AIP | Journal of Mathematical Physics

From integrable nets to integrable lattices

Manuel Mañas

Citation: Journal of Mathematical Physics **43**, 2523 (2002); doi: 10.1063/1.1454185
View online: http://dx.doi.org/10.1063/1.1454185
View Table of Contents: http://scitation.aip.org/content/aip/journal/jmp/43/5?ver=pdfcov
Published by the AIP Publishing

JOURNAL OF MATHEMATICAL PHYSICS VOLUME 43, NUMBER 5 MAY 2002

From integrable nets to integrable lattices

Manuel Mañas[a)]
Departamento de Física Teórica II, Universidad Complutense, E28040 Madrid, Spain

(Received 29 March 2001; accepted for publication 8 January 2002)

Inspired by the results of Jonas, Einsenhart, Demoulin, and Bianchi on the permutability property of classical geometrical transformations of conjugate nets and its reductions—of pseudo-orthogonal, pseudo-symmetric, and pseudo-Egorov types—dressing transformations of the N-component KP hierarchy (described within the Grassmannian) are used to generate quadrilateral lattices and its corresponding reductions. As a byproduct we get the corresponding discrete dressing transformations; in particular, we characterize the vectorial fundamental discrete transformations preserving the symmetric lattice. *© 2002 American Institute of Physics.*
[DOI: 10.1063/1.1454185]

I. INTRODUCTION

As was noticed in Ref. 1 Jonas and Einsenhart[2,3] knew that the iteration of fundamental transformations[2,4] of a conjugate net generate a lattice built up of quadrilaterals. More recently this idea of interpreting transformed integrable as points on the corresponding integrable lattices was considered in Refs. 5 and 6. In Ref. 7, following Demoulin[8] and Bianchi,[9] it was also observed that the circular lattice could be obtained by the iteration of Ribaucour–Bianchi transformations.[10,9]

On the one hand, in Refs. 11 and 12 we gave the τ-function formalism of conjugate and orthogonal nets and found the role of Miwa transformations in the construction of quadrilateral lattices. We extended further these ideas to the orthogonal net and the circular lattice.[13] On the other hand, we have a Grassmannian theory for conjugate and reduced nets as well as for their dressing transformations.[14,15] The aim of this article is to generate, using these dressing transformations, the corresponding lattices, expressing the elements of the lattices in terms of the so-called Cauchy propagator. Thus, this article could be considered as an analytical version—of a geometric nature—of the folklore ''Darboux transformations discretise continuous integrable system.''

In this article we study conjugate nets and their reductions of pseudo-orthogonal, pseudo-symmetric, and pseudo-Egorov types. In this respect we generalize the Grassmannian description[16,17] of nets in Euclidean space[14,15] to the pseudo-Euclidean case. We take advantage of this extension to present the characterization of the Baker functions, Cauchy propagator, and dressing transformations for these more general reductions. Although, the $\bar{\partial}$ dressing—as was discussed in Ref. 15—is not adequate to generate lattices from the reduced nets, we introduce a limiting case of it, which we call singular dressing transformation, in order to generate the corresponding lattices. As these lattices are generated from reduced nets in pseudo-Euclidean space they are not of the standard circular or Egorov types. Indeed, we get integrable lattices that extend these lattices and that we have named as pseudo-circular and pseudo-Egorov lattices.

The contents of the article are as follows. In Sec. II we remind the reader of the Grassmannian formulation of conjugate nets, their reductions (now in pseudo-Euclidean space), and the corresponding dressing transformations, $\bar{\partial}$ and singular (note that even for the $\bar{\partial}$-dressing the construction given here is more general than that of Ref. 15). In Sec. III we generate quadrilateral lattices and give their geometrical elements, including those of the backward lattice,[18] in terms of the Cauchy propagator. Section IV is devoted to the generation, by means of singular dressing, of the

[a)]Electronic mail: manuel@darboux.fis.ucm.es and http://darboux.fis.ucm.es

reduced lattices. We first generate pseudo-circular lattices, then symmetric and finally pseudo-Egorov lattices. Finally, in Sec. V we recover the dressing transformations of the lattices. The central idea is that of permutability: dressing transformations of the net permute. Hence, we could either first generate a lattice and then perform a general dressing, or first perform the general dressing of the net and then, through particular dressing transformations, generate a dressed lattice. We prove that the passage from the lattice to the dressed lattice is a discrete fundamental transformation.[19,1,20] We also show that for the reduced pseudo-circular case we get the generalization to this pseudo-Euclidean case of the discrete Ribaucour–Bianchi transformation.[20–22] We finally characterize the discrete fundamental transformations for the symmetric lattices of Ref. 18.

We now proceed to briefly review some basic facts regarding geometric nets, lattices, and their transformations.

A. Geometric nets

1. Conjugate nets

In recent years[23,24,11] it has been found that the theory of integrable systems is a useful tool to study **geometric nets** $\mathbf{x}=\mathbf{x}(u_1,\ldots,u_N)$ of **conjugate type** in $\mathbb{R}^N$.[25,9] They are characterized by the Laplace equations

$$\frac{\partial^2\mathbf{x}}{\partial u_i\partial u_j}=\frac{\partial \ln H_i}{\partial u_j}\frac{\partial \mathbf{x}}{\partial u_i}+\frac{\partial \ln H_j}{\partial u_i}\frac{\partial \mathbf{x}}{\partial u_j},\quad i,j=1,\ldots,N,\quad i\neq j, \tag{1}$$

where H_i are the so-called *Lamé coefficients*. The compatibility conditions for the above equations can be expressed in terms of the *rotation coefficients* β_{ij},

$$\beta_{ji}:=\frac{1}{H_j}\frac{\partial H_i}{\partial u_j},$$

as

$$\frac{\partial \beta_{ik}}{\partial u_k}=\beta_{ik}\beta_{kj},\quad i,j,\ \text{and}\ k\ \text{different}. \tag{2}$$

Each solution β_{ij} of (2) determines a family of parallel conjugate nets $\mathbf{x}$ given by the solutions of

$$\frac{\partial \mathbf{x}}{\partial u_i}=H_i\mathbf{X}_i.$$

Here $\mathbf{X}_i$ stands for the *renormalized tangent vectors* of the net defined by

$$\frac{\partial \mathbf{X}_i}{\partial u_j}=\beta_{ij}\mathbf{X}_j,\quad i\neq j. \tag{3}$$

2. Reduced nets

Let us consider now conjugate nets in the pseudo-Euclidean space $\mathbb{R}^N_{p,q}$ with $p+q=N$. Thus, we have a nondegenerate symmetric bilinear form

$$\langle \mathbf{X},\tilde{\mathbf{X}}\rangle:=\sum_{i=1}^N \epsilon_i X_i\tilde{X}_i,\quad \text{with}\quad \epsilon_i=\begin{cases}1, & i=1,\ldots,p,\\ -1, & i=p+1,\ldots,p+q,\end{cases}$$

which can be written as

$$\langle \mathbf{X},\tilde{\mathbf{X}}\rangle=(X_1,\dots,X_N)I_{p,q}\begin{pmatrix}\tilde{X}_1\\ \vdots\\ \tilde{X}_N\end{pmatrix},$$

with

$$I_{p,q}:=\operatorname{diag}(\epsilon_1,\dots,\epsilon_N).$$

Among the conjugate curvilinear coordinates systems in pseudo-Euclidean space $\mathbb{R}^N_{p,q}$ one finds the **pseudo-orthogonal** ones for which

$$\left\langle \frac{\partial \mathbf{x}}{\partial u_i},\frac{\partial \mathbf{x}}{\partial u_j}\right\rangle=0,\quad i\neq j.$$

The corresponding normalized tangent vectors

$$\mathbf{X}_i=\frac{1}{H_i}\frac{\partial \mathbf{x}}{\partial u_i},\quad H_i:=\sqrt{\left|\left\langle \frac{\partial \mathbf{x}}{\partial u_i},\frac{\partial \mathbf{x}}{\partial u_i}\right\rangle\right|},\quad i=1,\dots,N,$$

where we assume that the Lamé coefficients H_i, $i=1,\dots,N$, do not vanish; i.e., that none of the vectors $\partial\mathbf{x}/\partial u_i$ have zero norm. The linear system satisfied by these vectors is

$$\frac{\partial \mathbf{X}_i}{\partial u_j}-\beta_{ij}\mathbf{X}_j=0,\quad i,j=1,\dots,N,\ \ i\neq j, \tag{4}$$

$$\frac{\partial \mathbf{X}_i}{\partial u_i}+\sum_{\substack{k=1,\dots,N\\ k\neq i}}\epsilon_i\epsilon_k\beta_{ki}\mathbf{X}_k=0,\quad i=1,\dots,N, \tag{5}$$

where H_i and β_{ij} are the Lamé and rotation coefficients.

For the rotation coefficients one has

$$\frac{\partial \beta_{ij}}{\partial u_k}-\beta_{ik}\beta_{kj}=0,\quad i,j,k=1,\dots,N,\ \text{ with } i,j,k \text{ different}, \tag{6}$$

$$\epsilon_i\frac{\partial \beta_{ij}}{\partial u_i}+\epsilon_j\frac{\partial \beta_{ji}}{\partial u_j}+\sum_{\substack{k=1,\dots,N\\ k\neq i,j}}\epsilon_k\beta_{ki}\beta_{kj}=0,\quad i,j=1,\dots,N,\ \ i\neq j. \tag{7}$$

For $p=N$, $q=0$ we recover the Lamé system which characterize orthogonal nets.[26,27,9] It that case the metric $ds^2=\sum_{i=1}^N H_i^2 du_i^2$ is a flat diagonal metric, while in the pseudo-orthogonal case the flat metric is $ds^2=\sum_{i=1}^N\epsilon_i H_i^2 du_i^2$.

A different reduction is the pseudo-symmetric one. Now we request

$$\epsilon_i\beta_{ij}=\epsilon_j\beta_{ji}. \tag{8}$$

In this case we are dealing with *potential* conjugate nets; i.e., potentials functions exist such that

$$\epsilon_i H_i^2=\frac{\partial \Theta}{\partial u_i},\quad \epsilon_i\langle \mathbf{X}_i,\mathbf{X}_i\rangle=\frac{\partial V}{\partial u_i}. \tag{9}$$

for any symmetric bilinear form $\langle\cdot,\cdot\rangle$.

