

Vectorial Darboux Transformations for the Kadomtsev-Petviashvili Hierarchy

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Abstract

We consider the vectorial approach to the binary Darboux transformations for the Kadomtsev-Petviashvili hierarchy in its Zakharov-Shabat formulation. We obtain explicit formulae for the Darboux transformed potentials in terms of Grammian type determinants. We also study the n -th Gel'fand-Dickey hierarchy introducing spectral operators and obtaining similar results. We reduce the above mentioned results to the Kadomtsev-Petviashvili I and II real forms, obtaining corresponding vectorial Darboux transformations. In particular for the Kadomtsev-Petviashvili I hierarchy we get the line soliton, the lump solution and the Johnson-Thompson lump, and the corresponding determinant formulae for the non-linear superposition of several of them. For Kadomtsev-Petviashvili II apart from the line solitons we get singular rational solutions with its singularity set describing the motion of strings in the plane. We also consider the I and II real forms for the Gel'fand-Dickey hierarchies obtaining the vectorial Darboux transformation in both cases.

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1 Introduction

The Kadomtsev-Petviashvili (KP) hierarchy is a corner-stone in the theory of integrable systems, and its relevance in Mathematics and Physics is well established, see [1, 18, 21, 32] and references therein. The KP hierarchy is a set of commuting flows that receives its name from one of the equations contained in it: the KP equation. The KP equation was found for the first time in studies on plasma physics [20] and it is also relevant in the theory of surface water waves [3]. Moreover, the equation is the compatibility condition of two differential linear operators [41, 11] and one of them is precisely the non-stationary Schrödinger operator. There are two relevant real reductions, known as KPI and KP II, and the inverse spectral transform for KPI was performed in [23, 12] and for KP II in [4], for a more analytical point of view see [13]. For more details, such as type of solutions, hamiltonian structures, symmetry groups, etc., on the equation we refer the reader to [1] and references therein. Apart from its relevance in Physics, the KP equation is important in Mathematics as well. In particular, let us mention its deep connection with the theory of Riemann surfaces and theta functions, and the Novikov conjecture on the Schottky problem, see [38, 32, 18, 6]. Let us mention that recently Fokas and Zakharov [14] have extended the inverse scattering method and applied it, in particular, to the N -wave resonant interaction, Davey-Stewartson I and KPI equations. For this aim they consider a non-local Riemann-Hilbert problem. Further comparisons between this method and the results of our paper is a work in progress.

This hierarchy contains many relevant integrable equations such as the Korteweg-de Vries (KdV) and Boussinesq, which are particular examples of the Gel'fand-Dickey (GD) hierarchies [16]. The GD hierarchies appear also as particular reductions of the KP hierarchy [18]. The relevance of the KdV and Boussinesq equations in fluid dynamics [1] has been known for long time, but recently the KdV hierarchy has been found to play an important role in a non-perturbative analysis of 2 dimensional quantum gravity, and thus the consequent relevance in this field of the KP equation, see [9]. From here it stems the relevance of the KdV and KP hierarchies in the analysis of the moduli space for the intersection theory of complex curves [40].

At the end of the last century Darboux studied certain transformation of the Schrödinger equation that provided new potentials and wave functions at the same time [17]. This method has been very much extended and applied to a large number of integrable systems [28]. The Darboux transformation

for the KP equation was given by Matveev in [25], and the whole hierarchy, initially considered in [25], was recently considered, in its Lax form, by Oevel and Rogers [34] —see also [33, 10]. The binary extension of the Darboux transformation for the KdV equation was first introduced in [22, 26], but as was observed in [8] it is just an adequate composition of standard Darboux transformations.

One can iterate the binary Darboux transformations to obtain formulae of Gramm type for the KP hierarchy [35] (for the KP equation see [28, 8]). In this paper we observe that in fact there is an alternative way to obtain these results. The simple and basic idea is to replace the scalar character of the initial wave functions by a vectorial one. We will show that this replacement gives explicit compact expressions in terms of Gramm type determinants for all the potentials appeared in the zero-curvature representation of the KP hierarchy, avoiding in this manner the tedious iteration. Also we study how this technique reduces to the GD hierarchy. Finally, we show that one of the main advantages of the vectorial approach appears when real reduction is concerned. In fact, we give vectorial Darboux transformations for both KPI and KP II hierarchies, the real reductions of the KP hierarchy. Regarding the KPI hierarchy we obtain the following novel results:

