ in the corresponding spaces of Laurent series. Henceforth, it will be assumed that appropriate nonintersecting cuts connecting $p=\infty$ with the points q_i are made which allow us to define the logarithmic branches associated with Ω_{i0}. As several of these branches will appear simultaneously in certain equations, to avoid possible misunderstanding we introduce the notation convention $\log_i(p-q_i)$. For $M=0$ and $M=1$ these systems represent the dispersionless versions of the KP and Toda hierarchies, respectively.

In what follows Greek and Latin suffixes will be used to label indices of the sets $\{0, \ldots, M\}$ and $\{1, \ldots, M\}$, respectively. In our analysis we use an extended Lax formalism with Orlov functions

$$m_\alpha(z, \mathbf{t}) = \sum_{n=1}^{\infty} n t_{\alpha n} z_\alpha^{n-1} + \frac{t_{\alpha 0}}{z_\alpha} + \sum_{n \geqslant 2} \frac{v_{\alpha n}}{z_\alpha^n}, \quad t_{00} := -\sum_{i=1}^{M} t_{i0}, \tag{4}$$

such that

$$\{z_\alpha, m_\alpha\} = 1, \quad \forall\, \alpha,$$

and verifying the same Lax equations (2) as the variables z_α.

The basic notions about the Whitham hierarchy which are necessary for the subsequent discussion are introduced in Sec. II. String equations and symmetries are discussed in Sec. III, where the main results concerning the construction of solutions from meromorphic string equations and their Virasoro invariance are proved. Section IV presents a scheme for solving an special class of string equations, which is illustrated with several explicit examples. A formula for the correspond-

ing τ-function is given which generalizes the expression of the τ-function of analytic curves found in Ref. 10. Finally, we analyze the Virasoro symmetries associated to the string equations and obtain the corresponding Virasoro constraints in differential form.

II. THE WHITHAM HIERARCHY

In order to display the main features of the Whitham hierarchy it is convenient to use the following concise formulation in terms of the system of equations:

$$\mathrm{d}z_\alpha \wedge \mathrm{d}m_\alpha = \mathrm{d}\omega, \quad \forall\ \alpha, \tag{5}$$

where ω is the one-form defined by

$$\omega := \sum_{\mu,n} \Omega_{\mu n} \mathrm{d}t_{\mu n}. \tag{6}$$

To see how to get from system (5) to the Whitham hierarchy, note that by identifying the coefficients of $dp \wedge dt_{\mu n}$ and $dx \wedge dt_{\mu n}$ in (5) we obtain

$$\begin{aligned} \frac{\partial z_\alpha}{\partial p}\frac{\partial m_\alpha}{\partial t_{\mu n}} - \frac{\partial m_\alpha}{\partial p}\frac{\partial z_\alpha}{\partial t_{\mu n}} &= \frac{\partial \Omega_{\mu n}}{\partial p}, \\ \frac{\partial z_\alpha}{\partial x}\frac{\partial m_\alpha}{\partial t_{\mu n}} - \frac{\partial m_\alpha}{\partial x}\frac{\partial z_\alpha}{\partial t_{\mu n}} &= \frac{\partial \Omega_{\mu n}}{\partial x}. \end{aligned} \tag{7}$$

and, in particular, a $\Omega_{01}=p$, for $(\mu,n)=(0,1)$, system (7) implies

$$\{z_\alpha, m_\alpha\} = 1.$$

Thus, using this fact and solving (7) for $\partial z_\alpha / \partial t_{\mu n}$ and $\partial m_\alpha / \partial t_{\mu n}$, we deduce

$$\frac{\partial z_\alpha}{\partial t_{\mu n}} = \{\Omega_{\mu n}, z_\alpha\}, \quad \frac{\partial m_\alpha}{\partial t_{\mu n}} = \{\Omega_{\mu n}, m_\alpha\}.$$

It is now natural to introduce the S-functions of the Whitham hierarchy. Indeed as a consequence of (5) we find

$$\mathrm{d}\left(m_\alpha \mathrm{d}z_\alpha + \sum_{\mu,n} \Omega_{\mu n} \mathrm{d}t_{\mu n} \right) = 0, \quad \forall\ \alpha,$$

so that there exist functions $S_\alpha(z_\alpha, \boldsymbol{t})$ such that

$$\mathrm{d}S_\alpha = m_\alpha \mathrm{d}z_\alpha + \sum_{\mu,n} \Omega_{\mu n} \mathrm{d}t_{\mu n}, \quad \forall\ \alpha, \tag{8}$$

and from (4) we see that they admit expansions of the form

$$S_\alpha = \sum_{n\geqslant 1} z_\alpha^n t_{\alpha n} + \log z_\alpha t_{\alpha 0} - v_\alpha(\mathbf{t}) - \sum_{n\geqslant 1} \frac{v_{\alpha n+1}}{n}\frac{1}{z_\alpha^n}, \quad z_\alpha \to \infty. \tag{9}$$

It is important to notice that from (1)–(4) and (8) it follows that

$$\mathrm{d}S_0 = \sum_{n\geqslant 1} (n z_0^{n-1} t_{0n} \mathrm{d}z_0 + (z_0^n)_{(0,+)} \mathrm{d}t_{0n}) + \frac{t_{00}}{z_0}\mathrm{d}z_0 + \mathcal{O}\left(\frac{1}{z_0^2}\right)\mathrm{d}z_0, \quad z_0 \to \infty,$$

and consequently we may take

$$v_0(\mathbf{t}) \equiv 0.$$

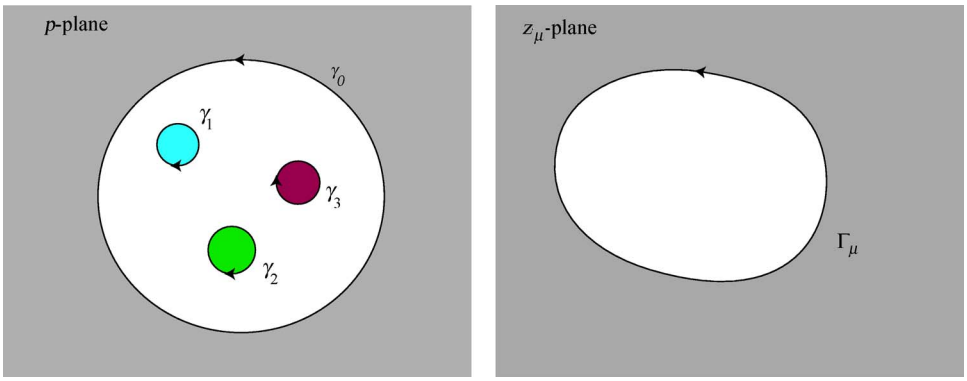

FIG. 1. (Color online) Right exteriors of γ_μ and Γ_μ.

To proceed further some analytic properties of the dynamical variables of the Whitham hierarchy are required. Thus we will henceforth suppose that there exist positively oriented closed curves Γ_μ in the complex planes of the variables z_μ such that each function $z_\mu(p)$ determines a conformal map of the right exterior of a circle $\gamma_\mu := z_\mu^{-1}(\Gamma_\mu)$ on the exterior of Γ_μ. We will assume that the circle γ_0 encircles all the $\gamma_i, (i=1,\ldots M)$ (see Fig. 1). Moreover, for each α the functions S_α and m_α will be assumed to be analytic in the exterior of Γ_α.

Under the previous conditions one can prove that

$$\partial_{\beta,m} v_{\alpha,n+1} = \partial_{\alpha,n} v_{\beta,m+1}, \quad \forall\, \alpha,\beta, \quad n,m \geqslant 0. \tag{10}$$

Here the functions v_{i1} are defined by

$$v_{i1} := v_i - \sum_{j<i} \log_{ji}(-1) t_{j0}, \tag{11}$$

and we are denoting

$$\log_{ji}(-1) := \log_j(q_i - q_j) - \log_i(q_j - q_i) = -\log_{ij}(-1).$$

In other words, it is ensured the existence of a *free-energy function* $F=F(\boldsymbol{t})$, the logarithm $F = \log \tau$ of the *dispersionless* τ-function, verifying

$$\mathrm{d}F = \sum_{(\alpha,n)\neq(0,0)} v_{\alpha n+1} \mathrm{d}t_{\alpha n}. \tag{12}$$

Let us first prove (10) for the case $(\alpha,n)=(i,0),(\beta,m)=(j,0)$. From the equations

$$\partial_{l0} S_k = -\log_l(p - q_l),$$

it follows that

$$\partial_{i0} v_j = \log_i(q_j - q_i), \quad \partial_{j0} v_i = \log_j(q_i - q_j),$$

so that the functions defined in (11) satisfy

$$\partial_{i0} v_{j1} = \partial_{j0} v_{i1}.$$

We indicate the strategy for proving the remaining cases of (10) by considering the choice $\alpha = i, \beta = j \geqslant 1, n,m \geqslant 1$ of (10). From (4) and (8) it follows that

$$v_{i,n+1} = \frac{1}{2\pi i}\oint_{\Gamma_i} z_i^n m_i \mathrm{d}z_i, \quad \partial_{j,m} S_i = (z_j^m)_{(j,+)},$$

so that

$$\partial_{j,m} v_{i,n+1} = \frac{1}{2\pi i}\oint_{\gamma_i} z_i^n \mathrm{d}(z_j^m)_{(j,+)} = \frac{1}{2\pi i}\oint_{\gamma_i} (z_i^n)_{(i,+)} \mathrm{d}(z_j^m)_{(j,+)} = \frac{1}{2\pi i}\oint_{\gamma_j} (z_j^m)_{(j,+)} \mathrm{d}(z_i^n)_{(i,+)} = \partial_{i,n} v_{j,m+1}, \tag{13}$$

where we have taken into account that $(z_i^n)_{(i,+)}\partial_p(z_j^m)_{(j,+)}$ is a rational function of p which has finite poles at q_i and q_j only and a zero residue at ∞.