A particularly important type of pseudo-orthogonal net is the **pseudo-Egorov net**.[27,28] Now, we have a pseudo-symmetric pseudo-orthogonal net. From (8) and (5) we deduce that the pseudo-orthonormal frame $\{\mathbf{X}_i\}_{i=1}^N$ is ∂-invariant

$$\partial X_i = 0, \quad i = 1,\dots,N,$$

with $\partial := \sum_{i=1}^{M} \partial/\partial u_i$, and using (8), (7) and then (6) we conclude that

$$\partial \beta_{i,j} = 0, \quad i,j = 1,\dots,N, \quad i \neq j. \tag{10}$$

In fact, the Lamé equations (6) and (7) reduce in the Egorov case to (6) and (10). The **∂-invariant Egorov nets**[28,29] are Egorov nets that further satisfy the ∂-invariance of the Lamé coefficients $\partial H_i = 0$, $i = 1,\dots,N$. These nets are relevant in relation with Frobenius manifolds and associativity equations.[29,30]

3. Transformations of conjugate nets

The fundamental transformation $\mathcal{F}(\mathbf{x})$ of a conjugate net $\mathbf{x}$ shares with $\mathbf{x}$ the same conjugate congruence; i.e., the lines $\langle \mathbf{x}, \mathcal{F}(\mathbf{x}) \rangle$ *intersect* both nets along the coordinate lines. The fundamental transformation is given by[2–4]

$$\mathcal{F}_i(\mathbf{x}) = \mathbf{x} - \Omega[X, \zeta^*] \frac{\Omega[\zeta, H]}{\Omega[\zeta, \zeta^*]};$$

here $\Omega[\zeta, \zeta^*]$ is a solution of

$$\frac{\partial \Omega}{\partial u_k} = \zeta_k \zeta_k^*, \quad k = 1,\dots,N,$$

where ζ_k are scalar solutions of (3).

4. Ribaucour transformation

The Ribaucour transformation,[10,27,9,4,31] is the reduction of the fundamental transformation which preserves the orthogonal character of the net. Essentially, both orthogonal nets are tangent to families of hiperspheres in $\mathbb{R}^N$, and can be recovered by studying nets in quadrics.[31] Given functions ζ_i such that

$$\frac{\partial \zeta_i}{\partial u_j} = \beta_{ij} \zeta_j, \quad i,j = 1,\dots,N, \quad i \neq j,$$

new orthogonal coordinates $\mathcal{R}(\mathbf{x})$ are given by

$$\mathcal{R}(\mathbf{x}) = \mathbf{x} - 2 \frac{1}{\sum_{k=1}^{N} \zeta_k^2} \Omega(\zeta, H) \sum_{k=1}^{N} \zeta_k \mathbf{X}_k.$$

B. Geometric lattices

1. Quadrilateral lattices

Among the N-dimensional lattices $\mathbf{x}: \mathbb{Z}^N \to \mathbb{R}^N$ there is a distinguished class for which the elementary quadrilaterals are planar.[32,33,1] The planarity condition can be expressed by the following linear equation for suitably renormalized tangent vectors $\mathfrak{C}_i(\mathbf{n}) \in \mathbb{R}^N$,

$$\Delta_j \mathfrak{C}_i = (T_j Q_{ij}) \mathfrak{C}_j, \quad i,j = 1,\dots,N, \quad i \neq j, \tag{11}$$

being its compatibility conditions, and the following discrete Darboux equations

$$\Delta_k Q_{ij} = (T_k Q_{ik}) Q_{kj}, \quad i,j \text{ and } k \text{ different}.$$

The points $\mathbf{x}$ of the lattice can be found by means of discrete integration of

$$\Delta_i \mathbf{x} = (T_i H_i)\mathfrak{C}_i, \quad i=1,...,N,$$

where H_i are solutions of equations

$$\Delta_i H_j = Q_{ij} T_i H_i, \quad i,j=1,...,N, \quad i \neq j. \tag{12}$$

In the above formulas, T_i is the translation operator in the discrete variable n_i:

$$T_i f(n_1,\ldots,n_i,\ldots,n_N) = f(n_1,\ldots,n_i+1,...,n_N),$$

and $\Delta_i = T_i - 1$ is the corresponding partial difference operator.

The discrete vectorial fundamental transformation[19,1] is given by

$$\hat{\mathfrak{C}}_i = \mathfrak{C}_i - \Omega(\mathfrak{C},\boldsymbol{\zeta}^*)\Omega(\boldsymbol{\zeta},\boldsymbol{\zeta}^*)^{-1}\boldsymbol{\zeta}_i, \quad i=1,...,N,$$

$$\hat{H}_i = H_i - \boldsymbol{\zeta}_i^*\Omega(\boldsymbol{\zeta},\boldsymbol{\zeta}^*)^{-1}\Omega(\boldsymbol{\zeta},H), \quad i=1,...,N,$$

$$\hat{Q}_{ij} = Q_{ij} - \langle \boldsymbol{\zeta}_j^*, \Omega(\boldsymbol{\zeta},\boldsymbol{\zeta}^*)^{-1}\boldsymbol{\zeta}_i\rangle,$$

$$\hat{\mathbf{x}} = \mathbf{x} - \Omega(\mathfrak{C},\boldsymbol{\zeta}^*)\Omega(\boldsymbol{\zeta},\boldsymbol{\zeta}^*)^{-1}\Omega(\boldsymbol{\zeta},H).$$

These are data for a new quadrilateral lattice $\hat{\mathbf{x}}$ as far as $\boldsymbol{\zeta}_i \in V$, V a linear space, and $\boldsymbol{\zeta}_i^* \in V^*$ being the dual of V, are solutions of (11) and (12), respectively. The linear operator $\Omega(\boldsymbol{\zeta},\boldsymbol{\zeta}^*): W \to V$ is defined by the compatible equations:

$$\Delta_i \Omega(\boldsymbol{\zeta},\boldsymbol{\zeta}^*) = \boldsymbol{\zeta}_i \otimes (T_i \boldsymbol{\zeta}_i^*), \quad i=1,...,N. \tag{13}$$

2. Reduced lattices

Quadrilateral lattices $\mathbf{x}: \mathbb{Z}^N \to \mathbb{R}^N$ for which each quadrilateral is inscribed in a circle are called circular or cyclid lattices.[34,35,7,36,20] It can be shown that for the tangent vectors the constraint

$$\mathfrak{C}_i \cdot T_i(\mathfrak{C}_j) + \mathfrak{C}_j \cdot T_j(\mathfrak{C}_i) = 0, \quad i \neq j,$$

is equivalent to the requirement that the lattice is circular. In Ref. 18 the symmetric and Egorov lattices were introduced—the Egorov lattice was also introduced by Schief. The symmetric lattice appears when backward and forward data of the lattice are related in a particular way. A circular, symmetric, and diagonal invariant lattice is called an Egorov lattice. It was proven that Egorov lattices are characterized by

$$\mathfrak{C}_i \cdot T_i(\mathfrak{C}_j) = 0, \quad i \neq j.$$

When the data defining the fundamental transformation fulfills

$$\boldsymbol{\zeta}_i := (\Omega(\mathfrak{C},\boldsymbol{\zeta}^*) + T_i\Omega(\mathfrak{C},\boldsymbol{\zeta}^*))^t \mathfrak{C}_i, \quad i=1,...,N,$$

$$\Omega(\boldsymbol{\zeta},\boldsymbol{\zeta}^*) + \Omega(\boldsymbol{\zeta},\boldsymbol{\zeta}^*)^t - 2\Omega(\mathfrak{C},\boldsymbol{\zeta}^*)^t\Omega(\mathfrak{C},\boldsymbol{\zeta}^*),$$

the transformation preserves the circular reduction.[20,37,22]

Comment: Continuous versions of these discrete vectorial fundamental transformations[37] appear in the continuous limit. The geometrical interpretation of these transformations can be found in Ref. 1.

II. GRASSMANNIANS, CAUCHY PROPAGATOR, NETS, REDUCTIONS, AND DRESSING

In this section we review the results of Refs. 14 and 15 and extend them to the pseudo-Euclidean reductions. We recall for the reader the Grassmannian scheme for conjugate nets as well as pivotal objects like the Cauchy propagator. We continue with the reductions in pseudo-Euclidean space and finish studying dressing transformations—of $\bar{\partial}$ and singular types—and the corresponding reductions.

A. Grassmannians

The KP formalism of conjugate nets can be conveniently formulated in terms of the two families $\mathrm{Gr}_{\gamma(r)}$, and $\mathrm{Gr}^*_{\gamma(r)}$ of infinite-dimensional Grassmannians ($\gamma(r):=\{z\in\mathbb{C}:|z|=r\}$).[14] The elements $W\in\mathrm{Gr}_{\gamma(r)}$ and $V\in\mathrm{Gr}^*_{\gamma(r)}$ are subsets of the space $H_{\gamma(r)}$ of Laurent series

$$\sum_{n=-\infty}^{\infty} a_n z^n,$$

with coefficients a_n in the algebra $M_N(\mathbb{C})$ of $N\times N$ complex matrices, which converge on the circle $\gamma(r)$ and such that the projection on $H^+_{\gamma(r)}$,

$$P_+:\sum_{n=-\infty}^{\infty} a_n z^n \mapsto \sum_{n=0}^{\infty} a_n z^n,$$

is a bijective map. Furthermore, W and V are assumed to be left and right modules, respectively, for the algebra $M_N(\mathbb{C})$.

Each $W\in\mathrm{Gr}_{\gamma(r)}$ has an associated $W^*\in\mathrm{Gr}^*_{\gamma(r)}$ defined as the set of those $v\in H_{\gamma(r)}$ satisfying

$$\int_{\gamma(r)} w(z)v(z)\,dz=0, \quad \forall w\in W. \tag{14}$$

Given $W\in\mathrm{Gr}_{\gamma(r)}$ and $V\in\mathrm{Gr}^*_{\gamma(r)}$ the action of the KP flows are implemented by the multiplication operators

$$W(\mathbf{u})=W\psi_0^{-1}(z,\mathbf{u}), \quad V(\mathbf{u})=\psi_0(z,\mathbf{u})V,$$

where $\mathbf{u}=(\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_N)$ denotes N infinite sequences

$$\mathbf{u}_i=(u_{i,1},u_{i,2},\ldots)\in\mathbb{C}^{N\cdot\infty},$$

and

$$\psi_0(z,\mathbf{u}):=\exp(\xi(z,\mathbf{u})), \quad \xi(z,\mathbf{u}):=\sum_{n\geqslant 1} z^n\left(\sum_{i=1}^{N} u_{i,n}E_i\right), \quad (E_i)_{jk}=\delta_{ij}\delta_{ik}.$$

B. Cauchy propagator

Given $W\in\mathrm{Gr}_{\gamma(r)}$, its associated Baker function is defined as the unique function $\psi=\psi(z,\mathbf{u})$ such that its restriction to $\gamma(r)$ is the element of W which admits a convergent expansion of the form

$$\psi=\chi(z,\mathbf{u})\psi_0(z,\mathbf{u}), \quad \chi(z,\mathbf{u})=I_N+\sum_{n\geqslant 1}\frac{a_n(\mathbf{u})}{z^n}. \tag{15}$$

Similarly, the adjoint Baker function associated to W is defined as the unique function $\psi^*=\psi^*(z,\mathbf{u})$ such that its restriction to $\gamma(r)$ is the element of W^* with a convergent expansion

$$\psi^* = \psi_0(z,\mathbf{u})^{-1}\chi^*(z,\mathbf{u}), \quad \chi^*(z,\mathbf{u}) = I_N + \sum_{n\geqslant 1}\frac{a_n^*(\mathbf{u})}{z^n}. \tag{16}$$

Finally, the corresponding Cauchy propagator (see Refs. 38, 39, 14, and 36) is a Green function for the $\bar{\partial}$ operator

$$\frac{\partial\Psi}{\partial\bar{z}}(z,z',\mathbf{u}) = \pi\delta(z-z'), \tag{17}$$

outside the disk $D(r) := \{z\in C : |z| < r\}$ which satisfies the following boundary conditions:

(1) The restriction of Ψ to $\gamma(r)$, as a function of z, is an element of W.
(2) As $z\to\infty$,

$$\Psi(z,z',\mathbf{u}) = \mathcal{O}\left(\frac{1}{z}\right)\psi_0(z,\mathbf{u}).$$

The relation between the Baker functions and the Cauchy propagator is given by[14]

$$\Psi(z,z',\mathbf{u}) = \begin{cases} -\dfrac{1}{z'}\psi^*(z',\mathbf{u})\psi(z,\mathbf{u}+[z']) & \text{for } |z|\leqslant|z'| \\[2ex] \dfrac{1}{2}\psi^*(z',\mathbf{u}-[z])\psi(z,\mathbf{u}) & \text{for } |z'|\leqslant|z| \end{cases} \tag{18}$$

$$[z] := ([z]_1,\dots,[z]_N), \quad [z]_i := \left(\frac{1}{z},\dots,\frac{1}{nz^n},\dots\right),$$

and by

$$\frac{\partial\Psi}{\partial u_i}(z,z',\mathbf{u}) = \psi^*(z',\mathbf{u})E_i\psi(z,\mathbf{u}). \tag{19}$$

C. Conjugate nets

We introduce the row matrix

$$e_i = (0,\dots,0,\underset{\substack{\uparrow\\ i\text{th place}}}{1},0,\dots,0), \quad i=1,\dots,N.$$

The Baker functions satisfy

$$\frac{\partial\psi_i}{\partial u_k} = \beta_{ik}\psi_k, \quad \frac{\partial\psi_i^*}{\partial u_k} = \psi_k^*\beta_{ki}, \quad i\neq k, \quad u_k := u_{k,1}, \tag{20}$$

where $\beta = \beta(\mathbf{u})$ is

$$\beta := a_1 = -a_1^*, \tag{21}$$

and

$$\psi_i := e_i\psi, \quad \psi_i^* := \psi^* e_i^t.$$

The compatibility of these linear systems implies the Darboux equations

$$\frac{\partial\beta_{ik}}{\partial u_k}=\beta_{ik}\beta_{kj}, \quad i,j \text{ and } k \text{ different.}$$

Thus, β_{ik} are the rotation coefficients for the family of parallel conjugate nets with tangent vectors and Lamé coefficients given by $(\mathbf{X}_i)_j=X_{ij}$ and the rows $H_i=H_{li}$, $(l=1,...,N)$, respectively. Here

$$X(\mathbf{u}):=\int_{\mathbb{C}}\psi(z,\mathbf{u})\mathcal{N}(z)\,\mathrm{d}^2z,$$

$$H(\mathbf{u}):=\int_{\mathbb{C}}\mathcal{M}(z)\,\psi^*(z,\mathbf{u})\,\mathrm{d}^2z,$$

where $\mathcal{N}(z)$ and $\mathcal{M}(z)$ are appropriate matrix distributions. The corresponding conjugate nets $\mathbf{x}$ are the rows $x_i=\mathbf{x}_{li}$, $(l=1,...,N)$ of[14]

$$\mathbf{x}(\mathbf{u}):=\int_{\mathbb{C}\times\mathbb{C}}\mathcal{M}(z')\Psi(z,z')\mathcal{N}(z)\,\mathrm{d}^2z\mathrm{d}^2z'. \tag{22}$$

D. Reductions

We introduce the following involution in the space of KP parameters:

$$e(\mathbf{u})=(e(\mathbf{u})_1,e(\mathbf{u})_2,\ldots,e(\mathbf{u})_N), \quad e(\mathbf{u})_{i,n}=(-1)^{n+1}\mathbf{u}_{i,n}.$$

The following notation convention is used:

$$\mathfrak{u}=(\mathfrak{u}_1,\ldots,\mathfrak{u}_N)\in\mathbb{C}^{N\cdot\infty}, \quad \begin{cases}\mathfrak{u}_{i,n}=0, & \text{for } n \text{ even,}\\ \mathfrak{u}_{i,n}=u_{i,n}, & \text{for } n \text{ odd.}\end{cases}$$

We should stress that the following propositions 1, 2 and 3 were originally obtained—in Euclidean space—in the context of the $\bar{\partial}$-formulation of integrable lattices in Refs. 36 and 18. Here we derived them following the Grassmannian scheme of Refs. 14 and 15.

1. Pseudo-orthogonal reduction

An element $W\in\mathrm{Gr}_{\gamma(r)}$ satisfies *the pseudo-orthogonal reduction* if for every $v\in W^*$ it follows that $\tilde{v}(z):=zv(-z)^{\mathrm{t}}I_{p,q}$ is an element of W. For the discrete case and $p=N$ (see Refs. 18 and 36) similar results are found. In the sense of Ref. 23 this is an F reduction.

Proposition 1: If $W\in\mathrm{Gr}_{\gamma(r)}$ satisfies the orthogonal reduction, then we have the following.

(1) The Baker function and the adjoint Baker function satisfy

$$z\psi^{*\mathrm{t}}(-z,e(\mathbf{u}))I_{p,q}=I_{p,q}\partial\psi(z,\mathbf{u})+(\beta^{\mathrm{t}}(e(\mathbf{u}))I_{p,q}-I_{p,q}\beta(\mathbf{u}))\,\psi(z,\mathbf{u}), \tag{23}$$

where $\partial:=\sum_i\partial/\partial u_i$.

(2) The following identity for the Cauchy propagator holds:

$$z'I_{p,q}\Psi(z,z',\mathbf{u})-z\Psi^{\mathrm{t}}(-z',-z,e(\mathbf{u}))I_{p,q}=-\psi^{\mathrm{t}}(-z',e(\mathbf{u}))I_{p,q}\psi(z,\mathbf{u}). \tag{24}$$

(3) The Baker function satisfies

$$\psi^{\mathrm{t}}(-z,e(\mathbf{u}))I_{p,q}\psi(z,\mathbf{u})=I_{p,q}. \tag{25}$$

(4) The Baker function at $z=0$ satisfies

$$\partial\psi(0,\mathbf{u}) = (\beta(\mathbf{u}) - I_{p,q}\beta^{\mathrm{t}}(e(\mathbf{u}))I_{p,q})\,\psi(0,\mathbf{u}),$$

$$\psi^{\mathrm{t}}(0,e(\mathbf{u}))I_{p,q}\psi(0,\mathbf{u}) = I_{p,q}.$$

Proof: (1) As W fulfills the pseudo-orthogonal reduction we can ensure that $z\psi^{*t}\times(-z,e(u))I_{p,q}\in W$. Moreover, its asymptotic expansion at $z=\infty$ is

$$z\psi^{*\mathrm{t}}(-z,e(\mathbf{u}))I_{p,q} = z\left(I_N + \beta^{\mathrm{t}}(e(\mathbf{u}))\frac{1}{z} + \mathcal{O}\left(\frac{1}{z^2}\right)\right)I_{p,q}\psi_0(z,\mathbf{u}),$$

which has the same normalization as

$$I_{p,q}\partial\psi(z,\mathbf{u}) + (\beta^{\mathrm{t}}(e(\mathbf{u}))I_{p,q} - I_{p,q}\beta(\mathbf{u}))\,\psi(z,\mathbf{u}).$$

(2) The function

$$\phi(z,z',\mathbf{u}) := I_{p,q}\Psi(z,z',\mathbf{u}) - \frac{z}{z'}\Psi^{\mathrm{t}}(-z',-z,e(\mathbf{u}))I_{p,q},$$

as a function of z, belongs to W and is homomorphic outside $D(r)$ up to a singularity at $z=z'$; however, this singularity is avoidable and therefore the asymptotic expansion at $z=\infty$,

$$\phi(z,z',\mathbf{u}) = \left(-\frac{1}{z'}\psi^{\mathrm{t}}(-z',e(\mathbf{u}))I_{p,q} + \mathcal{O}\left(\frac{1}{z}\right)\right)\psi_0(z,\mathbf{u})$$

can be extended to $\gamma(r)$. This implies the desired result.

(3) Just take $z'\to z$ in (24).

(4) Put $z=0$ in (23) and (25).

□

When $\mathcal{N}(z) = \mathcal{N}\delta(z)$, with $\mathcal{N}\in 0(p,q)$ a pseudo-orthogonal matrix, $\mathcal{N}^{\mathrm{t}}I_{p,q}\mathcal{N} = I_{p,q}$, the rows of $\mathbf{x}(\mathfrak{u})$ describe a set of parallel pseudo-orthogonal nets. This is a consequence of

$$X^{\mathrm{t}}(\mathfrak{u})I_{p,q}X(\mathfrak{u}) = \mathcal{N}^{\mathrm{t}}\psi^{\mathrm{t}}(0,\mathfrak{u})I_{p,q}\psi(0,\mathfrak{u})\mathcal{N} = \mathcal{N}^{\mathrm{t}}I_{p,q}\mathcal{N} = I_{p,q},$$

as it means that the rows of X, $\mathbf{X}_i$, from a pseudo-orthonormal basis. Moreover, from (23) we get

$$\partial_i\psi_i(0) = -\sum_{k\neq i}\epsilon_k\epsilon_i\beta_{ki}\psi_k(0)$$

that together with (20) implies for the $\mathbf{X}_i = \psi_i(0)\mathcal{N}$ the system formed by (4) and (5).

2. Pseudo-symmetric reduction

We say that an element $W\in\mathrm{Gr}_{\gamma(r)}$ satisfies *the pseudo-symmetric reduction* if for every $v\in W^*$, it follows that $\tilde{v}(z) := v(-z)^{\mathrm{t}}I_{p,q}\in W$.

Proposition 2: When $W\in\mathrm{Gr}_{\gamma(r)}$ fulfills the pseudo-symmetric reduction, then we have the following.

(1) The Baker function and the adjoint Baker function satisfy

$$\psi^{*\mathrm{t}}(-z,e(\mathbf{u}))I_{p,q} = I_{p,q}\psi(z,\mathbf{u}). \tag{26}$$

(2) For the Cauchy propagator one has

$$I_{p,q}\Psi(z,z',\mathbf{u})-\Psi^{\mathrm{t}}(-z',-z,e(\mathbf{u}))I_{p,q}=0. \tag{27}$$

Proof: (1) As W fulfills the pseudo-symmetric we know that $\psi^{*\mathrm{t}}(-z,e(u))I_{p,q}\in W$. The asymptotic expansion at ∞ is

$$\psi^{*\mathrm{t}}(-z,e(\mathbf{u}))I_{p,q}=\left(I_N+\beta^{\mathrm{t}}(e(\mathbf{u}))\frac{1}{z}+\mathcal{O}\left(\frac{1}{z^2}\right)\right)I_{p,q}\psi_0(z,\mathbf{u}),$$

which implies the stated result.