1. Departing from any solution we obtain vectorial Darboux transformations, expressed in terms of Gramm type determinants, that preserve the singularity structure of the solution; i. e., if the departing solution is singularity free so is the Darboux transformed one.
2. Next, we dress the vacuum solution using heat quasipolynomials. In doing so we obtain compact determinant formulae that contain both the N line soliton solutions and the lump solutions, and also mixture of them. In particular, determinant expressions are given for non-singular rational solutions, containing not only the standard lump solutions [24, 2] but also the more exotic lump solutions presented for the first time in [19] and recently studied by Ablowitz and Villarroel in [5].
3. The τ functions for these solutions, which are strictly positive as we shall prove, are given as determinant of Galilean shifted heat polynomials. For the scalar case we also provide an expression, a sum of square modulus of complex polynomials, which explicitly shows the positivity of the τ -function. This expression contains in particular the 1-lump

[24, 2], the Johnson-Thompson one [19] and the more involved solutions as those constructed with higher degree heat polynomials.

Let us mention that the use of heat polynomials for the construction of rational solutions of KPI was first propose in [27], see also [36].

For the KP II reduction we get a general vectorial Darboux transformation and as an example we find the line solitons and rational singular solutions. For the KP II equation these singularities are algebraic curves in the plane which evolve in time.

Finally we consider the two real reductions for the GD hierarchy, that is GDI and GDII hierarchies. We present for both cases different schemes providing vectorial Darboux transformations. We remark that the KdV hierarchy (the 1st GD hierarchy) is a very special case for which both KdVI and KdVII hierarchies coincide in the real KdV hierarchy, thus for the real KdV hierarchy one has at hand two unequivalent vectorial Darboux transformations. However, for the other GD hierarchies the two reductions are different as the Boussinesq (2nd GD hierarchy) example shows: one has two Boussinesq equations (however only the Boussinesq I has known physical applications); but for both we have vectorial Darboux transformations.

We need to emphasize that the vectorial Darboux transformation presented has at least two advantages: avoiding the iteration process and a simple implementation of the real reduction problem. Also, as any Darboux transformation it not only provides new potentials, it also gives new wave functions. The relevant feature of this is that the first linear equation is just the non-stationary Schrödinger equation. From here it steams the relevance of the results of this paper not only in relation with the KP equation but also with the non-stationary Schrödinger equation.

The layout of the paper is as follows: in §2 we recall the Zakharov-Shabat representation of the KP hierarchy and construct the corresponding vectorial Darboux transformation. We explicitly present expressions for all the Darboux transformed potential functions. Next, in §3, we reduce these results to the GD case. §4 is devoted to the real reductions KPI and KP II. The real reductions GDI and GDII are considered in §5.

2 Vectorial Darboux transformations for the KP hierarchy

In this section we study the vectorial extension of the binary Darboux transformation for the KP hierarchy. This hierarchy has several representations, and as for the vectorial Darboux transformation is concerned the most appropriate one is the Zakharov-Shabat or zero-curvature representation. Let us point out that the more usual one is the Lax representation, however it is a well-known fact that these two are actually equivalent [39].

We shall call KP hierarchy to the infinite set of compatible equations—Zakharov-Shabat equations or zero-curvature conditions—

$$\partial_n(B_m) - \partial_m(B_n) + [B_m, B_n] = 0, \quad m, n \in \mathbb{N}, \quad (1)$$

where B_m , $m \in \mathbb{N}$, are polynomials in $\partial := \partial/\partial x$; i. e., differential operators in the independent variable x , of the following form:

$$B_m = \partial^m + u_{m,m-2}\partial^{m-2} + \cdots + u_{m,0}, \quad (2)$$

where the dependent variables $u_{i,j}$ are functions of $t_1 := x, t_2, t_3 \dots$. Here we are using the notation $\partial_m = \partial/\partial t_m$. For example, the first non trivial case corresponds to $m = 2, n = 3$ for which $u_{2,0} =: u$, $\partial u_{3,1} = 3/2\partial u$, so that we can take $u_{3,1} = 3/2u + c$, c an arbitrary constant and $\partial(u_{3,0}) = 3/4(\partial_2(u) + \partial^2(u))$ and the well known KP equation appears:

$$\partial\left(\partial_3 u - \frac{1}{4}\partial^3 u - \left(\frac{3}{2}u + c\right)\partial u\right) = \frac{3}{4}\partial_2^2 u.$$

Observe that the KP hierarchy (1) is the set of compatibility conditions for the following linear system

$$\partial_m b = B_m(b), \quad m \in \mathbb{N}, \quad (3)$$

where b is usually assumed to be a scalar function of $\mathbf{t} := \{t_m\}_{m \in \mathbb{N}}$. However, this also holds when b takes values in an arbitrary linear space V . Note that the KP hierarchy is as well the compatibility condition for the adjoint linear system

$$\partial_m \beta = -\tilde{B}_m(\beta), \quad m \in \mathbb{N}, \quad (4)$$

where

$$\tilde{B}_m := (-1)^m \partial^m + (-1)^{m-2} \partial^{m-2} u_{m,m-2} + (-1)^{m-3} \partial^{m-3} u_{m,m-3} + \cdots + u_{m,0}.$$

As before, β is usually a scalar function of \mathbf{t} and once again we can give it a vector character, in particular we take it as a linear functional in V ; i. e., as an element in the dual space V^* of V .