III. STRING EQUATIONS AND SYMMETRIES

As it was shown in Ref. 26, the analysis of the factorization problem for the Whitham hierarchy shows that this hierarchy admits a natural formulation in terms of systems of string equations of the form

$$\begin{aligned} P_i(z_i,m_i) &= P_0(z_0,m_0), \\ Q_i(z_i,m_i) &= Q_0(z_0,m_0), \end{aligned} \quad i = 1,2,\ldots,M, \tag{14}$$

where $\{P_\alpha, Q_\alpha\}_{\alpha=0}^M$ are pairs of canonically conjugate variables

$$\{P_\alpha(p,x), Q_\alpha(p,x)\} = 1, \quad \forall\ \alpha. \tag{15}$$

In what follows we consider the problem of finding systems of form (14) which are appropriate to generate exact solutions of the Whitham hierarchy.

Given a solution $(z_\alpha(p,\boldsymbol{t}), m_\alpha(p,\boldsymbol{t}))$ of system (14), if we denote

$$\mathcal{P}_\alpha(p,\mathbf{t}) := P_\alpha(z_\alpha(p,\mathbf{t}), m_\alpha(p,\mathbf{t})), \quad \mathcal{Q}_\alpha(p,\mathbf{t}) := Q_\alpha(z_\alpha(p,\mathbf{t}), m_\alpha(p,\mathbf{t})),$$

then (14) and (15) imply

$$\mathrm{d}\mathcal{P}_\alpha \wedge \mathrm{d}\mathcal{Q}_\alpha = \mathrm{d}\mathcal{P}_\beta \wedge \mathrm{d}\mathcal{Q}_\beta, \quad \forall\ \alpha,\beta \tag{16}$$

and

$$\mathrm{d}\mathcal{P}_\alpha \wedge \mathrm{d}\mathcal{Q}_\alpha = \mathrm{d}z_\alpha \wedge \mathrm{d}m_\alpha, \quad \forall\ \alpha, \tag{17}$$

respectively. Hence solutions of the system of string equations verify

$$\mathrm{d}\mathcal{P}_\alpha \wedge \mathrm{d}\mathcal{Q}_\alpha = \mathrm{d}z_\beta \wedge \mathrm{d}m_\beta, \quad \forall\ \alpha,\beta. \tag{18}$$

The next result provides a convenient framework for our subsequent discussion of solutions of (14).

Theorem 1: *Let* $(z_\alpha(p,\boldsymbol{t}), m_\alpha(p,\boldsymbol{t}))$ *be a solution of* (14) *which admits expansions of forms* (1)–(4) *and such that the coefficients of the two-forms* (18) *are meromorphic functions of the complex variable p with finite poles at* $\{q_1,\ldots,q_M\}$ *only. Then* $(z_\alpha(p,\boldsymbol{t}), m_\alpha(p,\boldsymbol{t}))$ *is a solution of the Whitham hierarchy*.

Proof: In view of the hypothesis of the theorem the coefficients of the two-forms (18) with respect to the basis

$$\{\mathrm{d}p \wedge \mathrm{d}t_{\alpha n}, \quad \mathrm{d}t_{\alpha n} \wedge \mathrm{d}t_{\beta m}\}$$

are determined by their principal parts at $q_\mu, (\mu = 0,\ldots,M)$, so that by taking (18) into account we may write

$$\mathrm{d}z_\alpha \wedge \mathrm{d}m_\alpha = \sum_{\mu=0}^{M} (\mathrm{d}z_\mu \wedge \mathrm{d}m_\mu)_{(\mu,+)}, \quad \forall\, \alpha.$$

Moreover the terms in these decompositions can be found by using expansions (4) of the functions m_μ as follows:

$$\begin{aligned}\mathrm{d}z_\mu \wedge \mathrm{d}m_\mu &= \mathrm{d}z_\mu \wedge \left(\sum_{n=1}^{\infty} n z_\mu^{n-1} \mathrm{d}t_{\mu n} + \frac{\mathrm{d}t_{\mu 0}}{z_\mu} + \sum_{n\geqslant 2} \frac{\mathrm{d}v_{\mu n}}{z_\mu^n} \right) \\ &= \mathrm{d}\left(\sum_{n=1}^{\infty} z_\mu^n \mathrm{d}t_{\mu n} + \log z_\mu \mathrm{d}t_{\mu 0} - \sum_{n\geqslant 2} \frac{1}{n-1} \frac{\mathrm{d}v_{\mu n}}{z_\mu^{n-1}} \right),\end{aligned}$$

so that

$$(\mathrm{d}z_\mu \wedge \mathrm{d}m_\mu)_{(\mu,+)} = \mathrm{d}\left(\sum_{n=1}^{\infty} (z_\mu^n)_{(\mu,+)} \mathrm{d}t_{\mu n} - (1-\delta_{\mu 0}) \log(p - q_\mu) \mathrm{d}t_{\mu 0} \right) = \mathrm{d}\left(\sum_n \Omega_{\mu n} \mathrm{d}t_{\mu n} \right).$$

Thus we find

$$\mathrm{d}z_\alpha \wedge \mathrm{d}m_\alpha = \mathrm{d}\omega = \mathrm{d}\left(\sum_{\mu,n} \Omega_{\mu n} \mathrm{d}t_{\mu n} \right), \quad \forall\, \alpha,$$

and, consequently, this proves that the functions $(z_\alpha(p,\boldsymbol{t}), m_\alpha(p,\boldsymbol{t}))$ determine a solution of the Whitham hierarchy. ■

Following the dressing scheme of Refs. 19–22 it can be shown[26] that each solution of the Whitham hierarchy is determined by an associated system of string equations.

As it was shown in Ref. 26 a complete formulation of the symmetry group of the Whitham hierarchy is obtained by considering deformations of the associated factorization problem. On the other hand, a natural representation of this group is provided by the following symmetries of string equations implemented by Hamiltonian vector fields.

Theorem 2. *Given a vector function*

$$\mathbb{F} := (F_0(z_0,m_0), \ldots, F_M(z_M,m_M)), \tag{19}$$

the infinitesimal deformation

$$\begin{aligned}&\delta_{\mathbb{F}} P_\alpha := \{F_\alpha, P_\alpha\}, \quad \delta_{\mathbb{F}} Q_\alpha := \{F_\alpha, Q_\alpha\}, \\ &\delta_{\mathbb{F}} z_\alpha := \{z_\alpha, (F_\alpha)_-\}, \quad \delta_{\mathbb{F}} m_\alpha := \{m_\alpha, (F_\alpha)_-\},\end{aligned} \tag{20}$$

where

$$(F_\alpha)_- := F_\alpha - \sum_\beta (F_\beta)_{(\beta,+)},$$

determines a symmetry of the system of string equations (14).

Proof: We have to prove that given a solution (z_α, m_α) of (14), then at first order in ϵ

$$\begin{aligned}&(P_i + \epsilon\delta_{\mathbb{F}} P_i)(z_i + \epsilon\delta_{\mathbb{F}} z_i, m_i + \epsilon\delta_{\mathbb{F}} m_i) = (P_0 + \epsilon\delta_{\mathbb{F}} P_0)(z_0 + \epsilon\delta_{\mathbb{F}} z_0, m_0 + \epsilon\delta_{\mathbb{F}} m_0), \\ &(Q_i + \epsilon\delta_{\mathbb{F}} Q_i)(z_i + \epsilon\delta_{\mathbb{F}} z_i, m_i + \epsilon\delta_{\mathbb{F}} m_i) = (Q_0 + \delta_{\mathbb{F}} Q_0)(z_0 + \epsilon\delta_{\mathbb{F}} z_0, m_0 + \epsilon\delta_{\mathbb{F}} m_0),\end{aligned} \tag{21}$$

for all $i = 1, \ldots, M$. Let us consider the first group of equations of (21), they can be rewritten as

$$\frac{\partial P_i}{\partial z_i}\delta_{\mathbb{F}} z_i + \frac{\partial P_i}{\partial m_i}\delta_{\mathbb{F}} m_i + \{F_i, P_i\} = \frac{\partial P_0}{\partial z_0}\delta_{\mathbb{F}} z_0 + \frac{\partial P_0}{\partial m_0}\delta_{\mathbb{F}} m_0 + \{F_0, P_0\},$$

or, equivalently, by taking (20) into account, as

$$\{F_i - (F_i)_-, P_i\} = \{F_0 - (F_0)_-, P_0\}, \quad \forall\, i. \tag{22}$$

By hypothesis $P_i(z_i, m_i) = P_0(z_0, m_0)$. On the other hand

$$F_i - (F_i)_- = F_0 - (F_0)_- = \sum_\beta (F_\beta)_{(\beta,+)},$$

so that (22) is satisfied. The proof for the second group of equations of (21) is identical. □