(2) The function

$$\phi(z,z',\mathbf{u}):=I_{p,q}\Psi(z,z',\mathbf{u})-\Psi^{t}(-z',-z,e(\mathbf{u}))I_{p,q}$$

as a function of z belongs to W and is homomorphic outside $D(r)$ up to an avoidable singularity at $z=z'$; i.e., the asymptotic expansion at ∞

$$\phi(z,z',\mathbf{u})=\mathcal{O}\left(\frac{1}{z}\right)\psi_0(z,\mathbf{u})$$

extends to $\gamma(r)$ and the result follows.

□

From (26) we deduce $\beta^{\mathrm{t}}(e(\mathbf{u}))I_{p,q}=I_{p,q}\beta(\mathbf{u})$; i.e. $\epsilon_i\beta_{ij}=\epsilon_j\beta_{ji}$. Thus, parallel pseudo-symmetric nets are given by the rows of $\mathbf{x}(u)$ of (22). Moreover, the potentials in (9) are

$$\Theta=\left(\int_{\mathbb{C}\times\mathbb{C}}\mathcal{M}(z')\Psi(z,z')I_{p,q}\mathcal{M}^{\mathrm{t}}(-z)\,\mathrm{d}^2z\mathrm{d}^2z'\right)_u,$$

$$V=\mathrm{Tr}\left(\int_{\mathbb{C}\times\mathbb{C}}G\mathcal{N}^{\mathrm{t}}(-z')I_{p,q}\Psi(z,z')\mathcal{N}(z)\,\mathrm{d}^2z\mathrm{d}^2z'\right),$$

with G the symmetric matrix of the bilinear form $\langle\cdot,\cdot\rangle$.

3. Pseudo-Egorov reduction

The pseudo-Egorov reduction is the combination of pseudo-orthogonal and pseudo-symmetric reductions. Thus, an element $W\in\mathrm{Gr}_{\gamma(r)}$ satisfies *the pseudo-Egorov reduction* if

(i) for every $w\in W$, the function $\tilde{w}(z):=zw(z)$ is also in W, and
(ii) for every $v\in W^*$, the function $\tilde{v}(z):=v(-z)^{\mathrm{t}}$ is in W.

The pseudo-Egorov reduction implies the following properties for the Baker functions and the Cauchy propagators.

Proposition 3: If $W\in\mathrm{Gr}_{\gamma(r)}$ satisfies the pseudo-Egorov reduction, then we have the following.

(1) The Baker function and the adjoint Baker function satisfy

$$\psi^{*t}(-z,e(\mathbf{u}))I_{p,q}=I_{p,q}\psi(z,\mathbf{u}),\quad \partial\psi(z,\mathbf{u})=z\psi(z,\mathbf{u}). \tag{28}$$

(2) The Cauchy propagator is given by

$$\Psi(z,z',\mathbf{u})=\frac{I_{p,q}\psi^{\mathrm{t}}(-z',e(\mathbf{u}))I_{p,q}\psi(z,\mathbf{u})}{z-z'}. \tag{29}$$

E. Dressing methods

We review the dressing methods in the Grassmannian as presented in Refs. 14 and 15. We start with the $\bar{\partial}$-dressing and then define the singular dressing. We end describing how these transformations can be constrained in order to preserve the reductions.

1. $\bar{\partial}$-dressing

Let $D(r)$ and $D(\tilde{r})$ be two disks centered at the origin with $r<\tilde{r}$. Denote by $\gamma(r)$ and $\gamma(\tilde{r})$ their respective boundaries, and by A the annulus $D(\tilde{r})-D(r)$.

Given a matrix distribution $R=R(z,z')$ with support in $A\times A$, it determines a "dressing transformation"

$$T_R:\mathrm{Gr}_{\gamma(r)}\mapsto\mathrm{Gr}_{\gamma(\tilde{r})},\quad W\mapsto\tilde{W},\tag{30}$$

where for every $W\in\mathrm{Gr}_{\gamma(r)}$ the corresponding $\tilde{W}\in\mathrm{Gr}_{\gamma(\tilde{r})}$ is the set of boundary values on $\gamma(\tilde{r})$ of matrix functions $w=w(z)$ satisfying the $\bar{\partial}$-equation

$$\frac{\partial w}{\partial\bar{z}}(z)=\int_A w(z')R(z',z)\,\mathrm{d}^2z',\quad z\in A,$$

and such that the restriction of w to $\gamma(r)$ is an element of W. Observe that for $v\in W^*\subset\mathrm{Gr}^*_{\gamma(r)}$ we have

$$\frac{\partial v}{\partial\bar{z}}(z)=-\int_A R(z,z')v(z')\,\mathrm{d}^2z',\quad z\in A.$$

For separable kernels of the type,

$$R(z,z')=\pi\sum_{k=1}^{m}f_k(z)g_k(z')\tag{31}$$

with $\{f_k\}_{k=1}^m$, $\{g_k\}_{k=1}^m$ two sets of linearly independent $M_N(\mathbb{C})$-valued distributions, the Cauchy propagators Ψ and $\tilde{\Psi}$ associated with W and $\tilde{W}$, respectively, are related by

$$\tilde{\Psi}(z,z')=\Psi(z,z')+\boldsymbol{\mu}(z')(1-\boldsymbol{\omega})^{-1}\boldsymbol{\nu}(z),\tag{32}$$

where we have introduced the following notation:

$$\mu_k(z):=\int_A\Psi(z',z)f_k(z')\,\mathrm{d}^2z',\quad\boldsymbol{\mu}=(\mu_1,\ldots,\mu_m):A\to M_{N\times mN}(\mathbb{C}),$$

$$\nu_k(z):=\int_A g_k(z')\Psi(z,z')\,\mathrm{d}^2z',\quad k=1,\ldots,m,\quad\boldsymbol{\nu}=\begin{pmatrix}\nu_1\\\vdots\\\nu_m\end{pmatrix}:A\to M_{mN\times N}(\mathbb{C}),$$

$$\omega_{\ell k}:=\int_{A\times A}g_\ell(z'')\Psi(z',z'')f_k(z')\,\mathrm{d}^2z'\,\mathrm{d}^2z'',\quad k,\ell=1,\ldots,m,\quad\boldsymbol{\omega}=(\omega_{\ell k})\in M_{mN\times mN}(\mathbb{C}).\tag{33}$$

We also showed the following.[14]

(1) The dressing transformations for the Baker function $\psi(z)$, adjoint Baker functions $\psi^*(z)$, and the matrix of rotation coefficients β are

$$\tilde{\psi}(z)=\psi(z)+\boldsymbol{\varphi}(1-\boldsymbol{\omega})^{-1}\boldsymbol{\nu}(z),$$

$$\tilde{\psi}^*(z)=\psi^*(z)+\boldsymbol{\mu}(z)(1-\boldsymbol{\omega})^{-1}\boldsymbol{\varphi}^*,$$

$$\tilde{\beta}=\beta+\boldsymbol{\varphi}(1-\boldsymbol{\omega})^{-1}\boldsymbol{\varphi}^*,$$

with

$$\boldsymbol{\varphi}:=(\varphi_1,\ldots,\varphi_m),\quad \varphi_k=\int_A \psi(z) f_k(z)\,\mathrm{d}^2z,$$

$$\boldsymbol{\varphi}^*:=\begin{pmatrix}\varphi_1^*\\ \vdots\\ \varphi_n^*\end{pmatrix},\quad \varphi_\ell^*=\int_A g_\ell(z)\psi^*(z)\,\mathrm{d}^2z.$$

(2) The dressed nets are then given by the corresponding rows of the matrix,

$$\tilde{\mathbf{x}}:=\mathbf{x}+\boldsymbol{\mathcal{M}}(1-\boldsymbol{\omega})^{-1}\boldsymbol{\mathcal{N}}, \tag{34}$$

where

$$\boldsymbol{\mathcal{M}}=\int_{\mathbb{C}}\mathcal{M}(z)\boldsymbol{\mu}(z)\,\mathrm{d}^2z,\quad \boldsymbol{\mathcal{N}}=\int_{\mathbb{C}}\boldsymbol{\nu}(z)\mathcal{N}(z)\,\mathrm{d}^2z. \tag{35}$$

2. Singular dressing

The $\bar{\partial}$-dressing generated by the separable kernel (31) has the following singular form. For every $W\in \mathrm{Gr}_{\gamma(r)}$ the corresponding $\tilde{W}\in \mathrm{Gr}_{\gamma(\tilde{r})}$, $\tilde{W}^*\in \mathrm{Gr}^*_{\gamma(\tilde{r})}$ are the sets of boundary values on $\gamma(\tilde{r})$ of matrix functions $w=w(z)$, $v=v(z)$, respectively, satisfying

$$\int_A w(z')f_k(z')\,\mathrm{d}^2z'=0,\quad \int_A g_k(z')v(z')\,\mathrm{d}^2z'=0,\quad k=1,\ldots,m,$$

and such that the restriction of w, v to $\gamma(r)$ is an element of W, W^*, respectively. Observe that if in the $\bar{\partial}$-dressing with a separable kernel we scale $f_k\mapsto s f_k$, $g_k\mapsto s g_k$ and let $s\to\infty$, we get the just-introduced singular dressing. This observation justifies the name given to this type of dressing. The Cauchy propagators Ψ and $\tilde{\Psi}$ associated with W and $\tilde{W}$, respectively, are related by

$$\tilde{\Psi}(z,z')=\Psi(z,z')-\boldsymbol{\mu}(z')\boldsymbol{\omega}^{-1}\boldsymbol{\nu}(z). \tag{36}$$

Hence, all formulas for the nonsingular dressing translates into this singular case by just performing the replacement $(I-\boldsymbol{\omega})^{-1}\to-\boldsymbol{\omega}^{-1}$. In particular,

$$\tilde{\mathbf{x}}:=\mathbf{x}-\boldsymbol{\mathcal{M}}\boldsymbol{\omega}^{-1}\boldsymbol{\mathcal{N}}. \tag{37}$$

Comments:

(i) These two dressing methods can be identified with iterated fundamental transformations of conjugate nets.[3,4] In particular, the $\bar{\partial}$-dressing constitutes a spectral interpretation of these transformations.

(ii) The dressing transformations presented here are maps from $\mathrm{Gr}_{\gamma(r)}\to\mathrm{Gr}_{\gamma(\tilde r)}$. However, both Grassmannians are obviously isomorphic as a subspace $\tilde W\in\mathrm{Gr}_{\gamma(\tilde r)}$ determines a unique subspace $\hat W\in\mathrm{Gr}_{\gamma(r)}=\{w(zr/\tilde r):w(z)\in\tilde W\}$. Hence, with this at hand it is clear that we can iterate these dressing transformations.

3. Dressing reduced nets

(a) Pseudo-orthogonal reduction: For the $\bar\partial$-dressing when the dressing kernel satisfies the condition

$$zI_{p,q}R(z,z')-z'R^{\mathrm{t}}(-z',-z)I_{p,q}=0, \tag{38}$$

then its corresponding dressing transformation preserves the pseudo-orthogonal reduction. Examples of separable kernels of this type are

$$R(z,z')=-\pi z'\sum_{k,l=1}^{m} f_k(z)B_{kl}I_{p,q}f_l^{\mathrm{t}}(-z')I_{p,q}, \tag{39}$$

with $B_{kl}\in M_N(\mathbb{C})$ and $B_{kl}I_{p,q}+I_{p,q}B_{lk}^{\mathrm{t}}=0$, which corresponds to the following identification

$$g_k(z)=-z\sum_{l=1}^{m} B_{kl}I_{p,q}f_l^{\mathrm{t}}(-z)I_{p,q}.$$

This is precisely the choice to take within the singular dressing in order to ensure the pseudo-orthogonal reduction after the transformation. However, now there is an essential difference: now the matrix B_{kl} is not constrained in any way.