Fixed β , we now introduce a potential operator $\Omega(\mathbf{t}) \in \text{GL}(V)$, depending on an arbitrary wave function b , through the following compatible equations—this compatibility can be proven as in the scalar case [33]:

$$\partial_n(\Omega) = -\text{res}(\partial^{-1}b \otimes \tilde{B}_n \beta \partial^{-1}), \quad (5)$$

where the residue (res) of a pseudo-differential operator is the coefficient of ∂^{-1} .

Associated with b , β and Ω we now define fundamental scalar functions which will be needed in the construction of the Darboux transformation.

Definition 1. *Given a solution of the KP hierarchy $\{u_{n,m}\}$, a vector wave function b solving (3) and an adjoint wave function β solving (4), we introduce*

$$f_\ell := \langle \beta, \Omega^{-1} b^{(\ell)} \rangle. \quad (6)$$

In this paper we understand $f^{(\ell)} = \partial^\ell(f)$. Related to these quantities one has

$$g_\ell := \langle \beta, (\Omega^{-1} b)^{(\ell)} \rangle,$$

that can be expressed in terms of the f 's. To this end, we introduce

$$\mathcal{G}_{i,j} := \sum_{k+\ell=j-1} \binom{i-k-1}{\ell} f_k^{(\ell)}, \quad 0 < j \leq i, \quad \mathcal{G}_{i,0} = 1. \quad (7)$$

Proposition 1. *One can express g_ℓ as a differential polynomial in the f 's as follows:*

$$g_\ell = f_\ell - f_0 \mathcal{G}_{\ell,\ell} - \sum_{k=1}^{\ell-1} \left[\left(\mathcal{G}_{\ell,k} + \sum_{m=1}^{\ell-2} (-1)^m \sum_{\ell > k_1 > \cdots > k_m > k} \mathcal{G}_{\ell,k_1} \mathcal{G}_{k_1,k_2} \cdots \mathcal{G}_{k_m,k} \right) (f_k - f_0 \mathcal{G}_{k,k}) \right], \quad (8)$$

with $\mathcal{G}_{i,j}$ as in (7).

Proof. It is convenient to compute $\Omega^{-1}b^{(k)}$:

$$\Omega^{-1}\partial^k(b) = \Omega^{-1}\partial^k(\Omega\hat{b}) = (\Omega^{-1}\partial\Omega)^k(\hat{b}) = (\partial + \Omega^{-1}\partial(\Omega))^k(\hat{b}),$$

where $\hat{b} = \Omega^{-1}b$. Equation (5) implies in particular that

$$\partial(\Omega) = b \otimes \beta, \tag{9}$$

and so

$$\Omega^{-1}b^{(k)} = (\partial + \hat{b} \otimes \beta)^k(\hat{b}).$$

Therefore, we have

$$\Omega^{-1}b^{(k)} = \hat{b}^{(k)} + \sum_{j=1}^k A_{k,j} \hat{b}^{(k-j)}. \tag{10}$$

Using this formula one arrives at some recursion relations among the coefficients $A_{m,j}$, which are uniquely satisfied by the choice $A_{i,j} = \mathcal{G}_{i,j}$ of (7). Contracting this expression with the linear functional β one obtains the following inhomogeneous linear system

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ \mathcal{G}_{2,1} & 1 & 0 & \dots & 0 \\ \mathcal{G}_{3,2} & \mathcal{G}_{3,1} & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \mathcal{G}_{m,m-1} & \mathcal{G}_{m,m-2} & \dots & \mathcal{G}_{m,1} & 1 \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ \vdots \\ g_m \end{pmatrix} = \begin{pmatrix} f_1 - f_0 \mathcal{G}_{1,1} \\ f_2 - f_0 \mathcal{G}_{2,2} \\ f_3 - f_0 \mathcal{G}_{3,3} \\ \vdots \\ f_m - f_0 \mathcal{G}_{m,m} \end{pmatrix}.$$

Solving the above linear system we get the desired result. \square

Observe that the functions f 's are expressed in terms of the contraction of β , Ω^{-1} and x -derivatives of b . From (9), if Ω has a determinant, one deduces that

$$f_0 = \partial \ln \det \Omega,$$

and, in the finite-dimensional case, the Cramer's rule implies

$$f_j = \sum_k \beta_k \frac{\det \Omega^{[k,j]}}{\det \Omega},$$

where $\Omega^{[k,j]}$ is obtained from the corresponding matrix of Ω by replacing the k -th column by the column given by $b^{(j)}$.