We note that the condition for a solution (z_α, m_α) of the string equations (14) to be invariant under a symmetry (20) is

$$(F_\alpha(z_\alpha, m_\alpha))_- = 0, \quad \forall\, \alpha, \tag{23}$$

or equivalently

$$F_\alpha = \sum_\mu (F_\mu)_{(\mu,+)}, \quad \forall\, \alpha. \tag{24}$$

In other words, the functions $F_\alpha(z_\alpha, m_\alpha)$ must reduce to a unique meromorphic function of p with finite poles at the punctures q_i only. As a consequence it follows that, under the hypothesis of Theorem 1, solutions of the Whitham hierarchy satisfying a system of string equations (14) are invariant under the symmetries generated by

$$\mathbb{P} = (P_0(z_0, m_0), \ldots, P_M(z_M, m_M)), \quad \mathbb{Q} = (Q_0(z_0, m_0), \ldots, Q_M(z_M, m_M)),$$

and, more generally, they are invariant under the symmetries generated by

$$\mathbb{V}_{rs} = (P_0^{r+1} Q_0^{s+1}, \ldots, P_M^{r+1} Q_M^{s+1}), \quad r \geqslant -1, \quad s \geqslant 0, \tag{25}$$

which determine a Poisson Lie algebra $\mathcal{W}$ of symmetries

$$\{\mathbb{V}_{rs}, \mathbb{V}_{r's'}\} = ((r+1)(s'+1) - (r'+1)(s+1))V_{r+r's+s'}.$$

In particular the functions $\mathbb{V}_r := \mathbb{V}_{r0}$ and $\widetilde{\mathbb{V}}_s := -\mathbb{V}_{0s}$ generate two Virasoro algebras.

$$\{\mathbb{V}_r, \mathbb{V}_{r'}\} = (r - r')\mathbb{V}_{r+r'}, \quad \{\widetilde{\mathbb{V}}_s, \widetilde{\mathbb{V}}_{s'}\} = (s - s')\widetilde{\mathbb{V}}_{s+s'}.$$

IV. A SOLVABLE CLASS OF STRING EQUATIONS

In Ref. 26 a class of string equations was introduced which manifests special properties with respect to the group of dressing transformations. We next provide a scheme of solution for this class.

Let us consider systems of string equations associated to splittings

$$\{1, \ldots, M\} = I \cup J, \quad I \cap J = \emptyset$$

of the form

$$i \in I \begin{cases} z_i^{n_i} = z_0^{n_0} \\ \dfrac{1}{n_i} \dfrac{m_i}{z_i^{n_i-1}} = \dfrac{1}{n_0} \dfrac{m_0}{z_0^{n_0-1}}, \end{cases} \qquad j \in J \begin{cases} -\dfrac{n_0}{n_j} \dfrac{m_j}{z_j^{n_j-1}} = z_0^{n_0} \\ \dfrac{1}{n_0} z_j^{n_j} = \dfrac{1}{n_0} \dfrac{m_0}{z_0^{n_0-1}}, \end{cases} \tag{26}$$

where n_α are arbitrary positive integers. For $J=\emptyset$ these systems furnish the solutions describing the algebraic orbits of the Whitham hierarchy,[5] while the case $I=\emptyset$ includes the systems of string equations considered by Takasaki[18] and Wiegmann-Zabrodin[10–14] in their applications of the dToda hierarchy. The discussion of our scheme for solving (26) requires the consideration of the two cases $J=\emptyset$ and $J\neq\emptyset$ separately. In what follows we consider solutions with only a finite number of times $t_{\mu n}$ different from zero.

A. The case $J=\emptyset$

The corresponding system is given by

$$\begin{aligned} &z_i^{n_i} = z_0^{n_0}, \\ &\frac{1}{n_i} \frac{m_i}{z_i^{n_i-1}} = \frac{1}{n_0} \frac{m_0}{z_0^{n_0-1}}, \qquad i = 1, \ldots, M. \end{aligned} \tag{27}$$

The first group of equations (27) is satisfied by setting

$$z_i^{n_i} = z_0^{n_0} = E(p) := p^{n_0} + u_{n_0-2} p^{n_0-2} + \cdots + u_0 + \sum_{i=1}^{M} \sum_{s=1}^{n_i} \frac{v_{is}}{(p-q_i)^s}, \tag{28}$$

and, obviously, appropriate branches of z_μ can be defined which are compatible with the required asymptotic expansions (1). On the other hand, notice that the remaining string equations in (27) can be rewritten as

$$m_i = m_0 \frac{\mathrm{d}z_0}{\mathrm{d}z_i}, \tag{29}$$

so that they are verified by taking

$$m_\alpha = \frac{\partial S}{\partial z_\alpha}, \quad \forall\ \alpha, \tag{30}$$

for a given function $S(p,\mathbf{t})$, which means that all the S_α are equal to S. Moreover, it is straightforward to prove that the expansions (4) are satisfied if we set

$$S = \sum_{n=1}^{N_0} t_{0n} (z_0^n)_{(0,+)} + \sum_{j=1}^{M} \left(\sum_{n=1}^{N_j} t_{jn} (z_j^n)_{(j,+)} - t_{j0} \ln_j (p-q_j) \right). \tag{31}$$

In order to satisfy the hypothesis of Theorem 1, the functions $z_0^{n_0}$ and $m_0/z_0^{n_0-1}$ must be rational functions of p with possible finite poles at the points q_i only. In view of (28) this condition is verified by z_0^n. On the other hand, (27) and (30) imply that

$$\frac{1}{n_0} \frac{m_0}{z_0^{n_0-1}} = \frac{\partial_p S}{\partial_p E}.$$

Therefore, the requirements of Theorem 1 are satisfied provided that

$$\partial_p S(p_r) = 0, \tag{32}$$

where p_r are the zeros of

$$\partial_p E(p_r) = 0.$$

We observe that the number of zeros p_r is

$$M + n_0 - 1 + \sum_{i=1}^{M} n_i,$$

which equals the number of unknowns $\{q_i, u_0, \ldots, u_{n_0-2}, v_{is}\}$. Equation (32) coincides with those formulated by Krichever in Ref. 5 for determining the algebraic orbits of the Whitham hierarchy.

Example: $M=2$, $n_0=n_1=n_2=1$, $N_0=2$, $N_1=N_2=1$.

In this case (28) reads

$$z_\alpha = E(p) = p + \frac{v_1}{p-q_1} + \frac{v_2}{p-q_2}, \quad 0 \leqslant \alpha \leqslant 2,$$

and (32) leads to

$$-t_{10} - t_{20} + 2t_{02}v_1 + 2t_{02}v_2 = 0,$$

$$q_1 t_{10} + 2q_2 t_{10} + 2q_1 t_{20} + q_2 t_{20} + t_{01}v_1 - 4q_2 t_{02}v_1 - t_{11}v_1 + t_{01}v_2 - 4q_1 t_{02}v_2 - t_{21}v_2 = 0,$$

$$-2q_1 q_2 t_{10} - {q_2}^2 t_{10} - {q_1}^2 t_{20} - 2q_1 q_2 t_{20} - 2q_2 t_{01}v_1 + 2{q_2}^2 t_{02}v_1 + 2q_2 t_{11}v_1 - 2q_1 t_{01}v_2 + 2{q_1}^2 t_{02}v_2 + 2q_1 t_{21}v_2 = 0,$$

$$q_1 {q_2}^2 t_{10} + {q_1}^2 q_2 t_{20} + {q_2}^2 t_{01}v_1 - {q_2}^2 t_{11}v_1 + {q_1}^2 t_{01}v_2 - {q_1}^2 t_{21}v_2 = 0.$$

By solving this system we obtain

$$z_\alpha = E(p) = p + \frac{2t_{10} + t_{20}}{4t_{02}\left(p + \dfrac{2t_{01}t_{10} - 2t_{10}t_{11} + 2t_{01}t_{20} - t_{11}t_{20} - t_{20}t_{21}}{4t_{02}(t_{10}+t_{20})}\right)} + \frac{t_{20}}{4t_{02}\left(p - \dfrac{-2t_{01}(t_{10}+t_{20}) + 2t_{20}t_{21} + t_{10}(t_{11}+t_{21})}{4t_{02}(t_{10}+t_{20})}\right)}, \quad 0 \leqslant \alpha \leqslant 2.$$

B. The case $J \neq \emptyset$

Now we consider system (26) for the generic case $J \neq \emptyset$. We look for functions m_α of the form

$$m_\alpha(z,t) = \sum_{n=1}^{N_\alpha} n t_{\alpha n} z_\alpha^{n-1} + \frac{t_{\alpha 0}}{z_\alpha} + \sum_{n \geqslant 2} \frac{v_{\alpha n}}{z_\alpha^n}, \quad t_{00} = -\sum_{i=1}^{M} t_{i0}, \tag{33}$$

for arbitrary positive integers N_α. In order to verify the hypothesis of Theorem 1 and expansions (1), we set

$$z_0^{n_0} = z_i^{n_i} = E_1(p) := p^{n_0} + u_{n_0-2}p^{n_0-2} + \cdots + u_0 + \sum_{l \in I}\sum_{n=1}^{n_l} \frac{a_{ln}}{(p-q_l)^n} + \sum_{k \in J}\sum_{n=1}^{n_{0k}} \frac{b_{kn}}{(p-q_k)^n}, \quad \forall\, i \in I, \tag{34}$$

$$z_j^{n_j} = E_2(p) := \sum_{n=0}^{n_{00}} c_n p^n + \sum_{l\in I}\sum_{n=1}^{n_{0l}} \frac{\widetilde{a}_{ln}}{(p-q_l)^n} + \sum_{k\in J}\sum_{n=1}^{n_k} \frac{\widetilde{b}_{kn}}{(p-q_k)^n}, \quad \forall\, j \in J, \tag{35}$$

where n_{00}, n_{0l}, n_{0k} $(l \in I, k \in J)$, the poles q_i and the coefficients of E_1 and E_2 are to be determined.