Now we apply the different relations characterizing the pseudo-orthogonal reduction. First, from the differential relation (23) between Baker and adjoint Baker functions we deduce that we can relate the following *transformation data*

$$\varphi_k:=\int_A\psi(z)f_k(z)\,\mathrm{d}^2z \quad\text{and}\quad \varphi_k^*=-\int_A zI_{p,q}f_k^{\mathrm{t}}(-z)I_{p,q}\psi^*(z)\,\mathrm{d}^2z,$$

as follows:

$$\varphi_k^{*\mathrm{t}}I_{p,q}=I_{p,q}\partial\varphi_k+(\beta^{\mathrm{t}}I_{p,q}-I_{p,q}\beta)\varphi_k.$$

With the notation

$$\zeta_k(z):=-\int_A z'I_{p,q}f_{k'}^{\mathrm{t}}(-z')I_{p,q}\Psi(z,z')\,\mathrm{d}^2z',$$

$$\varpi_{kl}:=-\int_{A\times A} z'I_{p,q}f_k^{\mathrm{t}}(-z')I_{p,q}\Psi(z,z')f_l(z)\,\mathrm{d}^2z\mathrm{d}^2z',$$

we can write

$$\mu_k(z):=\int_A\Psi(z',z)f_k(z')\mathrm{d}^2z',\quad \nu_k(z):=-\sum_{k'=1}^{m}B_{kk'}\zeta_{k'}(z),$$

$$\psi_{kl}:=\sum_{k'=1}^{m}B_{kk'}\varpi_{k'l}.$$

Then, (24) implies the following relations:

$$z\boldsymbol{\mu}^{\rm t}(-z)I_{p,q}+I_{p,q}\boldsymbol{\zeta}(z)=\boldsymbol{\varphi}^{\rm t}I_{p,q}\psi(z), \tag{40}$$

$$I_{p,q}\boldsymbol{\varpi}+\boldsymbol{\varpi}^{\rm t}I_{p,q}=\boldsymbol{\varphi}^{\rm t}I_{p,q}\boldsymbol{\varphi}. \tag{41}$$

If we set $z=0$ in (40), then

$$\boldsymbol{\varphi}=\psi(0)I_{p,q}\boldsymbol{\zeta}^{\rm t}(0)I_{p,q}, \tag{42}$$

so that

$$\boldsymbol{\varphi}^{\rm t}I_{p,q}\boldsymbol{\varphi}=I_{p,q}\boldsymbol{\zeta}(0)I_{p,q}\boldsymbol{\zeta}^{\rm t}(0)I_{p,q}, \tag{43}$$

and we conclude

$$\boldsymbol{\varpi}I_{p,q}+I_{p,q}\boldsymbol{\varpi}^{\rm t}=\boldsymbol{\mathcal{N}}'I_{p,q}\boldsymbol{\mathcal{N}}'^{\rm t},\quad \boldsymbol{\mathcal{N}}'=:\int_{\mathbb{C}}\boldsymbol{\zeta}(z)\mathcal{N}(z)\,\mathrm{d}^2z=\boldsymbol{\zeta}(0)\mathcal{N}. \tag{44}$$

Then, the dressing is

$$\tilde{\mathbf{x}}:=\mathbf{x}+\boldsymbol{\mathcal{M}}(1-\mathbf{B}\boldsymbol{\varpi})^{-1}\mathbf{B}\boldsymbol{\mathcal{N}}',\quad \mathbf{B}=(B_{kl}).$$

From the above expressions we conclude the elementary singular dressing (corresponding to $m=1$ and $f_1=fP_i$ with f a scalar function) coincides with the well-known Ribaucour–Bianchi transformation of orthogonal nets[10,9] and both $\bar{\partial}$-dressing and singular dressing are generalizations of it.

(b) Symmetric reduction: When the dressing kernel satisfies the condition

$$I_{p,q}R(z,z')-R^{\rm t}(-z',-z)I_{p,q}=0, \tag{45}$$

its corresponding dressing transformation preserves the symmetric reduction. Separable kernels of this type are

$$R(z,z')=\pi\sum_{k,l=1}^{m}f_k(z)C_{kl}I_{p,q}f_l^{\rm t}(-z')I_{p,q}, \tag{46}$$

with $C_{kl}\in M_N(\mathbb{C})$ and $C_{kl}I_{p,q}-I_{p,q}C_{lk}^{\rm t}=0$, which corresponds to the following identification:

$$g_k(z)=\sum_{l=1}^{m}C_{kl}I_{p,q}f_l^{\rm t}(-z)I_{p,q}.$$

This is the proper choice to perform when dealing with the singular dressing in order to preserve the symmetric reduction but now the constant matrices C_{kl} are completely free.

As for the pseudo-orthogonal reduction we use the relations characterizing the symmetric reduction. First, with the differential relation (26) between Baker and adjoint Baker functions we derive for the following *transformation data*

$$\varphi_k:=\int_A\psi(z)f_k(z)\,\mathrm{d}^2z\quad\text{and}\quad\varphi_k^*=\int_A I_{p,q}f_k^{\rm t}(-z)I_{p,q}\psi^*(z)\,\mathrm{d}^2z,$$

where

$$\varphi_k^{*\rm t}I_{p,q}=I_{p,q}\varphi_k.$$

The notation

$$\zeta_k(z) := \int_A I_{p,q} f^{\mathrm{t}}_{k'}(-z') I_{p,q} \Psi(z,z')\, \mathrm{d}^2 z',$$

$$\varpi_{kl} := \int_{A\times A} I_{p,q} f^{\mathrm{t}}_{k}(-z') I_{p,q} \Psi(z,z') f_l(z)\, \mathrm{d}^2 z \mathrm{d}^2 z',$$

allows us to write

$$\mu_k(z) := \int_A \Psi(z',z) f_k(z')\, \mathrm{d}^2 z', \quad \nu_k(z) := \sum_{k'=1}^{m} C_{kk'} \zeta_{k'}(z),$$

$$\omega_{kl} := \sum_{k'=1}^{m} C_{kk'} \varpi_{k'l}.$$

Thus, (27) gives

$$I_{p,q} \boldsymbol{\mu}^{\mathrm{t}}(-z) = \boldsymbol{\zeta}(z) I_{p,q}, \tag{47}$$

$$I_{p,q} \boldsymbol{\varpi} = \boldsymbol{\varpi}^{\mathrm{t}} I_{p,q}. \tag{48}$$

Finally, the dressing is

$$\tilde{\mathbf{x}} := \mathbf{x} + \boldsymbol{\mathcal{M}}(1 - \mathbf{C}\boldsymbol{\varpi})^{-1} \mathbf{C} \boldsymbol{\mathcal{N}}', \quad \mathbf{C} = (C_{kl}), \boldsymbol{\mathcal{N}}' = \int_{\mathbb{C}} \boldsymbol{\zeta}(z) \boldsymbol{\mathcal{N}}(z)\, \mathrm{d}^2 z.$$

Comment: Observe that, apparently, in the singular dressing there is more freedom in the matrices B,C. However, this is not true in general, for if we assume, for example in the symmetric case, that the matrix $\mathbf{C} = (C_{kl})$ is invertible, then in the dressed Cauchy propagator there is no trace of these matrices; i.e., it is built up only in terms of the functions $f_k(z)$, while in the $\bar{\partial}$-dressing these constant matrices do not disappear at all. Of course, the situation changes when the matrix $\mathbf{C}$ is not invertible.

III. QUADRILATERAL LATTICES GENERATED BY DRESSING TRANSFORMATIONS

We discuss here how the dressing transformations for conjugate nets generate quadrilateral lattices. We finish discussing the backward representation[18] of the lattice in this context.

A. $\bar{\partial}$-dressing and quadrilateral lattices

We shall consider within the N-component KP hierarchy a set of N $\bar{\partial}$-dressing transformations with the jth transformation, say T_j, being generated by the following separable kernel:

$$R^{(i)}(z,z') := \pi a^{(i)}(z) P_i b^{(i)}(z'),$$

with $a^{(i)}$ and $b^{(i)}$ scalar spectral distributions. The Cauchy propagator $\Psi(z,z')$ satisfies

$$T_i \Psi(z,z') = \Psi(z,z') + \boldsymbol{\alpha}^{(i)}(z') \frac{1}{1 - \Omega^{(ii)}_{ii}} \boldsymbol{\beta}^{(i)}(z)$$

with

$$\boldsymbol{\alpha}^{(i)}(z) := \int_A \Psi(z',z) a^{(i)}(z')\, \mathrm{d}^2 z' e^{\mathrm{t}}_i,$$

$$\boldsymbol{\beta}^{(i)}(z) := e_i \int_A b^{(i)}(z')\Psi(z,z')\,\mathrm{d}^2 z',$$

$$\Omega^{(ij)} := \int_{A\times A} b^{(i)}(z')\Psi(z,z')a^{(j)}(z)\,\mathrm{d}^2 z\mathrm{d}^2 z',$$

or, in terms of the shift operator $\Delta_i := T_i - 1$,

$$\Delta_i \Psi(z,z') = \boldsymbol{\alpha}^{(i)}(z')\frac{1}{1-\Omega_{ii}^{(ii)}}\boldsymbol{\beta}^{(i)}(z). \tag{49}$$

The transformations T_i are fundamental transformations and, following Jonas and Eisenhart,[2,3,1] their composition must generate a quadrilateral lattice. We can show explicit expressions of the geometrical objects defining the net in terms of the Cauchy propagator. For that aim we introduce the following definition.

Definition 1:

$$T_j Q_{ij} := \frac{\Omega_{ij}^{(ij)}}{1-\Omega_{jj}^{(jj)}}, \quad T_i \mathbf{h}_i := \frac{1}{1-\Omega_{ii}^{(ii)}}\int_{\mathbb{C}} \mathcal{M}(z)\boldsymbol{\alpha}^{(i)}(z)\,\mathrm{d}^2 z, \quad \mathfrak{C}_i := \int_{\mathbb{C}} \boldsymbol{\beta}^{(i)}(z)\mathcal{N}(z)\,\mathrm{d}^2 z.$$

Then, we have the following proposition.

Proposition 4: The following equation holds:

$$\Delta_i \mathbf{x} = (T_i \mathbf{h}_i)\mathfrak{C}_i. \tag{50}$$

The vectors $\mathfrak{C}_i$ are the Comberscure vectors of the fundamental transformation T_i. Assuming that the set of Comberscure vectors $\{\mathfrak{C}_i\}_{i=1}^N$ is linearly independent and also that $T_i f = 0$ implies $f=0$, we have the following.