The operator Ω resembles a Gramm matrix. In fact, from (9) we have

$$\Omega = C + \partial^{-1}(b \otimes \beta),$$

with ∂^{-1} a primitive of ∂ and $C(\mathbf{t})$ a linear operator with $\partial(C) = 0$, for vanishing initial data $x = -\infty$ the operator C could be chosen as a constant operator.

To proceed further we introduce some convenient notation:

Definition 2.

$$h_{n,k} := \sum_{\ell=k}^n \sum_{i=0}^{\ell-k} (-1)^{i+\ell} \binom{\ell-k}{i} (u_{n,\ell} g_i)^{(\ell-k-i)}, \quad 0 \leq k \leq n,$$

$$h_{n,n+1} := 0.$$

With this at hand we can show:

Lemma 1. *Given a solution of the KP hierarchy $\{u_{n,m}\}$, a function b solving (3) and an adjoint wave function β solving (4), then the functions $\hat{b} := \Omega^{-1}b$ and $\hat{\beta} := \beta\Omega^{-1}$ satisfy*

$$\begin{aligned} \partial_n \hat{b} &= \hat{B}_n(\hat{b}), \\ \partial_n \hat{\beta} &= -\tilde{B}_n(\hat{\beta}), \end{aligned}$$

where

$$\hat{B}_n = \sum_{m=0}^n \hat{u}_{n,m} \partial^m, \quad \hat{u}_{n,m} := \sum_{j=m}^n \mathcal{G}_{j,m}(u_{n,j} + (-1)^j h_{n,j+1}). \quad (11)$$

Proof. From $b = \Omega \hat{b}$ it follows that

$$\partial_n \hat{b} = \Omega^{-1} \left(B_n(\Omega \hat{b}) + \text{res}(\partial^{-1} b \otimes \tilde{B}_n \beta \partial^{-1}) \hat{b} \right), \quad (12)$$

and using identity (10) we arrive to

$$\Omega^{-1} B_n(\Omega \hat{b}) = \sum_{m=0}^n \left(\sum_{j=m}^n u_{n,j} \mathcal{G}_{j,j-m} \right) \hat{b}^{(m)},$$

and

$$\Omega^{-1} \text{res}(\partial^{-1}b \otimes \tilde{B}_n \beta \partial^{-1}) \hat{b} = \sum_{m=0}^{n-1} \left(\sum_{j=m}^{n-1} h_{n,j+1} \mathcal{G}_{j,j-m} \right) \hat{b}^{(m)}.$$

The substitution of the last two formulae into (12) proves the lemma. \square

Observe that the differential operator has the same form as B_n ; i. e., the highest coefficient is 1 and the next to highest vanishes. The first non-trivial coefficient in \hat{B}_n is

$$\hat{u}_{n,n-2} = u_{n,n-2} + n\partial(f_0).$$

We are ready to prove the following result

Theorem 1. *The functions*

$$\hat{u}_{n,m} := \sum_{j=m}^n \mathcal{G}_{j,m} (u_{n,j} + (-1)^j h_{n,j+1}),$$

with

$$\begin{aligned} \mathcal{G}_{i,j} &:= \sum_{k+\ell=j-1} \binom{i-k-1}{\ell} f_k^{(\ell)}, \quad 0 < j \leq i, \quad \mathcal{G}_{i,0} = 1, \\ h_{n,k} &:= \begin{cases} \sum_{\ell=k}^n \sum_{i=0}^{\ell-k} (-1)^{i+\ell} \binom{\ell-k}{i} (u_{n,\ell} g_i)^{(\ell-k-i)}, & 0 \leq k \leq n, \\ 0, & k = n+1, \end{cases} \\ f_\ell &:= \langle \beta, \Omega^{-1} b^{(\ell)} \rangle, \\ g_\ell &:= \langle \beta, (\Omega^{-1} b)^{(\ell)} \rangle, \end{aligned}$$

solve the KP hierarchy.