By introducing the functions

$$\mathcal{M}_\alpha := m_\alpha z_\alpha, \tag{36}$$

and taking (34) and (35) into account it follows that the system of string equations (26) reduces to

$$\mathcal{M}_0 = \frac{n_0}{n_i}\mathcal{M}_i = -\frac{n_0}{n_j}\mathcal{M}_j = E_1(p)E_2(p), \quad \forall\, i \in I, \quad j \in J. \tag{37}$$

On the other hand, due to their rational character, the functions $\mathcal{M}_\alpha$ can be written in terms of their principal parts at the poles q_β

$$\mathcal{M}_\alpha = \sum_{\beta=0}^{M} (\mathcal{M}_\alpha)_{(\beta,+)},$$

and by taking (4) into account we get

$$(\mathcal{M}_0)_{(0,+)} = \sum_{n=1}^{N_0} n t_{0n} (z_0^n)_{(0,+)} + t_{00}, \quad t_{00} = -\sum_{i=1}^{M} t_{i0},$$

$$(\mathcal{M}_i)_{(i,+)} = \sum_{n=1}^{N_i} n t_{in} (z_i^n)_{(i,+)}.$$

Therefore (37) is satisfied by

$$\begin{gathered}\mathcal{M}_0 = \sum_{n=1}^{N_0} n t_{0n} (z_0^n)_{(0,+)} + t_{00} + \sum_{i\in I} \frac{n_0}{n_i} \sum_{n=1}^{N_i} n t_{in} (z_i^n)_{(i,+)} - \sum_{j\in J} \frac{n_0}{n_j} \sum_{n=1}^{N_j} n t_{jn} (z_j^n)_{(j,+)}, \quad t_{00} = -\sum_{j=1}^{M} t_{j0} \\ \mathcal{M}_i = \frac{n_i}{n}\mathcal{M}_0, \quad \mathcal{M}_j = -\frac{n_j}{n}\mathcal{M}_0, \quad \forall\, i \in I, \quad j \in J,\end{gathered} \tag{38}$$

provided $\mathcal{M}_0$ verifies the equation

$$\mathcal{M}_0 = E_1(p)E_2(p). \tag{39}$$

At this point notice that from (34), (35), and (37) it follows that (39) is the only equation to be satisfied in order to solve the system of string equations (26). Both sides of (39) are rational functions of p with finite poles at $\{q_1, \ldots, q_M\}$ only, so that (39) holds if and only if the principal parts of both members at $\{q_0, q_1, \ldots, q_M\}$ coincide. Now we have that:

(i) At $q_0 = \infty$, the function $\mathcal{M}_0$ has a pole of order N_0, while $E_1(p)E_2(p)$ has a pole of order $n_{00}+n_0$, consequently (39) requires that $n_{00} = N_0 - n_0$, so that identifying the principal parts at q_0 represents N_0+1 equations.

(ii) At q_i, $(i \in I)$, the function $\mathcal{M}_0$ has a pole of order N_i and $E_1(p)E_2(p)$ has a pole of order n_i+n_{0i}. Hence $n_{0i} = N_i - n_i$ and identifying the corresponding principal parts leads to N_i equations.

(iii) At q_j, $(j \in J)$, the function $\mathcal{M}_0$ has a pole of order N_j and $E_1(p)E_2(p)$ has a pole of order $n_{0j}+n_j$. Hence $n_{0j} = N_j - n_j$ and identifying the corresponding principal parts leads to N_j equations.

Thus, Eq. (39) leads to $N_0+\Sigma_{i=1}^{M}N_i+1$ equations. On the other hand we have $N_0+\Sigma_{i=1}^{M}N_i+M$ unknown coefficients given by

$$q_i, \quad i=1,2,\ldots,M,$$

$$a_{i1},\ldots,a_{in_i}, \quad \tilde{a}_{i1},\ldots,\tilde{a}_{iN_i-n_i}, \quad i\in I,$$

$$b_{j1},\ldots,b_{jN_j-n_j}, \quad \tilde{b}_{j1},\ldots,\tilde{b}_{jn_j}, \quad j\in J, \tag{40}$$

$$u_0,\ldots,u_{n_0-2},$$

$$c_0,\ldots,c_{N_0-n_0}.$$

The additional $M-1$ equations required for determining these coefficients arise by imposing the asymptotic behavior (1)–(4) to (z_α,m_α). Note that (34) and (35) imply that the functions z_α have the asymptotic form (1). In what concerns the functions m_α, from the expression (38) for $\mathcal{M}_0$ it follows that

$$\mathcal{M}_0=\sum_{n=1}^{N_0}nt_{0n}z_0^n+t_{00}+\mathcal{O}\left(\frac{1}{z_0}\right), \quad z_0\to\infty,$$

so that m_0 satisfies an expansion of form (4). But in order for m_i $(i=1,2,\ldots,M)$ to satisfy (4) we must impose that

$$\mathrm{Res}(m_i,z_i=\infty)=t_{i0}, \quad i=1,2,\ldots,M. \tag{41}$$

However, let us see that as a consequence of the string equations (26) it follows that

$$\sum_{\alpha=0}^{M}\mathrm{Res}(m_\alpha,z_\alpha=\infty)=0, \tag{42}$$

and, consequently, only $M-1$ of Eq. (41) need to be imposed. Indeed, we have

$$2\pi i\sum_{\alpha=0}^{M}\mathrm{Res}(m_\alpha,z_\alpha=\infty)=\sum_{\alpha=0}^{M}\oint_{\Gamma_\alpha}m_\alpha \mathrm{d}z_\alpha=\sum_{\alpha=0}^{M}\oint_{\gamma_\alpha}m_\alpha\partial_p z_\alpha \mathrm{d}p.$$

On the other hand (34), (35), and (37) imply

$$m_i\partial_p z_i=m_0\partial_p z_0, \quad i\in I,$$

$$m_j\partial_p z_j=m_0\partial_p z_0-\frac{1}{n_0}\partial_p(E_1(p)E_2(p)), \quad j\in J,$$

so that

$$2\pi i\sum_{\alpha=0}^{M}\mathrm{Res}(m_\alpha,z_\alpha=\infty)=\oint_{\gamma}m_0\partial_p z_0\mathrm{d}p=0, \quad \gamma:=\sum_{\alpha=0}^{M}\gamma_\alpha,$$

where we have taken into account that

$$m_0\partial_p z_0 = \frac{1}{n_0}E_2(p)\partial_p E_1(p),$$

is a rational function of p with finite poles at q_i only, and the fact that

$$\gamma \sim 0 \quad \text{in} \quad \mathbb{C}\setminus\{q_1,\ldots,q_M\}.$$

In this way we have a system of

$$N_0 + \sum_{i=1}^{M} N_i + M,$$

equations to determine the same number of unknown coefficients. Therefore, according to Theorem 1, this method furnishes solutions of the Whitham hierarchy.

C. Examples

(1) $M=1$, $I=\emptyset$, $n_0=2$, $n_1=1$, $N_0=N_1=3$. Note that in this case all the equations come from (39). We set

$$z_0^2 = p^2 + u_0 + \frac{a_1}{p-q} + \frac{a_2}{(p-q)^2}, \quad z_1 = \frac{b_1}{p-q} + c_0 + c_1 p.$$