Proposition 5: The Comberscure vectors $\mathfrak{C}_i$, $i=1,\ldots,N$, satisfy

$$\Delta_j \mathfrak{C}_i = (T_j Q_{ij})\mathfrak{C}_j, \quad i,j=1,\ldots,N, \quad i\neq j;$$

the coefficients Q_{ij} satisfy the discrete Darboux equations

$$\Delta_k Q_{ij} = (T_k Q_{ik})Q_{kj}, \quad i,j,k=1,\ldots,N, \quad i,j,k \text{ different.}$$

and $\mathbf{h}_j$ fulfill

$$\Delta_j \mathbf{h}_i = (T_j \mathbf{h}_j)Q_{ji}, \quad i,j=1,\ldots,N, \quad i\neq j.$$

Proof: The first assertion follows from

$$\begin{aligned}\Delta_j \mathfrak{C}_i &= e_i \int_{\mathbb{C}\times A} b^{(i)}(z')(\Delta_j \Psi(z,z'))\mathcal{N}(z)\,\mathrm{d}^2 z\mathrm{d}^2 z' \\ &= \left[\int_A \frac{b^{(i)}(z')\alpha_i^{(j)}(z')}{1-\Omega_{jj}^{(jj)}}\mathrm{d}^2 z'\right]\left[\int_{\mathbb{C}} \beta^{(j)}(z)\mathcal{N}(z)\,\mathrm{d}^2 z\right] = (T_j Q_{ij})\mathfrak{C}_j.\end{aligned}$$

The second is the compatibility condition for the just-deduced equation. That is, $\Delta_k(\Delta_j \mathfrak{C}_i) = \Delta_j(\Delta_k \mathfrak{C}_i)$ so that

$$[T_k((\Delta_j Q_{ij}) - (T_j Q_{ij})Q_{jk})]\mathfrak{C}_k + [T_j((\Delta_k Q_{ik}) - (T_k Q_{ik})Q_{kj})]\mathfrak{C}_j = 0,$$

and using the linear independence of $\mathfrak{C}_j$, $\mathfrak{C}_k$ and the invertibility of T_j and T_k, we deduce the discrete Darboux equations. The compatibility of (50), $\Delta_k(\Delta_j \mathbf{x}) = \Delta_j(\Delta_k \mathbf{x})$, implies

$$[T_k((\Delta_j\mathbf{h}_k)-(T_j\mathbf{h}_j)Q_{jk})]\mathfrak{C}_k+[T_j((\Delta_k\mathbf{h}_j)-(T_k\mathbf{h}_k)Q_{kj})]\mathfrak{C}_j=0,$$

and as before we deduce the linear system for $\mathbf{h}$. □

We have just shown that the lth row $\mathbf{x}_l$ of $\mathbf{x}$ has the property that the vectors $\{\mathbf{x}_l, T_i\mathbf{x}_l, T_j\mathbf{x}_l, T_iT_j\mathbf{x}_l\}$ form a planar quadrilateral. To build up the quadrilateral lattice we need to give a meaning to $T_i^{n_i}$ with $n_i\in\mathbb{Z}$: first, for $n_i=1$ we identify $T_i^1=T_i$; second, to construct $T_i^{n_i}\mathbf{x}_{(l)}$ we just iterate $|n_i|$ $\bar\partial$-dressing transformations with kernels $\pi a_j^{(i)}(z)P_ib_j^{(i)}(z')$, $j=1,...,n_i$. In each step we are sure that we are constructing a quadrilateral strip of the lattice. Thus, we deduce the following.

Theorem 1: *The rows of the matrix*

$$\mathbf{x}(n_1,\dots,n_N):=(T_1^{n_1}\dots T_N^{n_N})\mathbf{x}(\mathbf{t}_0),\quad n_1,\dots,n_N\in\mathbb{Z}$$

are parallel quadrilateral lattices with renormalized tangent vectors $\{\mathfrak{C}_i\}_{i=1}^N$, *Lamé coefficients of the* l*th lattice given by* $(\mathbf{h}_i)_l$ *and rotation coefficients* Q_{ij}.

B. Singular dressing and quadrilateral lattices

Analogous considerations as above lead to the same conclusions; i.e., the generation of quadrilateral lattices, as in the $\bar\partial$-dressing but with the new definitions.

Definition 2: Discrete rotation coefficients and Lamé coefficients are given by

$$T_jQ_{ij}:=-\frac{\Omega_{ij}^{(ij)}}{\Omega_{jj}^{(jj)}},\quad T_i\mathbf{h}_i:=-\frac{1}{\Omega_{ii}^{(ii)}}\int_{\mathbb{C}}\mathcal{M}(z)\boldsymbol{\alpha}^{(i)}(z)\,\mathrm{d}^2z,\quad \mathfrak{C}_i:=\int_{\mathbb{C}}\boldsymbol{\beta}^{(i)}(z)\mathcal{N}(z)\,\mathrm{d}^2z.$$

It is particularly relevant, as we shall see, in relation with the pseudo-orthogonal, pseudo-symmetric, and pseudo-Egorov reductions.

C. Backward representation of the generated quadrilateral lattice

For the $\bar\partial$-dressing generated quadrilateral lattice we introduce the following.

Definition 3: Discrete backward rotation coefficients and Lamé coefficients are given by

$$T_i\hat{Q}_{ji}:=\frac{\Omega_{ij}^{(ij)}}{1-Q_{ii}^{(ii)}},\quad \hat{\mathbf{h}}_i:=-\int_{\mathbb{C}}\mathcal{M}(z)\boldsymbol{\alpha}^{(i)}(z)\,\mathrm{d}^2z,\quad T_i\hat{\mathfrak{C}}_i:=-\frac{1}{1-\Omega_{ii}^{(ii)}}\int_{\mathbb{C}}\boldsymbol{\beta}^{(i)}(z)\mathcal{N}(z)\,\mathrm{d}^2z,$$

and then

Proposition 6: The backward coefficients just introduced are subject to the following relations:

$$\Delta_k\hat{Q}_{ij}=(T_k\hat{Q}_{ik})\hat{Q}_{kj},\quad i,j,k=1,...,N, i,j,k\ \ \textit{different},$$

$$\Delta_j\hat{\mathbf{h}}_i=\hat{\mathbf{h}}_j(T_j\hat{Q}_{ij}),\quad i,j=1,...,N, i\neq j,\quad \Delta_j\hat{\mathfrak{C}}_i=\hat{Q}_{ji}T_j\hat{\mathfrak{C}}_j,\quad i,j=1,...,N, i\neq j.$$

Proof: First we compute

$$\begin{aligned}\Delta_j\hat{\mathbf{h}}_i&=\int_{\mathbb{C}\times A}\mathcal{M}(z')(\Delta_j\Psi(z,z'))a^{(i)}(z)\,\mathrm{d}^2z'\,\mathrm{d}^2z e_i^{\mathrm{t}}\\&=\left[\int_{\mathbb{C}}\mathcal{M}(z')\boldsymbol{\alpha}^{(j)}(z')\mathrm{d}^2z'\right]\left[\int_{A\times A}\frac{b^{(j)}(z')\Psi_{ji}(z,z')a^{(i)}(z)}{1-\Omega_{ii}^{(ii)}}\mathrm{d}^2z\right]\\&=(T_j\hat{Q}_{ij})\hat{\mathbf{h}}_j.\end{aligned}$$

Then, the compatibility of this equation, $\Delta_i\Delta_j\mathbf{h}_k=\Delta_j\Delta_i\mathbf{h}_k$, gives the discrete Darboux equations for $\hat{Q}_{ij}$. We also observe that

$$\Delta_i\mathbf{x}=\hat{\mathbf{h}}_i T_i\hat{\mathfrak{C}}_i$$

and its compatibility gives the linear system for $\hat{\mathfrak{C}}_i$. □

This is the backward representation of the quadrilateral lattice as explained in Ref. 18. Moreover, the functions ρ_i introduced there can be identified with

$$\rho_i=\Omega_{ii}^{(ii)}-1. \tag{51}$$

If instead of the $\bar{\partial}$-dressing one uses the singular dressing, one only needs to perform the replacement $1-\Omega_{ii}^{(ii)}\to-\Omega_{ii}^{(ii)}$.

For a further check we can evaluate

$$\begin{aligned}T_j\rho_i&=\rho_i+\int_{A\times A}b^{(i)}(z')(\Delta_j\Psi_{ii}(z,z'))a^{(i)}(z)\,\mathrm{d}^2z\mathrm{d}^2z'\\&=\rho_i\bigg[\int_A b^{(i)}(z')\alpha_i^{(j)}(z')\,\mathrm{d}^2z'\bigg]\frac{1}{\rho_j}\bigg[\int_A\beta_i^{(j)}(z)a^{(i)}(z)\,\mathrm{d}^2z\bigg]\\&=\rho_i-\Omega_{ij}^{(ij)}\frac{1}{\rho_j}\Omega_{ji}^{(ji)}=\rho_i-(T_jQ_{ji})\rho_i(T_iQ_{ij}),\end{aligned}$$

so that we obtain (2.8) of Ref. 18, from where it is derived the τ-function expression $\rho_i=T_i\tau/\tau$.[18]

IV. PSEUDO-CIRCULAR, PSEUDO-SYMMETRIC, AND PSEUDO-EGOROV LATTICES GENERATED BY SINGULAR DRESSING TRANSFORMATIONS

The quadrilateral lattice generated by $\bar{\partial}$-dressing does not admit the pseudo-circular or pseudo-symmetric reductions. This was the main motivation for us to consider the singular dressing which indeed allows for both reductions. Thus, we will use it to generate reduced lattices and show that they are of the pseudo-circular, symmetric, and pseudo-Egorov types, respectively. For the standard case $I_{p,q}=I_N$ this was anticipated by Demoulin[8] and Bianchi[9] and we are following the path suggested in Ref. 7.

A. Singular dressing and circular lattices

We consider the pseudo-orthogonal reduction and take the singular dressing for generating quadrilateral lattices with

$$b^{(i)}(z)=-za^{(i)}(-z).$$

In this way it is ensured that the dressing transformation is a Ribaucour–Bianchi transformation of the corresponding pseudo-orthogonal net.