Proof. Given an arbitrary solution $\psi(\mathbf{t}) \in V$ of (3) we take the following solution of (3)

$$\underline{b} := \begin{pmatrix} \psi \\ b \end{pmatrix} \in V \oplus V,$$

and a solution of (4) as

$$\underline{\beta} := (0, \beta) \in V^* \oplus V^*,$$

then a potential operator $\underline{\Omega}$, defined by \underline{b} and $\underline{\beta}$ as prescribed by (5), is given by

$$\underline{\Omega} = \begin{pmatrix} 1 & \check{\Omega} \\ 0 & \Omega \end{pmatrix},$$

where $\check{\Omega}$ is defined by (5) replacing b by ψ . Now, the transformed wave function $\hat{\underline{b}}$ is

$$\hat{\underline{b}} = \underline{\Omega}^{-1} \begin{pmatrix} \psi \\ b \end{pmatrix} = \begin{pmatrix} \psi - \check{\Omega}\Omega^{-1}b \\ \Omega^{-1}b \end{pmatrix},$$

while

$$f_j = \langle \underline{\beta}, \underline{\Omega}^{-1}\hat{\underline{b}}^{(j)} \rangle = \langle \beta, \Omega^{-1}b^{(j)} \rangle$$

is not changed. So, we can use the above lemma, that tells us that $\hat{\underline{b}}$ does satisfy (3) with the coefficients $\hat{u}_{n,m}$, and deduce that so does $\psi - \check{\Omega}\Omega^{-1}b$ for an arbitrary departing wave function ψ . Therefore, following standard arguments [28], we conclude that \hat{B}_n must satisfy (1). □

From the proof of the theorem we can extract the following:

Corollary 1. *Given an arbitrary initial wave function ψ the vectorial Darboux transformation associated with the data b, β and Ω as in the previous theorem provides a new wave function $\hat{\psi}$ satisfying the new linear system. Namely:*

$$\hat{\psi} := \psi - \check{\Omega}\Omega^{-1}b,$$

with $\check{\Omega}$ defined by (5) replacing b by ψ .

3 Reduction to the Gel'fand-Dickey hierarchy

It follows from the equivalence between the Lax and Zakharov-Shabat representations of the KP hierarchy that the n -th GD reduction appears when the solutions $\{u_{i,j}\}$ do not depend on t_n . In fact, if this is the case, they

do not depend on the times $\{t_{nm}\}_{m \in \mathbb{N}}$. The other flows describes isospectral transformations of the differential operator

$$\mathcal{L} := B_n = \partial^n + u_{n-2}\partial^{n-2} + \cdots + u_0,$$

where we use the notation

$$u_j = u_{n,j},$$

in terms of which all the $u_{m,i}$ are expressed as differential polynomials.

Regarding the vectorial Darboux transformation the point is to choose b , β and Ω in an appropriate way so that the GD reduction is preserved. That is, given a solution $\{u_{i,j}\}$ of the KP hierarchy not depending on t_n choose those Darboux transformations for which the Darboux transformed solution, given by $\{\hat{u}_{i,j}\}$ as in (11), does not depend on t_n . Because the new solution \hat{u} is constructed in terms of the u and the functions f 's, defined by (6), this aim will be achieved if the latter functions do actually not depend on t_n .

The pertinent choice in this case, as we shall proof later, for b and β is given by

$$\partial_n b = \mathcal{L}(b) = Lb, \quad \partial_n \beta = -\mathcal{L}^*(\beta) = -\beta\Lambda, \quad (13)$$

where L and Λ are arbitrary linear operators in V . These two linear operators could be considered as noncommutative extensions of spectral parameters. Observe also that in general they can not be got from an standard iteration of the scalar binary Darboux transformation: to get non-diagonalizable spectral operators one needs to perform, apart from the iteration, suitable coalescence of eigen-values.

When b and β are chosen in this particular form one has

Proposition 2. *If $\{u_{i,j}\}$ is a solution of the KP hierarchy not depending on t_n , and b and β are chosen as solutions of (13), the operator Ω satisfies*

$$\partial_m(\partial_n \Omega - L\Omega + \Omega\Lambda) = 0, \quad m \in \mathbb{N}.$$

Proof. From the compatibility condition $\partial_m \partial_n \Omega = \partial_n \partial_m \Omega$ and (5) one easily derives

$$\begin{aligned} \partial_m \partial_n \Omega &= -\partial_n \left(\text{res}(\partial^{-1} b \otimes \tilde{B}_m \beta \partial^{-1}) \right) \\ &= -\text{res} \left(\partial^{-1} \left[(\partial_n b) \otimes \tilde{B}_m \beta + b \otimes (\partial_n \tilde{B}_m) \beta + b \otimes \tilde{B}_m (\partial_n \beta) \right] \partial^{-1} \right). \end{aligned}$$