From (39) one obtains the system

$$p^3: \quad 3t_{03} = c_1,$$

$$p^2: \quad 2t_{02} = c_0,$$

$$p^1: \quad t_{01} + \frac{9t_{03}u_0}{2} = b_1 + c_1 u_0,$$

$$p^0: \quad \frac{9a_1 t_{03}}{2} - t_{10} + 2t_{02}u_0 = a_1 c_1 + b_1 q + c_0 u_0,$$

$$(p-q)^{-3}: \quad -6{b_1}^2 t_{13} = a_2,$$

$$(p-q)^{-2}: \quad -2{b_1}^2(2t_{12} + 9(c_0 + c_1 q)t_{13}) = a_1 b_1 + a_2(c_0 + c_1 q),$$

$$(p-q)^{-1}: \quad -2b_1(t_{11} + 4(c_0 + c_1 q)t_{12} + 9(c_0^2 + 2c_0 c_1 q + c_1 b_1 + c_1^2 q^2)t_{13}) = a_2 c_1 + a_1(c_0 + c_1 q) + b_1(q^2 + u_0),$$

and by solving these equations we find

$$z_0^2 = p^2 - \frac{2(qt_{01} + t_{10} + 6t_{01}t_{03}t_{12} + 36t_{01}t_{02}t_{03}t_{13} + 54qt_{01}{t_{03}}^2 t_{13})}{3t_{03}(q + 6t_{03}t_{12} + 36t_{02}t_{03}t_{13} + 54q{t_{03}}^2 t_{13})} + \frac{4t_{10}(t_{12} + 6t_{02}t_{13} + 9qt_{03}t_{13})}{(p-q)(q + 54q{t_{03}}^2 t_{13} + 6t_{03}(t_{12} + 6t_{02}t_{13}))} - \frac{6{t_{10}}^2 t_{13}}{(p-q)^2(q + 54q{t_{03}}^2 t_{13} + 6t_{03}(t_{12} + 6t_{02}t_{13}))^2},$$

$$z_1 = -\frac{t_{10}}{(p-q)(q+6t_{03}t_{12}+36t_{02}t_{03}t_{13}+54q{t_{03}}^2t_{13})} + 2t_{02} + 3pt_{03},$$

where q is determined by the implicit equation

$$\begin{aligned}
&-2qt_{01}+3q^3t_{03}-2t_{10}+6qt_{03}t_{11}-12t_{01}t_{03}t_{12}+24qt_{02}t_{03}t_{12}+54q^2{t_{03}}^2t_{12}+36{t_{03}}^2t_{11}t_{12}\\
&\quad+144t_{02}{t_{03}}^2{t_{12}}^2+216q{t_{03}}^3{t_{12}}^2-72t_{01}t_{02}t_{03}t_{13}+72q{t_{02}}^2t_{03}t_{13}-108qt_{01}{t_{03}}^2t_{13}+324q^2t_{02}{t_{03}}^2t_{13}\\
&\quad+324q^3{t_{03}}^3t_{13}-108{t_{03}}^2t_{10}t_{13}+216t_{02}{t_{03}}^2t_{11}t_{13}+324q{t_{03}}^3t_{11}t_{13}+1296{t_{02}}^2{t_{03}}^2t_{12}t_{13}\\
&\quad+3888q{t_{02}}{t_{03}}^3t_{12}t_{13}+2916q^2{t_{03}}^4t_{12}t_{13}+2592{t_{02}}^3{t_{03}}^2{t_{13}}^2+11664q{t_{02}}^2{t_{03}}^3{t_{13}}^2\\
&\quad+17496q^2t_{02}{t_{03}}^4{t_{13}}^2+8748q^3{t_{03}}^5{t_{13}}^2=0.
\end{aligned}$$

(2) $M=2$, $I=\emptyset$, $n_0=n_1=n_2=1$, $N_0=N_1=2$, $N_2=1$. In this case there are three punctures $\{q_0=\infty, q_1, q_2\}$ and we have to impose Eq. (41) for $i=1$. We take

$$z_0 = p + \frac{a_1}{p-q_1}, \quad z_1 = z_2 = \frac{b_1}{p-q_1} + \frac{b_2}{p-q_2} + c_0 + c_1p.$$

Then, by identifying powers of p, $(p-q_1)^{-1}$ and $(p-q_2)^{-1}$ in (39) the following system of equations arises

$$p^2: \quad 2t_{02} = c_1,$$

$$p^1: \quad t_{01} = c_0,$$

$$p^0: \quad 4a_1t_{02} - t_{10} - t_{20} = b_1 + b_2 + a_1c_1,$$

$$(p-q_1)^{-2}: \quad -2b_1t_{12} = a_1,$$

$$(p-q_1)^{-1}: \quad -b_1t_{11} - 4b_1\left(c_0 + c_1q_1 + \frac{b_2}{q_1-q_2}\right)t_{12} = b_1q_1 + a_1\left(c_0 + c_1q_1 + \frac{b_2}{q_1-q_2}\right),$$

$$(p-q_2)^{-1}: \quad -t_{21} = q_2 - \frac{a_1}{q_1-q_2}.$$

Moreover, by taking (38) into account, from (41) we get

$$\begin{aligned}
&-q_1t_{01} - 2(2a_1+{q_1}^2)t_{02} - \left(c_0 + c_1q_1 + \frac{b_2}{q_1-q_2}\right)t_{11}\\
&\quad -2\left(2b_1\left(c_1 - \frac{b_2}{(q_1-q_2)^2}\right) + \left(c_0 + c_1q_1 + \frac{b_2}{q_1-q_2}\right)^2\right)t_{12} + t_{20} + \frac{b_2t_{21}}{q_1-q_2} = 0.
\end{aligned}$$

These equations lead to

$$z_0 = p - \frac{1}{2(1+4t_{02}t_{12})(p-q_1)}(r^2(1+4t_{02}t_{12}) - 2t_{12}(t_{10}+t_{20}) + r(t_{11}+2t_{01}t_{12}-t_{21}-4t_{02}t_{12}t_{21})),$$

$$z_1 = \frac{1}{4t_{12}(1+4t_{02}t_{12})(p-q_1)}(r^2(1+4t_{02}t_{12}) - 2t_{12}(t_{10}+t_{20}) + r(t_{11}+2t_{01}t_{12} - t_{21} - 4t_{02}t_{12}t_{21}))$$
$$+ \frac{1}{4t_{12}(p-q_2)}(-r^2(1+4t_{02}t_{12}) - 2t_{12}(t_{10}+t_{20}) + r(-t_{11} - 2t_{01}t_{12} + t_{21} + 4t_{02}t_{12}t_{21}))$$
$$+ t_{01} + 2pt_{02},$$

where

$$q_1 = \frac{1}{2r(1+4t_{02}t_{12})}(r^2(1+4t_{02}t_{12}) + 2t_{12}(t_{10}+t_{20}) - r(t_{11}+2t_{01}t_{12}+t_{21}+4t_{02}t_{12}t_{21})),$$

$$q_2 = -\frac{1}{2r(1+4t_{02}t_{12})}(r^2(1+4t_{02}t_{12}) - 2t_{12}(t_{10}+t_{20}) + r(t_{11}+2t_{01}t_{12}+t_{21}+4t_{02}t_{12}t_{21})),$$

and r is determined by

$$3r^4(1+4t_{02}t_{12})^2 - 4{t_{12}}^2(t_{10}+t_{20})^2 + 4r^3(1+4t_{02}t_{12})(t_{11}+2t_{01}t_{12} - (1+4t_{02}t_{12})t_{21}) + r^2({t_{11}}^2$$
$$+ 4{t_{01}}^2{t_{12}}^2 + 4t_{10}t_{12}(1+4t_{02}t_{12}) - 4t_{12}t_{20} - 16t_{02}{t_{12}}^2t_{20} - 4t_{01}t_{12}t_{21} - 16t_{01}t_{02}{t_{12}}^2t_{21} + {t_{21}}^2$$
$$+ 8t_{02}t_{12}{t_{21}}^2 + 16{t_{02}}^2{t_{12}}^2{t_{21}}^2 + 2t_{11}(2t_{01}t_{12} - (1+4t_{02}t_{12})t_{21})) = 0.$$

(3) $M=2$, $I=\{1\}$, $J=\{2\}$, $n_0=n_1=n_2=N_0=1$, $N_1=N_2=2$. In this case we take

$$z_0 = z_1 = E_1(p) = p + \frac{v_{11}}{p-q_1} + \frac{v_{21}}{p-q_2}, \quad z_2 = E_2(p) = \frac{w_{21}}{p-q_2} + \frac{w_{11}}{p-q_1} + c_0.$$

By equating the coefficients of p^1, p^0, $(p-q_i)^{-j}$, $i,j=1,2$ in (39) one finds

$$p^1: \quad t_{01} = c_0,$$

$$p^0: \quad -t_{10} - t_{20} = w_{11} + w_{21},$$

$$(p-q_1)^{-2}: \quad 2t_{12}{v_{11}}^2 = v_{11}w_{11},$$

$$(p-q_1)^{-1}: \quad v_{11}\left(t_{11} + 4t_{12}\left(q_1 + \frac{v_{21}}{q_1-q_2}\right)\right) = \left(q_1 + \frac{v_{21}}{q_1-q_2}\right)w_{11} + v_{11}\left(c_0 + \frac{w_{21}}{q_1-q_2}\right),$$

$$(p-q_2)^{-2}: \quad -2t_{22}{w_{21}}^2 = v_{21}w_{21},$$

$$(p-q_2)^{-1}: \quad -(t_{21}w_{21}) - 4t_{22}\left(c_0 + \frac{w_{11}}{-q_1+q_2}\right)w_{21} = c_0v_{21} + \frac{v_{21}w_{11} + (-(q_1q_2) + {q_2}^2 + v_{11})w_{21}}{-q_1+q_2},$$

and (41) leads to

$$-q_2t_{01} + t_{10} - \frac{t_{11}v_{11}}{-q_1+q_2} - 2t_{12}\left(\frac{{v_{11}}^2}{(-q_1+q_2)^2} + \frac{2v_{11}\left(q_1 + \frac{v_{21}}{q_1-q_2}\right)}{-q_1+q_2}\right) - t_{21}\left(c_0 + \frac{w_{11}}{-q_1+q_2}\right)$$
$$- 2t_{22}\left(\left(c_0 + \frac{w_{11}}{-q_1+q_2}\right)^2 - \frac{2w_{11}w_{21}}{(q_1-q_2)^2}\right) = 0.$$