From (24) we deduce

$$I_{p,q}\Omega^{(ij)}+\Omega^{(ji)\mathrm{t}}I_{p,q}=\phi^{(i)\mathrm{t}}I_{p,q}\phi^{(i)}=I_{p,q}\bigg[\int_A b^{(i)}(z)\Psi(0,z)\,\mathrm{d}^2z\bigg]I_{p,q}\bigg[\int_A b^{(i)}(z)\Psi(0,z)\,\mathrm{d}^2z\bigg]^{\mathrm{t}}I_{p,q}, \tag{52}$$

where

$$\phi^{(i)}=\int_A\psi(z)a^{(i)}(z)\,\mathrm{d}^2z.$$

As we choose $\mathcal{N}(z)=\mathcal{N}\delta(z)$, $\mathcal{N}\in O(p,q)$, we have for the Comberscure vectors

$$\mathfrak{C}_i=\beta^{(i)}(0)\mathcal{N}.$$

Proposition 7: The Comberscure $\{\mathfrak{C}_i\}_{i=1}^N$ *vectors fulfill*

$$\langle\mathfrak{C}_i,T_i\mathfrak{C}_j\rangle+\langle\mathfrak{C}_j,T_j\mathfrak{C}_i\rangle=0,\quad i\neq j.$$

Proof: From (52) it follows that

$$\Omega_{ij}^{(ij)}\epsilon_j+\epsilon_i\Omega_{ji}^{(ji)}=\langle\mathfrak{C}_i,\mathfrak{C}_j\rangle. \tag{53}$$

As we also have

$$T_j\mathfrak{C}_i=\mathfrak{C}_i-2\,\frac{\Omega_{ij}^{(ij)}}{\epsilon_j\langle\mathfrak{C}_j,\mathfrak{C}_j\rangle}\,\mathfrak{C}_j,$$

we conclude

$$\langle\mathfrak{C}_i,T_i\mathfrak{C}_j\rangle+\langle\mathfrak{C}_j,T_j\mathfrak{C}_i\rangle=2(\langle\mathfrak{C}_i,\mathfrak{C}_j\rangle-\Omega_{ij}^{(ij)}\epsilon_j-\epsilon_i\Omega_{ji}^{(ji)})=0.$$

□

Hence, following Ref. 18, for $p=N$ and $q=0$ (the Euclidean case) the **corresponding quadrilateral lattice is circular**. Thus, we are dealing with extensions of the circular lattice to pseudo-Euclidean spaces.

Observe that (53) implies which, in particular implies, for the ρ_i function introduced in (51),

$$\rho_i=\tfrac{1}{2}\langle\mathfrak{C}_i,\mathfrak{C}_i\rangle.$$

As in the construction of the lattice we need to have a well-defined inverse ρ_i^{-1}. We require that

$$\langle\mathfrak{C}_i,\mathfrak{C}_i\rangle\neq 0.$$

B. Singular dressing and symmetric lattices

For the symmetric reduction the associated quadrilateral lattice is generated taking

$$b^{(i)}(z)=a^{(i)}(-z),$$

which ensures that in each step the symmetric reduction is preserved. In this case we have

$$I_{p,q}\Omega^{(ij)}=\Omega^{(ji)\mathrm{t}}I_{p,q}.$$

In particular,

$$\epsilon_i\Omega_{ij}^{(ij)}=\Omega_{ji}^{(ji)}\epsilon_j,$$

which gives, using Definition 2, the relation

$$\epsilon_i\rho_jT_jQ_{ij}=\epsilon_j\rho_iT_iQ_{ji},$$

or

$$\epsilon_j\hat{Q}_{ij}=Q_{ij}\epsilon_i.$$

After rescaling of the backward coefficients[18] $\hat{Q}_{ij}\rightarrow\epsilon_j\epsilon_j\hat{Q}_{ij}$ we get a **symmetric lattice** as introduced in Ref. 18.

A further check of this fact is the following.

Proposition 8: The discrete rotation coefficients do satisfy

$$(T_iQ_{ji})(T_jQ_{kj})(T_kQ_{ik})=(T_jQ_{ij})(T_iQ_{ki})(T_kQ_{jk}),\quad i,j \text{ and } k \text{ different.}$$

Proof: Replacing the explicit form of the discrete rotation coefficients given in Definition 2 we have

$$\begin{aligned}&(T_iQ_{ji})(T_jQ_{kj})(T_kQ_{ik})-(T_jQ_{ij})(T_iQ_{ki})(T_kQ_{jk})\\&=\left(-\frac{\Omega_{ji}^{(ji)}}{\Omega_{ii}^{(ii)}}\right)\left(-\frac{\Omega_{kj}^{(kj)}}{\Omega_{jj}^{(jj)}}\right)\left(-\frac{\Omega_{ik}^{(ik)}}{\Omega_{kk}^{(kk)}}\right)-\left(-\frac{\Omega_{ij}^{(ij)}}{\Omega_{jj}^{(jj)}}\right)\left(-\frac{\Omega_{ki}^{(ki)}}{\Omega_{ii}^{(ii)}}\right)\left(-\frac{\Omega_{jk}^{(jk)}}{\Omega_{kk}^{(kk)}}\right),\end{aligned}$$

which, using (8), is easily shown to vanish as desired. □

C. Singular dressing and pseudo-Egorov lattices

We now take the pseudo-Egorov reduction and we generate the quadrilateral lattice with

$$a^{(i)}(z)=\delta(z-1),\quad b^{(i)}(z)=\delta(z+1),$$

which ensures that Egorov reduction is preserved—this corresponds to Miwa transformations of the CBKP hierarchy. In this case we have

$$2\epsilon_j\Omega_{ij}^{(ij)}=\langle\mathfrak{C}_i,\mathfrak{C}_j\rangle. \tag{54}$$

Proposition 9: The Comberscure $\{\mathfrak{C}_i\}_{i=1}^N$ vectors fulfill

$$\langle\mathfrak{C}_i,T_i\mathfrak{C}_j\rangle=0,\quad i\neq j. \tag{55}$$

Proof: From

$$T_j\mathfrak{C}_i=\mathfrak{C}_i-2\epsilon_j\frac{\Omega_{ij}^{(ij)}}{\langle\mathfrak{C}_j,\mathfrak{C}_j\rangle}$$

and (54) we derive

$$T_j\mathfrak{C}_i=\mathfrak{C}_i-\frac{\langle\mathfrak{C}_i,\mathfrak{C}_j\rangle}{\langle\mathfrak{C}_j,\mathfrak{C}_j\rangle}\mathfrak{C}_j.$$

Hence, the desired statement follows. □

Again, for the case $p=N$ and $q=0$, we conclude that **the quadrilateral lattice happens to be of Egorov type** (following Ref. 18 and Schief). Hence, (55) is our definition of **pseudo-Egorov lattice**.

V. INDUCING DRESSING TRANSFORMATIONS ON LATTICES

Either the $\bar{\partial}$-dressing or the singular dressing, of the Cauchy propagator (33), (32), and (36), as fundamental transformations—say $\mathcal{F}$—provide new conjugate nets $\tilde{\mathbf{x}}(\mathbf{t})$ from known ones $\mathbf{x}(\mathbf{t})$. We also have shown how to generate, using basic dressing transformations, quadrilateral lattices from them, say $\tilde{\mathbf{x}}(\mathbf{n},\mathbf{t})$ and $\mathbf{x}(\mathbf{n},\mathbf{t})$. The important permutability property of these fundamental transformations gives the following commutative diagram

$$\begin{array}{ccc}\mathbf{x}(\mathbf{t}) & \xrightarrow{\mathcal{F}} & \tilde{\mathbf{x}}(\mathbf{t})\\ {\scriptstyle T_i}\big\downarrow & & {\scriptstyle T_i}\big\downarrow\\ \mathbf{x}(\mathbf{n},\mathbf{t}) & \xrightarrow{\mathcal{F}} & \tilde{\mathbf{x}}(\mathbf{n},\mathbf{t})\end{array}.$$

Furthermore, the transformation $\mathbf{x}(\mathbf{n},\mathbf{t}) - \tilde{\mathbf{x}}(\mathbf{n},\mathbf{t})$, as we shall see, is a discrete fundamental transformation in the sense of Refs. 1 and 19. To check this point, first recall, either with the $\bar{\partial}$-dressing or with the singular dressing, formulas (34), (35), and (37). Then, (49) implies for the *transformation data*,

$$\Phi_{ij} := \int_A \beta^{(i)}(z) f_j(z) \mathrm{d}^2 z, \quad T_i \Phi^*_{ji} := \begin{cases} \displaystyle\int_A g_j(z) \frac{\alpha^{(i)}(z)}{1-\Omega^{(ii)}_{ii}} \mathrm{d}^2 z, & \bar{\partial}\text{-dressing,} \\ \displaystyle -\int_A g_j(z) \frac{\alpha^{(i)}(z)}{\Omega^{(ii)}_{ii}} \mathrm{d}^2 z, & \text{singular dressing,} \end{cases}$$

the following:

Proposition 10: The transformation data Φ_{ij}, Φ_{ji} *and the potentials* $\mathcal{M}_j$, $\mathcal{N}_j$ *and* ω_{ij} *are linked by*

$$\Delta_j \Phi_{ik} = (T_j Q_{ij}) \Phi_{jk}, \quad \Delta_j \Phi^*_{ki} = Q_{ji} T_j \Phi^*_{kj},$$

$$\Delta_i \mathcal{M}_j = (T_i \mathbf{h}_i) \Phi_{ij}, \quad \Delta_i \mathcal{N}_j = (T_i \Phi^*_{ji}) \mathfrak{C}_i, \quad \Delta_i \omega_{kl} = (T_i \Phi^*_{ki}) \Phi_{il}.$$

Proof: Although the proof given here is for the $\bar{\partial}$-dressing, a similar proof holds for the singular dressing case.

We first evaluate

$$\begin{aligned} \Delta_j \Phi_{ik} &= e_i \int_{A\times A} b^{(i)}(z') (\Delta_j \Psi(z,z')) f_k(z) \, \mathrm{d}^2 z \mathrm{d}^2 z' \\ &= \left[\int_A \frac{b^{(i)}(z') \alpha_i^{(j)}(z')}{1-\Omega^{(jj)}_{jj}} \right] \left[\int_A \beta^{(j)}(z) f_k(z) \right] = (T_j Q_{ij}) \Phi_{jk}. \end{aligned}$$

We continue with

$$\begin{aligned} \Delta_i \mathcal{M}_j &= \int_A \mathcal{M}(z) (\Delta_i \Psi(z',z)) f_j(z') \, \mathrm{d}^2 z \mathrm{d}^2 z' \\ &= \left[\int_A \frac{\mathcal{M}(z)}{1-\Omega^{(ii)}_{ii}} \mathrm{d}^2 z \right] \left[\int_A \beta^{(i)}(z') f_j(z') \, \mathrm{d}^2 z' \right] = (T_i \mathbf{h}_i) \Phi_{ij}, \end{aligned}$$

and

$$\begin{aligned} \Delta_i \mathcal{N}_j &= \int_A g_j(z') (\Delta_i \Psi(z,z')) \mathcal{N}(z) \, \mathrm{d}^2 z \mathrm{d}^2 z' \\ &= \left[\int_A \frac{g_j(z') \alpha^{(i)}(z')}{1-\Omega^{(ii)}_{ii}} \mathrm{d}^2 z \right] \left[\int_A \beta^{(i)}(z') \mathcal{N}(z') \, \mathrm{d}^2 z' \right] = (T_i \Phi^*_{ji}) \mathfrak{C}_i. \end{aligned}$$

Now, from the compatibility condition $\Delta_i \Delta_j \mathcal{N}_k = \Delta_j \Delta_i \mathcal{N}_k$ it follows that

$$\Delta_j \Phi^*_{ki} = Q_{ji} T_j \Phi^*_{kj}.$$

Finally,

$$\Delta_i\omega_{kl}=\int_{A\times A}g_k(z)(\Delta_i\Psi(z,z'))f_l(z')\,\mathrm{d}^2z\mathrm{d}^2z'$$

$$=\left[\int_A\frac{g_k(z)\alpha^{(i)}(z)}{1-\Omega_{ii}^{(ii)}}\mathrm{d}^2z\right]\left[\int_A\beta^{(i)}(z')f_l(z')\,\mathrm{d}^2z'\right]=(T_i\Phi_{ki}^*)\Phi_{il}\,.$$

□

As a corollary we conclude that the induced transformations **are discrete vectorial fundamental transformations of the quadrilateral lattice**.[1,19]

A. Pseudo-circular lattices

Given a pseudo-orthogonal net we know how to generate a pseudo-circular lattice and also how construct new pseudo-orthogonal nets via dressing, which happens to be a generalized Ribaucour–Bianchi transformation. The permutability of these transformations lead us to conclude that the generalized Ribaucour–Bianchi transformation generates a transformation preserving the pseudo-circularity property. In fact, as we shall show here, **we are dealing with a generalized discrete Ribaucour–Bianchi transformation**, in the sense of Refs. 20, 37, and 22. For that aim it is enough to check the following.