As we are dealing with a solution that reduces to the n -th GD hierarchy we have $\partial_n(B_m) = 0$, and because (13) holds we find that

$$\begin{aligned}\partial_m \partial_n \Omega &= -L \operatorname{res}(\partial^{-1} b \otimes \tilde{B}_m \beta \partial^{-1}) + \operatorname{res}(\partial^{-1} b \otimes \tilde{B}_m \beta \partial^{-1}) \Lambda \\ &= \partial_m (L\Omega - \Omega\Lambda),\end{aligned}$$

where we have used again (5), which gives the desired conclusion. \square

Therefore, the value of

$$\partial_n \Omega - L\Omega + \Omega\Lambda$$

is a constant that does not vary with the KP flows. In particular, choosing adequate initial conditions, we could take it as zero; i. e.,

$$-\operatorname{res}(\partial^{-1} b \otimes \tilde{\mathcal{L}} \beta \partial^{-1}) = L\Omega - \Omega\Lambda. \quad (14)$$

Now, is easy to prove

Theorem 2. *The vectorial Darboux transformation (11) of a solution of the n -th GD hierarchy defined by b and β as in (13) and Ω satisfying (14) gives again a solution of the n -th GD hierarchy.*

Proof. As we have already commented we only need to check that the f 's are independent of t_n . This follows from

$$\begin{aligned}\partial_n(f_j) &= \langle \partial_n \beta, \Omega^{-1} b^{(j)} \rangle - \langle \beta, \Omega^{-1} (\partial_n \Omega) \Omega^{-1} b^{(j)} \rangle + \langle \beta, \Omega^{-1} (\partial_n b^{(j)}) \rangle \\ &= - \langle \beta, \Omega^{-1} (\Omega\Lambda + \partial_n \Omega - L\Omega) \Omega^{-1} b^{(j)} \rangle \\ &= 0.\end{aligned}$$

\square

4 Reductions to the KPI and KP II hierarchies

There are two main real reductions of the KP equation known as the KPI and KP II equations, both with physical relevance. We show here how the vectorial Darboux scheme presented above reduces to these two real reductions. First, let us introduce the KPI and KP II hierarchies:

- The KPI hierarchy appears when

$$t_{2n} \in i\mathbb{R}, \quad t_{2n-1} \in \mathbb{R}, \quad n \in \mathbb{N},$$

and $u_{2,0}(\mathbf{t}) \in \mathbb{R}$ takes real values. From the classical results on the KP hierarchy one concludes that this is equivalent to the following identity:

$$(B_n)^* = (-1)^n \tilde{B}_n,$$

between the complex conjugate of the Zakharov-Shabat operators and its adjoint.

- The KPII case corresponds to real dependent and independent variables $t_n \in \mathbb{R}$, $n \in \mathbb{N}$, and $u_{2,0}$ real, that means real B 's and real \tilde{B} 's.

As for the GD reduction, the idea here is to choose b , β and Ω in an appropriate way.

4.1 KPI hierarchy

For the KPI case we assume that our linear space V is a Hilbert space so that it has an adjoint operation \dagger . The adjoint linear system for β can be written

$$(\partial_n)^* \beta = B_n^*(\beta),$$

where we have used that $(\partial_n)^* = (-1)^{n+1} \partial_n$. So, if b solves the direct problem then $b^\dagger H$ solves the adjoint one, where H is any linear operator in V . This motivates us to choose β precisely in this form, and then we have for Ω :

$$\begin{aligned} H^\dagger \partial_n \Omega &= -H^\dagger \text{res}(\partial^{-1} b \otimes \tilde{B}_n b^\dagger \partial^{-1}) H, \\ \partial_n \Omega^\dagger H &= -H^\dagger \text{res}(\partial^{-1} b \otimes (-1)^n B_n^* b^\dagger \partial^{-1}) H \\ &= -H^\dagger \text{res}(\partial^{-1} b \otimes \tilde{B}_n b^\dagger \partial^{-1}) H, \end{aligned}$$

and so we find the relation

$$\partial_n (H^\dagger \Omega - \Omega^\dagger H) = 0, \quad n \in \mathbb{N}.$$

With this we consider the imaginary part of f_0 :

$$f_0 - f_0^* = \langle b^\dagger, (\Omega^\dagger)^{-1} (\Omega^\dagger H - H^\dagger \Omega) \Omega^{-1} b \rangle,$$

so if the constant operator $\Omega^\dagger H - H^\dagger \Omega$ is chosen as zero; i. e., take the integration constant Ω_0 for Ω such that $\Omega_0^\dagger H = H^\dagger \Omega_0$, the function f_0 is real and therefore so is $\hat{u}_{2,0}$ and the vectorial Darboux transformation preserves the KPI hierarchy reduction.