By solving these equations one obtains

$$E_1(p) = p - \frac{2r_1{}^2 t_{12} + (t_{10}+t_{20})(1+4t_{12}t_{22}) - r_1(t_{01}-t_{11}+2t_{12}t_{21}+4t_{01}t_{12}t_{22})}{4(p-q_1)t_{12}(1+4t_{12}t_{22})}$$
$$- \frac{t_{22}(2r_1{}^2 t_{12} - (t_{10}+t_{20})(1+4t_{12}t_{22}) - r_1(t_{01}-t_{11}+2t_{12}t_{21}+4t_{01}t_{12}t_{22}))}{(p-q_2)(1+4t_{12}t_{22})},$$

$$E_2(p) = t_{01} - \frac{2r_1{}^2 t_{12} + (t_{10}+t_{20})(1+4t_{12}t_{22}) - r_1(t_{01}-t_{11}+2t_{12}t_{21}+4t_{01}t_{12}t_{22})}{2(p-q_1)(1+4t_{12}t_{22})}$$
$$+ \frac{2r_1{}^2 t_{12} - (t_{10}+t_{20})(1+4t_{12}t_{22}) - r_1(t_{01}-t_{11}+2t_{12}t_{21}+4t_{01}t_{12}t_{22})}{2(p-q_2)(1+4t_{12}t_{22})},$$

where

$$q_1 = \frac{r_1}{2} - \frac{(t_{10}+t_{20})(1+4t_{12}t_{22}) + r_1(t_{11}+2t_{12}t_{21}+t_{01}(-1+4t_{12}t_{22}))}{4r_1 t_{12}},$$

$$q_2 = -\frac{r_1}{2} - \frac{(t_{10}+t_{20})(1+4t_{12}t_{22}) + r_1(t_{11}+2t_{12}t_{21}+t_{01}(-1+4t_{12}t_{22}))}{4r_1 t_{12}},$$

and r_1 satisfies

$$-12r_1{}^4 t_{12}{}^2 + (t_{10}+t_{20})^2(1+4t_{12}t_{22})^2 + 8r_1{}^3 t_{12}(t_{01}-t_{11}+2t_{12}t_{21}+4t_{01}t_{12}t_{22}) - r_1{}^2(t_{11}{}^2 - 4t_{11}t_{12}t_{21}$$
$$- 2t_{01}(t_{11}-2t_{12}t_{21})(1+4t_{12}t_{22}) + (t_{01}+4t_{01}t_{12}t_{22})^2 + 4t_{12}(-t_{10}+t_{20}+t_{12}t_{21}{}^2 - 4t_{10}t_{12}t_{22}$$
$$+ 4t_{12}t_{20}t_{22})) = 0.$$

D. *S*-functions

According to the identities

$$\partial_p S_\alpha = \mathcal{M}_\alpha \partial_p \log z_\alpha, \quad \mathcal{M}_\alpha = m_\alpha z_\alpha,$$

it follows at once from (34), (35), and (37) that the functions $\partial_p S_\alpha$ are rational functions of p with finite poles at the points q_i $(i=1,\ldots,M)$ only. Thus we may decompose the functions $\partial_p S_\alpha$ into their principal parts

$$\partial_p S_\alpha = \sum_\beta (\partial_p S_\alpha)_{(\beta,+)}, \tag{43}$$

and, in view of asymptotic behavior (9), we may write

$$(\partial_p S_\alpha)_{(\alpha,+)} = \partial_p R_\alpha, \tag{44}$$

where

$$R_\alpha := \sum_{n\geqslant 1} (z_\alpha^n)_{(\alpha,+)} t_{\alpha n} - (1-\delta_{\alpha 0}) t_{\alpha 0} \log_\alpha (p-q_\alpha). \tag{45}$$

Further, from (34), (35), and (37) we obtain

$$\partial_p S_i = \partial_p S_0, \quad \forall\, i \in I,$$

$$\partial_p S_j = \partial_p S_0 - \frac{1}{n_0}\partial_p(E_1E_2), \quad \forall\, j \in J, \tag{46}$$

which leads to

$$S_\alpha = \begin{cases} \sum_\beta R_\beta + \frac{1}{n_0}\sum_{j\in J}(E_1E_2)_{(j,+)}, & \alpha \in \{0\}\cup I \\ \sum_\beta R_\beta - \frac{1}{n_0}(E_1E_2)_{(0,+)} - \frac{1}{n_0}\sum_{i\in I}(E_1E_2)_{(i,+)}, & \alpha \in J. \end{cases} \tag{47}$$

In principle, (46) implies the expressions (47) plus additional p-independent terms $w_\alpha(t)$. However these terms can be removed by using (8) and (9). Indeed, assymptotic behavior (9) for S_0 requires $w_0=0$. On the other hand, (8) says that

$$\mathrm{d}S_i - \mathrm{d}S_0 = m_i \mathrm{d}z_i - m_0 \mathrm{d}z_0,$$

so that by using the string equations (26) we deduce

$$\mathrm{d}w_i = \begin{cases} \mathrm{d}S_i - \mathrm{d}S_0 = 0, & i \in I \\ \mathrm{d}S_i - \mathrm{d}S_0 + \frac{1}{n_0}\mathrm{d}(E_1E_2) = 0, & i \in J. \end{cases}$$

E. τ-functions

Theorem 3: *The τ-function for the solutions of the Whitham hierarchy associated with the class of string equations* (26) *is given by*

$$\begin{aligned} 2\log\tau &= \sum_\alpha \frac{1}{2\pi i}\oint_{\Gamma_\alpha}\left(\sum_{n\geqslant 1} z_\alpha^n t_{\alpha n}\right) m_\alpha \mathrm{d}z_\alpha - \sum_{j\in J}\frac{1}{4\pi i n_j}\oint_{\Gamma_j} z_j m_j^2 \mathrm{d}z_j + \sum_i t_{i0}v_{i1} \\ &= \sum_\alpha \sum_{n\geqslant 0} t_{\alpha n} v_{\alpha n+1} - \sum_{j\in J}\frac{1}{n_j}\sum_{n\geqslant 1} n t_{jn} v_{jn+1} - \sum_{j\in J}\frac{t_{j0}^2}{2n_j} \end{aligned} \tag{48}$$

Proof: Our strategy to prove (48) is to start from the free-energy function for the algebraic orbits of the Whitham hierarchy[5]

$$F_0 := \sum_\alpha \frac{1}{4\pi i}\oint_{\Gamma_\alpha}\left(\sum_{n\geqslant 1} z_\alpha^n t_{\alpha n}\right) m_\alpha \mathrm{d}z_\alpha + \frac{1}{2}\sum_i t_{i0}v_{i1}, \tag{49}$$

and determine the appropriate modifications to get the free-energy function for the solutions of the class of string equations (26).

By differentiating F_0 with respect to t_{ln}, $n\geqslant 1$ we get

$$\partial_{l,n}F_0 = \frac{1}{4\pi i}\oint_{\Gamma_l} z_l^n m_l \mathrm{d}z_l + \sum_\alpha \frac{1}{4\pi i}\oint_{\gamma_\alpha}\left(\sum_{m\geqslant 1} z_\alpha^m t_{\alpha m}\right)\partial_p((z_l^n)_{(l,+)})\mathrm{d}p + \frac{1}{2}\sum_i t_{i0}\partial_{l,n}v_{i1}, \tag{50}$$

and arguing as in the derivation of (13), we find that

$$\oint_{\gamma_\alpha}\left(\sum_{m\geqslant 1} z_\alpha^m t_{\alpha m}\right)\partial_p((z_l^n)_{(l,+)})\mathrm{d}p = \oint_{\gamma_l} z_l^n \partial_p\left(\sum_{m\geqslant 1}(z_\alpha^m)_{(\alpha,+)} t_{\alpha m}\right)\mathrm{d}p.$$

On the other hand, for $i\neq l$ we have

$$\partial_{l,n}S_i = (z_l^n)_{(l,+)} = -\partial_{l,n}v_{i1} + \mathcal{O}\left(\frac{1}{z_i}\right), \quad p \to q_i,$$

so that

$$\begin{aligned}\partial_{l,n}v_{i1} &= -(z_l^n)_{(l,+)}(q_i)\\ &= \frac{1}{2\pi i}\oint_{\gamma_i}\frac{(z_l^n)_{(l,+)}}{p-q_i}\mathrm{d}p = -\frac{1}{2\pi i}\oint_{\gamma_l}(z_l^n)_{(l,+)}\partial_p\log(p-q_i)\mathrm{d}p\\ &= -\frac{1}{2\pi i}\oint_{\gamma_l} z_l^n\partial_p\log(p-q_i)\mathrm{d}p, \quad i \neq l,\end{aligned}$$

and the same expression turns out to hold for $i=l$. Hence, (50) can be rewritten as

$$\begin{aligned}\partial_{l,n}F_0 &= \frac{1}{2}v_{ln+1} + \frac{1}{4\pi i}\oint_{\gamma_l} z_l^n\sum_\alpha \partial_p\left(\sum_{m\geqslant 1}(z_\alpha^m)_{(\alpha,+)}t_{\alpha m} - (1-\delta_{\alpha 0})t_{\alpha 0}\log(p-q_\alpha)\right)\mathrm{d}p\\ &= v_{ln+1} + \frac{1}{4\pi i}\oint_{\gamma_l} z_l^n\left(\sum_\alpha \partial_p R_\alpha - \partial_p S_l\right)\mathrm{d}p\\ &= v_{ln+1} + \sum_\alpha \frac{1}{4\pi i}\oint_{\gamma_l}(z_l^n)_{(l,+)}(\partial_p R_\alpha - (\partial_p S_l)_{(\alpha,+)})\mathrm{d}p. \end{aligned} \tag{51}$$