Proposition 11: The transformation data Φ_{ij}, Comberscure vectors $\mathfrak{C}_j$, and the potentials $\mathcal{N}_j'$ and ϖ_{ij} are linked by

$$\Phi_{ij}=\tfrac{1}{2}\mathfrak{C}_iI_{p,q}(\mathcal{N}_j'+T_i\mathcal{N}_j')^{\mathrm{t}}I_{p,q}\,,$$

$$\varpi_{lk}I_{p,q}+I_{p,q}\varpi_{kl}^{\mathrm{t}}=\mathcal{N}_l'I_{p,q}\mathcal{N}_k'^{\,t}\,.$$

Proof: The latter equality is (44). For the former we first notice that

$$\mathfrak{C}_iI_{p,q}(\mathcal{N}_j'+T_i\mathcal{N}_j')^{\mathrm{t}}=\boldsymbol{\beta}^{(i)}(0)I_{p,q}(\zeta_j(0)+T_i\mathfrak{s}_j(0))^{\mathrm{t}}$$

and

$$\boldsymbol{\beta}^{(i)}(0)=\left[e_i\int_A I_{p,q}\psi^{\mathrm{t}}(z)a^{(i)}(z)\,\mathrm{d}^2z\right]I_{p,q}\psi(0),\qquad \mathfrak{s}_j^{\mathrm{t}}(0)=I_{p,q}\psi^{\mathrm{t}}(0)\int_A\psi(z)f_j(z)\,\mathrm{d}^2zI_{p,q}\,,$$

so that

$$\boldsymbol{\beta}^{(i)}(0)I_{p,q}\mathfrak{s}_j^{\mathrm{t}}(0)=\left[e_iI_{p,q}\int_A\psi^{\mathrm{t}}(z)a^{(i)}(z)\,\mathrm{d}^2z\right]I_{p,q}\left[\int_A\psi(z)f_j(z)\,\mathrm{d}^2z\right]I_{p,q}\,.$$

Second, using again (24) we have

$$\Phi_{ij}=\left[e_iI_{p,q}\int_A\psi^{\mathrm{t}}(z)a^{(i)}(z)\right]I_{p,q}\left[\int_A\psi(z)f_j(z)\right]$$

$$-e_i\left[\int_{A\times A}(-zI_{p,q}f_j^{\mathrm{t}}(-z)I_{p,q})\Psi(z',z)a^{(i)}(z')\,\mathrm{d}^2z\mathrm{d}^2z'\right]^{\mathrm{t}}I_{p,q}$$

$$=\boldsymbol{\beta}^{(i)}(0)I_{p,q}\mathfrak{s}_j^{\mathrm{t}}(0)I_{p,q}+\rho_i(T_i\Phi_{ji}^*)^{\mathrm{t}}I_{p,q}\,,$$

where, in this reduced case, we define

$$T_i\Phi_{ji}^*:=\int_A(-zI_{p,q}f_j^{\mathrm{t}}(-z)I_{p,q})\frac{\alpha^{(i)}(z)}{\Omega_{ii}^{(ii)}}\mathrm{d}^2z.$$

In third place, recalling that $\Delta_i \mathfrak{s}_j^t(0)=\boldsymbol{\beta}^{(i)}(T_i\Phi_{ji}^*)^t$, we find

$$\tfrac{1}{2}\boldsymbol{\beta}^{(i)}(0)I_{p,q}\Delta_j\mathfrak{s}_j'(0)=\rho_i(T_i\Phi_{ji}^*)^t.$$

Finally, collecting these results we arrive at

$$\Phi_{ij}=\boldsymbol{\beta}^{(i)}(0)I_{p,q}[\zeta_j^{\mathrm{t}}(0)+\tfrac{1}{2}(T_i\mathfrak{s}_j^{\mathrm{t}}(0)-\mathfrak{s}_j^{\mathrm{t}}(0))],$$

from where we conclude the first relation. □

B. Symmetric lattices

The same arguments as for the pseudo-circular lattice lead us to conclude that we are generating a discrete fundamental transformation that preserves the symmetric constraint.

In fact, we can derive the following characteristic equations

Proposition 12: The transformation data Φ_{ij}, Φ_{ji} *and the potentials* ρ_j *and* ϖ_{ij} *are linked by*

$$\Phi_{ij}=\rho_i(T_i\Phi_{ji}^*)^{\mathrm{t}},$$

$$\varpi_{kl}=\varpi_{lk}^{\mathrm{t}}.$$

Proof: The second equation is just (48). The first derives from the definition of the transformation data and (27) □

Indeed, one can show that **these are the only two requirements to impose to the fundamental transformation to preserve the symmetric reduction**.

ACKNOWLEDGMENT

This work was partially supported by Direction General de Enseñanza Superior e Investigación Científica proyecto PB98-0821

[1] A. Doliwa, P. M. Santini, and M. Mañas, J. Math. Phys. **41**, 944 (2000).
[2] H. Jonas, Berl. Math. Ges. Ber. Sitzungsber. **14**, 96 (1915).
[3] L. P. Eisenhart, Trans. Am. Math. Soc. **18**, 111 (1917).
[4] L. P. Eisenhart, *Transformations of Surfaces* (Princeton University Press, Princeton, NJ, 1923) [reprint (Chelsea, New York, 1962)].
[5] D. Levi and R. Benguria, Proc. Natl. Acad. Sci. U.S.A. **77**, 5025 (1980).
[6] J. J. C. Nimmo and W. K. Schief, Proc. R. Soc. London, Ser. A **453**, 255 (1997).
[7] J. Cieśliński, A. Doliwa, and P. M. Santini, Phys. Lett. A **235**, 480 (1997).
[8] M. A. Demoulin, C. R. Acad. Sci. Paris **150**, 156 (1910).
[9] L. Bianchi, *Lezioni di Geometria Differenziale* 3rd ed. (Zanichelli, Bologna, 1924).
[10] A. Ribaucour, C. R. Acad. Sci. Paris **74**, 1489 (1869).
[11] A. Doliwa, M. Mañas, L. Martínez Alonso, E. Medina, and P. M. Santini, J. Phys. A **32**, 1197 (1999).
[12] M. Mañas and L. Martínez Alonso, Phys. Lett. B **436**, 316 (1998).
[13] A. Doliwa, M. Mañas, and Luis Martínez Alonso, Phys. Lett. A **262**, 330 (1999).
[14] M. Mañas, L. Martínez Alonso, and E. Medina, J. Phys. A **33**, 2871 (2000).
[15] M. Mañas, L. Martínez Alonso, and E. Medina, J. Phys. A **33**, 7181 (2000).
[16] G. Segal and G. Wilson, Publ. Math. I. H. E. S. **61**, 5 (1985).
[17] E. Witten, Commun. Math. Phys. **113**, 529 (1988).
[18] A. Doliwa and P. M. Santini, J. Geom. Phys. **36**, 60 (2000).
[19] M. Mañas, A. Doliwa, and P. M. Santini, Phys. Lett. A **232**, 365 (1997).
[20] B. G. Konopelchenko and W. K. Schief, Proc. R. Soc. London, Ser. A **454**, 3075 (1998).
[21] Q. P. Liu and M. Mañas, J. Phys. A **31**, L193 (1998).
[22] A. Doliwa, J. Geom. Phys. **30**, 169 (2000).
[23] V. E. Zakharov and S. E. Manakov, Funct. Anal. Appl. **19**, 89 (1985).
[24] V. E. Zakharov, Duke Math. J. **94**, 103 (1998); V. E. Zakharov and S. E. Manakov, Dokl. Math. **57**, 471 (1998).
[25] G. Darboux, *Leçons sur la théorie générale des surfaces IV* (Gauthier-Villars, Paris, 1896) [reprint (Chelsea, New York, 1972)].
[26] G. Lamé, *Leçons sur la théorie des coordenées curvilignes et leurs diverses applications* (Mallet-Bachalier, Paris 1859).
[27] G. Darboux, *Leçons sur les systèmes orthogonaux et les coordenées curvilignes (deuxième édition)* (Gauthier-Villars, Paris, 1910) (the first edition was in 1897) [reprint (Éditions Jacques Gabay, Sceaux 1993)].
[28] D.-Th. Egorov, C. R. Acad. Sci. Paris **131**, 668 (1900); **132**, 174 (1901).

[29] B. Dubrovin, Nucl. Phys. B **79**, 627 (1992); B. Dubrovin and Y. Zhang, Commun. Math. Phys. **198**, 311 (1998).
[30] E. Witten, Nucl. Phys. B **340**, 281 (1990).
[31] E. P. Lane, *Projective Differential Geometry of Curves and Surfaces* (University of Chicago, Chicago, 1932).
[32] R. Sauer, *Differenzengeometrie* (Springer-Verlag, Berlin, 1970).
[33] A. Doliwa and P. M. Santini, Phys. Lett. A **233**, 365 (1997).
[34] R. R. Martin, in *The Mathematics of Surfaces*, edited by J. Gregory (Clarendon, Oxford, 1986); A. W. Nutborne, in *The Mathematics of Surfaces*, edited by J. Gregory (Clarendon, Oxford, 1986).
[35] A. Bobenko in *Symmetries and Integrability of Difference Equations II*, edited by P. Clarckson and F. Nijhoff (Cambridge University Press, Cambridge, 1996); A. Bobenko and U. Pinkall, J. Diff. Geom. **43**, 527 (1996); J. Reine Angew. Math. **475**, 187 (1996); A. Bobenko and W. K. Schief, in *Discrete Integrable Geometry and Physics*, edited by A. Bobenko and R. Seiler (Oxford University Press, Oxford, 1998).
[36] A. Doliwa, S. V. Manakov, and P. M. Santini, Commun. Math. Phys. **196**, 1 (1998).
[37] Q. P. Liu and M. Mañas, Phys. Lett. A **249**, 424 (1998).
[38] P. G. Grinevich and A. Yu. Orlov, *Virasoro Action on Riemann Surfaces, Grassmannians*, det $\overline{\partial}_J$*. and Segal-Wilson* τ*-Function in Problems of Modern Quantum Field Theory*, edited by A. A. Belavin, A. V. Klimyk, and A. B. Zamolodchikov (Springer, Berlin, 1989).
[39] P. G. Grinevich and A. Yu. Orlov, Funct. Anal. Appl. **24**, 61 (1990); L. V. Bogdanov and B. G. Konopelchenko, J. Math. Phys. **39**, 4701 (1998).