This can be resumed in the following:

Proposition 3. *Let B_n be a solution of the KPI hierarchy, $b \in V$ be a vectorial solution of*

$$\partial_n b = B_n(b),$$

H be a linear operator and $\Omega \in \text{GL}(V)$ be an invertible linear operator defined by the compatible equations

$$\partial_n \Omega = (-1)^{n+1} \text{res}(\partial^{-1} b \otimes B_n^* b^\dagger \partial^{-1}) H,$$

with initial condition Ω_0 such that

$$\Omega_0^\dagger H = H^\dagger \Omega_0,$$

then the new \hat{B}_n also solves the KPI hierarchy.

Apart from the reality condition just shown, we are also interested in those vectorial Darboux transformations that preserve the singularity structure of the solution; i. e., $\det \Omega \neq 0$. From now on we take $H = \mathbb{I}$ so that Ω must be Hermitian; i. e., $\Omega^\dagger = \Omega$, and $\dim V < \infty$.

Proposition 4. *Let b be as in Proposition 3 with its components be linearly independent and such that the improper integral*

$$\int_{-\infty}^x dx b(x) \otimes b^\dagger(x),$$

converges for all $x \in \mathbb{R}$. Then Ω can be taken as

$$\Omega(x) = C + \int_{-\infty}^x dx b(x) \otimes b^\dagger(x),$$

where C is a hermitian constant operator. Moreover, if C is positive semi-definite, we have

$$\det \Omega > 0.$$

Proof. By construction $\Omega(x) = C + \int_{-\infty}^x dx b(x) \otimes b(x)^\dagger$, where the constant hermitian operator C is positive semi-definite (all its eigen-values non-negative). Now, the matrix $\int_{-\infty}^x dx b(x) \otimes b(x)^\dagger$ is precisely a Gramm matrix for functions in the interval $(-\infty, x]$. An elementary result from linear algebra [15] tells us that it must be positive semi-definite and the determinant vanishes only if the functions $\{b_1, \dots, b_N\}$ are linearly dependent. \square

Let us remark that for zero-background similar solutions can be found in [32], in fact there the solutions to the heat hierarchy are expressed in integral form. However, there is no proof on the absence of singularities.

Examples: line solitons, standard and exotic lumps. For the zero background $u_{2,0} = 0$, that is $B_n = \partial^n$, the linear system for b is just the free heat hierarchy: $\partial_m b = \partial^m(b)$. An obvious solution is $\exp(\sum_{m \geq 1} t_m k^m)$ where $k \in \mathbb{C}$ is an arbitrary complex number. Polynomial solutions can be expressed as finite linear combination of the Schur or heat polynomials $s_n(\mathbf{t})$ which can be constructed from

$$\exp\left(\sum_{n \geq 1} t_n k^n\right) = \sum_{m \geq 0} s_m(\mathbf{t}) k^m,$$

or equivalently form the recursion relation

$$s_n(\mathbf{t}) = \frac{1}{n} \sum_{j=1}^n j t_j s_{n-j}(\mathbf{t}), \quad s_0(\mathbf{t}) = 1,$$

being the first few:

$$\begin{aligned} s_0(\mathbf{t}) &= 1, \\ s_1(\mathbf{t}) &= t_1, \\ s_2(\mathbf{t}) &= t_2 + \frac{1}{2} t_1^2, \\ s_3(\mathbf{t}) &= t_3 + t_2 t_1 + \frac{1}{6} t_1^3, \\ s_4(\mathbf{t}) &= t_4 + t_3 t_1 + \frac{1}{2} t_2 (t_2 + t_1^2) + \frac{1}{24} t_1^4. \end{aligned}$$

We recall the Galilean transformation: if $b(\mathbf{t})$ is a solution so is

$$b(\tilde{\mathbf{t}}(k)) \exp\left(\sum_{n \geq 1} t_n k^n\right), \quad \tilde{t}_n(k) := \sum_{m \geq n} \binom{m}{n} k^{m-n} t_m.$$

Thus, performing a Galilean transformation on the heat polynomials one gets *heat quasipolynomials*.