Further, we have that

$$\begin{aligned}\frac{1}{4\pi i}\oint_{\gamma_l}(z_l^n)_{(l,+)}(\partial_p R_\alpha - (\partial_p S_l)_{(\alpha,+)})\mathrm{d}p &= -\frac{1}{4\pi i}\oint_{\gamma_\alpha}(z_l^n)_{(l,+)}(\partial_p R_\alpha - (\partial_p S_l)_{(\alpha,+)})\mathrm{d}p\\ &= -\frac{1}{4\pi i}\oint_{\gamma_\alpha}(z_l^n)_{(l,+)}(\partial_p S_\alpha - \partial_p S_l)\mathrm{d}p,\end{aligned}$$

so that from (46) and by taking into account

$$\oint_{\gamma_0}(z_l^n)_{(l,+)}\partial_p(E_1E_2)\mathrm{d}p = -\sum_{i=1}^M\oint_{\gamma_i}(z_l^n)_{(l,+)}\partial_p(E_1E_2)\mathrm{d}p,$$

we get

$$\begin{aligned}\partial_{l,n}F_0 &= v_{ln+1} + \frac{1}{n_0}\sum_{j\in J}\frac{1}{4\pi i}\oint_{\gamma_j}(z_l^n)_{(l,+)}\partial_p(E_1E_2)\mathrm{d}p = v_{ln+1} + \sum_{j\in J}\frac{1}{4\pi i n_j}\oint_{\Gamma_j} z_j m_j\partial_{l,n}m_j\mathrm{d}z_j\\ &= v_{ln+1} + \partial_{l,n}\left(\sum_{j\in J}\frac{1}{8\pi i n_j}\oint_{\Gamma_j} z_j m_j^2\mathrm{d}z_j\right),\end{aligned} \tag{52}$$

which shows that

$$\partial_{l,n}\log\tau = v_{ln+1}.$$

By a similar procedure one finds

$$\partial_{0,n}\log\tau = v_{0n+1}, \quad n \geqslant 1.$$

Nevertheless, proving that

$$\partial_{l,0}\log\tau = v_{l1}, \quad l = 1, \ldots, M \tag{53}$$

requires a more involved analysis. First we differentiate F_0 with respect to t_{l0}

$$\partial_{l,0}F_0 = \sum_\alpha \frac{1}{4\pi i}\oint_{\gamma_\alpha}\left(\sum_{m\geqslant 1} z_\alpha^m t_{\alpha m}\right)\partial_p(-\log(p-q_l))\mathrm{d}p + \frac{1}{2}\sum_i t_{i0}\partial_{l,0}v_{i1} + \frac{1}{2}v_{l1}, \tag{54}$$

and use the following relations:

$$\oint_{\gamma_\alpha}\left(\sum_{m\geqslant 1} z_\alpha^m t_{\alpha m}\right)\frac{1}{p-q_l}\mathrm{d}p = -\oint_{\gamma_l}\left(\sum_{m\geqslant 1}(z_\alpha^m)_{(\alpha,+)}t_{\alpha m}\right)\frac{1}{p-q_l}\mathrm{d}p, \quad \alpha \neq l,$$

$$\oint_{\gamma_l}\left(\sum_{m\geqslant 1} z_l^m t_{lm}\right)\frac{1}{p-q_l}\mathrm{d}p = -2\pi i\lim_{p\to q_l}\left(\sum_{m\geqslant 1}(z_l^m - (z_l^m)_{(l,+)})t_{lm}\right),$$

$$\sum_{i\neq l} t_{i0}\partial_{l0}v_{i1} = \sum_{i\neq l} t_{i0}\log_l(q_i - q_l) - \sum_{i>l} t_{i0}\log_{li}(-1) = \sum_{i\neq l} t_{i0}\log_i(q_l - q_i) - \sum_{i<l} t_{i0}\log_{il}(-1),$$

$$\partial_{l,0}v_{l1} = \lim_{p\to q_l}\log(z_l(p-q_l)).$$

Then (54) becomes

$$\begin{aligned}\partial_{l,0}F_0 = {} & \frac{1}{2}v_{l1} + \frac{1}{4\pi i}\oint_{\gamma_l}\frac{\mathrm{d}p}{p-q_l}\sum_{\alpha\neq l}\left(\sum_{m\geqslant 1}(z_\alpha^m)_{(\alpha,+)}t_{\alpha m} - (1-\delta_{\alpha 0})t_{\alpha 0}\log_\alpha(p-q_\alpha)\right) \\ & + \frac{1}{2}\lim_{p\to q_l}\left(\sum_{m\geqslant 1}(z_l^m - (z_l^m)_{(l,+)})t_{lm} + t_{l0}\log z_l + t_{l0}\log_l(p-q_l)\right) - \frac{1}{2}\sum_{i<l} t_{i0}\log_{il}(-1).\end{aligned} \tag{55}$$

Further, from asymptotic expansion (9) of S_l we have that

$$v_l = \lim_{p\to q_l}\left(\sum_{m\geqslant 1} z_l^m t_{lm} + t_{l0}\log z_l - S_l\right),$$

which allows us to rewrite (55) in the form

$$\partial_{l,0}F_0 = v_{l1} + \frac{1}{4\pi i}\oint_{\gamma_l}\frac{\mathrm{d}p}{p-q_l}\sum_{\alpha\neq l} R_\alpha + \frac{1}{2}\lim_{p\to q_l}(S_l - R_l) = v_{l1} + \frac{1}{4\pi i}\oint_{\gamma_l}\frac{\mathrm{d}p}{p-q_l}\left(\sum_\alpha R_\alpha - S_l\right). \tag{56}$$

Now by using (47) it is straightforward to see that

$$\frac{1}{4\pi i}\oint_{\gamma_l}\frac{\mathrm{d}p}{p-q_l}\left(\sum_\alpha R_\alpha - S_l\right) = \frac{1}{n_0}\sum_{j\in J}\frac{1}{4\pi i}\oint_{\gamma_j}\frac{\mathrm{d}p}{p-q_l}E_1E_2 = \sum_{j\in J}\frac{1}{4\pi i n_j}\oint_{\Gamma_j} z_j m_j \partial_{l,0}m_j \mathrm{d}z_j, \tag{57}$$

which shows that (56) is equivalent to (53). □

F. Conformal maps dynamics

We will outline how our scheme applies for characterizing dToda dynamics of conformal maps, and, in particular, how (48) gives rise to the expression of the τ-function of analytic curves found in Ref. 10.

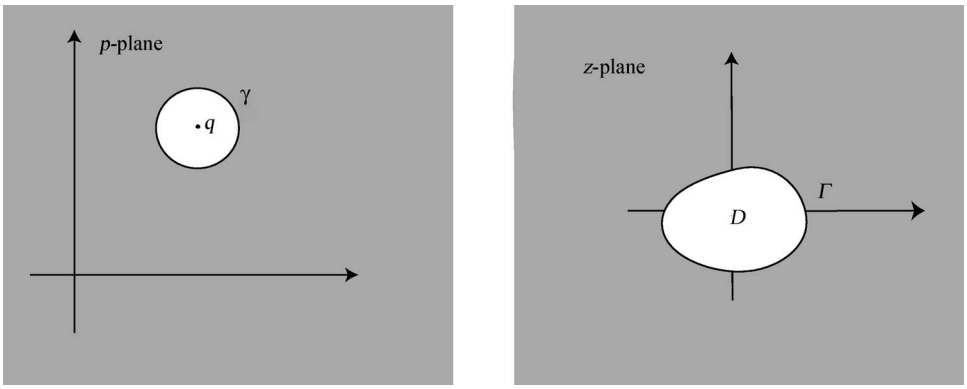

FIG. 2. Conformal map $z=z(p)$.

Given a simply connected domain D bounded by a closed path Γ in the z-plane, there exists[27] a unique circle γ in the p-plane and a unique conformal map $z=z(p)$ satisfying

$$z(p) = p + \sum_{n=1}^{\infty} \frac{d_n}{p^n}, \quad z \to \infty, \tag{58}$$

such that $z=z(p)$ transforms the exterior of γ into the exterior of Γ (see Fig. 2). Note that the conformal map used by Wiegmann-Zabrodin[10] is given by $z(rp+q)$, where q and r are the center and the radius of γ, respectively.

Let us define the function

$$\overline{z}(p) = \overline{z(\mathcal{I}_\gamma(p))}, \tag{59}$$

where $\mathcal{I}_\gamma$ denotes the inversion with respect to the circle γ

$$\mathcal{I}_\gamma(p) := q + \frac{r^2}{\overline{p} - \overline{q}}.$$

It is clear that

$$\overline{z}(p) = \overline{z(p)}, \quad \forall\, p \in \gamma.$$

If Γ is asssumed to be an analytic curve, then it can be described by an equation of the form

$$\overline{z} = \mathcal{S}(z), \tag{60}$$

where $\mathcal{S}(z) := \overline{z}(p(z))$ (the Schwarz function) is analytic in a neigborhood of γ. Thus, if Γ encircles the origin, $\mathcal{S}(z)$ can be expanded as

$$\mathcal{S}(z) = \sum_{n\geqslant 1} n t_n z^{n-1} + \frac{t_0}{z} + \sum_{n\geqslant 1} \frac{v_n(t)}{z^{n+1}}, \tag{61}$$

where the coefficients t_n $(n \geqslant 0)$, the exterior harmonic moments of Γ, determine the curve Γ and conformal map (58). Note, in particular, that the coefficient t_0,

$$t_0 = \frac{1}{2\pi i} \oint_\Gamma \overline{z} \mathrm{d}z = \frac{1}{\pi} \int_D \mathrm{d}x\mathrm{d}y,$$

represents the area of D.

In this way, by considering the harmonic moments as independent complex parameters, if we define

$$q_1 = q, \quad z_0(p) = z(p), \quad z_1(p) = \bar{z}(p),$$

$$t_{0n} := t_n, \quad t_{1n} = -\bar{t}_n, \quad n \geqslant 0,$$

$$m_0 = \mathcal{S}(z_0), \quad m_1 = -z(p(z_1)),$$

we obtain a solution of the system of string equations

$$-m_1 = z_0, \quad z_1 = m_0. \tag{62}$$

Further, by taking into account that

$$v_{1n+1} = -\bar{v}_{0n+1} = -\bar{v}_n, \quad n \geqslant 1,$$

and the identity

$$t_0^2 + 2\sum_{n\geqslant 1} n t_n v_n = \frac{1}{2\pi i}\oint_\Gamma z\mathcal{S}(z)^2 \mathrm{d}z = \frac{1}{2\pi i}\oint_\Gamma z\bar{z}^2 \mathrm{d}z = \frac{1}{\pi}\int_D |z|^2 \mathrm{d}x\mathrm{d}y = \bar{t}_0^2 + 2\sum_{n\geqslant 1} n\bar{t}_n\bar{v}_n,$$

we see that (48) reduces to

$$2\log\tau = \frac{1}{2}\sum_{n\geqslant 1}(2-n)(t_n v_n + \bar{t}_n\bar{v}_n) + t_0 v_0 - \frac{t_0^2}{2}, \tag{63}$$

where $v_0 := -v_{11}$, which is the expression for the τ-function associated to analytic curves obtained in Ref. 10. Notice that (62) is the simplest nontrivial case ($I=\emptyset$, $J=\{1\}$, $n_0=n_1=1$) of the class of string equations (26).

G. Symmetry constraints

As we proved earlier, solutions (z_α, m_α) of systems of string equations (14) are invariant under the symmetries

$$\mathbb{V}_{rs} = (P_0^{r+1}Q_0^{s+1}, \ldots, P_M^{r+1}Q_M^{s+1}), \quad r \geqslant -1, s \geqslant 0.$$

Moreover, as a consequence of (14) we have that the following identities hold:

$$P_0^{r+1}Q_0^{s+1} = P_i^{r+1}Q_i^{s+1} = P_j^{r+1}Q_j^{s+1} = P_0^{r+1}Q_j^{s+1}, \quad \forall\, i \in I, j \in J, \tag{64}$$

for the values of the functions P_μ and Q_μ at a solution (z_α, m_α). In particular these identities lead to the following expressions for the constraints arising from the invariance of (26) under the action of $\mathbb{V}_{rs}$.

Theorem 4: *If* (z_α, m_α) *is a solution of the string equations* (26) *then it satisfies the identities*

$$\sum_{\alpha\in\{0\}\cup I}\oint_{\Gamma_\alpha}\left(\frac{z_\alpha}{n_\alpha}\right)^s z_\alpha^{(r-s)n_\alpha} m_\alpha^{s+1}\mathrm{d}z_\alpha + (-1)^r\frac{s+1}{r+1}n_0^{r-s}\sum_{j\in J}\oint_{\Gamma_j}\left(\frac{z_j}{n_j}\right)^r z_j^{(s-r)n_j} m_j^{r+1}\mathrm{d}z_j = 0, \tag{65}$$

for all $r, s \geqslant 0$.

Proof: From (34)–(37) we find that (64) takes the form

$$z_0^{n_0(r+1)}\left(\frac{1}{n_0}\frac{m_0}{z_0^{n_0-1}}\right)^{s+1}=z_i^{n_i(r+1)}\left(\frac{1}{n_i}\frac{m_i}{z_i^{n_i-1}}\right)^{s+1}=\left(\frac{z_j^{n_j}}{n_0}\right)^{s+1}\left(-\frac{n_0}{n_j}\frac{m_j}{z_j^{n_j-1}}\right)^{r+1}$$

$$=\frac{1}{n_0^{s+1}}E_1^{r+1}E_2^{s+1},\quad \forall\, i\in I, j\in J, \tag{66}$$

and we have that

$$z_0^{n_0(r+1)}\left(\frac{1}{n_0}\frac{m_0}{z_0^{n_0-1}}\right)^{s+1}\partial_p\log E_1=z_i^{n_i(r+1)}\left(\frac{1}{n_i}\frac{m_i}{z_i^{n_i-1}}\right)^{s+1}\partial_p\log E_1=\frac{(\partial_p E_1^{r+1})E_2^{s+1}}{(r+1)n_0^{s+1}}, \tag{67}$$

$$\left(\frac{z_j^{n_j}}{n_0}\right)^{s+1}\left(-\frac{n_0}{n_j}\frac{m_j}{z_j^{n_j-1}}\right)^{r+1}\partial_p\log E_2=-\frac{r+1}{s+1}z_0^{n_0(r+1)}\left(\frac{1}{n_0}\frac{m_0}{z_0^{n_0-1}}\right)^{s+1}\partial_p\log E_1+\partial_p\left(\frac{E_1^{r+1}E_2^{s+1}}{(s+1)n_0^{s+1}}\right),$$

for all $r,s\geqslant 0$ and $i\in I$, $j\in J$. Hence if we proceed as in the proof of (42) and take into account that

$$\partial_p\log E_1=n_0\partial_p\log z_0=n_i\partial_p\log z_i,\quad \forall\, i\in I,$$

$$\partial_p\log E_2=n_j\partial_p\log z_j,\quad \forall\, j\in J,$$

it is straightforward to prove that

$$\sum_{\alpha\in\{0\}\cup I}\oint_{\Gamma_\alpha}z_\alpha^{n_\alpha(r+1)}\left(\frac{1}{n_\alpha}\frac{m_\alpha}{z_\alpha^{n_\alpha-1}}\right)^{s+1}n_\alpha\frac{\mathrm{d}z_\alpha}{z_\alpha}-\sum_{j\in J}\frac{s+1}{r+1}\oint_{\Gamma_j}\left(\frac{z_j^{n_j}}{n_0}\right)^{s+1}\left(-\frac{n_0}{n_j}\frac{m_j}{z_j^{n_j-1}}\right)^{r+1}n_j\frac{\mathrm{d}z_j}{z_j}$$

$$=\oint_\gamma z_0^{n_0(r+1)}\left(\frac{1}{n_0}\frac{m_0}{z_0^{n_0-1}}\right)^{s+1}\partial_p\log E_1\mathrm{d}p=\oint_\gamma\frac{(\partial_p E_1^{r+1})E_2^{s+1}}{(r+1)n_0^{s+1}}\mathrm{d}p=0, \tag{68}$$

where $\gamma:=\Sigma_{\alpha=0}^M\gamma_\alpha$. This proves that the identities (65) hold. □

By evaluating the integrals of the left-hand side we obtain the symmetry constraints in terms of differential equations for the free-energy function $F=\log\tau$.

Examples: For $r=s=0$, Eq. (68) reduces to (42), so that it implies

$$\sum_\alpha t_{\alpha 0}=0.$$

The cases $(r,s)=(1,0)$ and $(r,s)=(2,0)$ correspond to the Virasoro constraints induced by $\mathbb{V}_1$ and $\mathbb{V}_2$, respectively, and lead to the identities

$$\sum_{\alpha\in\{0\}\cup I}\partial_{\alpha,n_\alpha}F-\sum_{j\in J}\frac{n_0}{n_j}\left(\sum_{n-n'=n_j-1}nt_{jn}\partial_{j,n'-1}F+n_jt_{jn_j}t_{j0}+\frac{1}{2}\sum_{n+n'=n_j}nn't_{jn}t_{jn'}\right)=0,$$

$$\sum_{\alpha\in\{0\}\cup I}\partial_{\alpha,2n_\alpha}F+\sum_{j\in J}\left(\frac{n_0}{n_j}\right)^2\left(\sum_{n-n'-n''=2n_j-2}nt_{jn}(\partial_{j,n'-1}F)(\partial_{j,n''-1}F)+2\sum_{n-n'=2n_j-1}nt_{jn}t_{j0}\partial_{j,n'-1}F\right.$$

$$+\sum_{n+n'-n''=2n_j-1}nn't_{jn}t_{jn'}\partial_{j,n''-1}F+2n_jt_{j2n_j}t_{j0}^2+\sum_{n+n'=2n_j}nn't_{jn}t_{jn'}t_{j0}$$

$$\left.+\frac{1}{3}\sum_{n+n'+n''=2n_j}nn'n''t_{jn}t_{jn'}t_{jn''}\right)=0.$$

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