We assume V to be N -dimensional and take $b = (b_1, \dots, b_N)^t$ with $b_j(\mathbf{t}) = p_j(\tilde{\mathbf{t}}(k_j)) \exp(\sum_{n \geq 1} k_j^n t_n)$ a heat quasipolynomial, that is $p_j(\mathbf{t})$ is a heat polynomial of order n_j ; here we take distinct k_j with $\operatorname{Re} k_j > 0$. Observe that this choice for b fulfils the requirements of Proposition 4 and therefore the associated solution will be non-singular. As a result we get the following family of non-singular solutions of the KPI hierarchy:

Proposition 5. *Let Ω be the following hermitian matrix*

$$\Omega = C + \mathcal{E} \omega \mathcal{E}^\dagger,$$

with C a non-negative definite hermitian matrix and

$$\mathcal{E} := \operatorname{diag}\left(\exp\left(\sum_{n \geq 1} t_n k_1^n\right), \dots, \exp\left(\sum_{n \geq 1} t_n k_N^n\right)\right),$$

and

$$\omega_{ij}(\mathbf{t}) = \frac{1}{k_i + k_j^*} \sum_{\ell=0}^{n_i+n_j} \left(-\frac{1}{k_i + k_j^*} \partial\right)^\ell (p_i(\tilde{\mathbf{t}}(k_i)) p_j^*(\tilde{\mathbf{t}}(k_j))),$$

where p_j is a heat polynomial of degree n_j and $\tilde{t}_n(k) = \sum_{m \geq n} \binom{m}{n} k^{m-n} t_m$. Then,

$$u = 2\partial^2 \ln \det \Omega,$$

is a non-singular solution of the KPI hierarchy. When $C = 0$ we get the following non-singular rational solution

$$u = 2\partial^2 \ln \det \omega.$$

The N -line soliton solution of KPI equation appears once we take the simplest heat polynomial; i. e., $p_i = a_i$ a complex constant and $\omega_{ij} =$

$a_i a_j^*/(k_i + k_j^*)$, and $C = \mathbb{I}$. If one allows higher heat polynomials we get non-singular rational deformation of the line solitons whenever $C \neq 0$. See [30, 29].

However, when $C = 0$ the situation changes drastically. Then we obtain non-singular rational solutions of the KPI equation. In fact, for $N = 1$ and $n_1 = 1$ we get the 1-lump solution [24, 2, 37] for KPI and for $n_1 = 2$ the Johnson-Thompson solution [19] recently studied in [5]. For $N = 1$ we can increase the degree n_1 of the heat polynomial p_1 , as is already suggested [19], and get more involved rational behaviour showing, as is studied in [5], non-trivial interaction of localized lumps. For $N > 1$ the τ -function $\det \omega$ could be understood as the non-linear composition of N of these objects.

For the 1-lump solution [24, 2] we choose b as

$$b = (x + 2iky + 3k^2t) \exp(kx + ik^2y + k^3t),$$

with $k = k_{\mathbb{R}} + i k_{\mathbb{I}} \in \mathbb{C}$, $k_{\mathbb{R}}, k_{\mathbb{I}} \in \mathbb{R}$ by direct calculation, we have the following τ -function

$$\omega = \frac{1}{2k_{\mathbb{R}}} \left[\left(z(t) - \frac{1}{2k_{\mathbb{R}}} \right)^2 + k_{\mathbb{R}}^2 y(t)^2 + \frac{1}{4k_{\mathbb{R}}^2} \right],$$

with $x(t) := x + 3(k_{\mathbb{R}}^2 + k_{\mathbb{I}}^2)t$, $y(t) := y + 3k_{\mathbb{I}}t$ and $z(t) := x(t) - 2k_{\mathbb{I}}y(t)$.

For the Johnson-Thompson solution [19] we take b as

$$b = [(x + 2iky + 3k^2t)^2 + 2(iy + 3kt)] \exp(kx + ik^2y + k^3t),$$

again by direct calculation, we obtain

$$\begin{aligned} \omega = \frac{1}{2k_{\mathbb{R}}} & \left[\left\{ \left(z(t) - \frac{1}{2k_{\mathbb{R}}} \right)^2 + \frac{1}{4k_{\mathbb{R}}^2} + 6k_{\mathbb{R}}t - 4k_{\mathbb{R}}^2 y(t)^2 \right\}^2 \right. \\ & \left. + 16k_{\mathbb{R}}^2 y(t)^2 z(t)^2 + \frac{1}{k_{\mathbb{R}}^2} \left(z(t) - \frac{1}{2k_{\mathbb{R}}} \right)^2 + 4y(t)^2 + \frac{1}{4k_{\mathbb{R}}^2} \right]. \end{aligned}$$

After a translation in the x -coordinate this goes to the solution presented in [19].

Given a polynomial p with degree $\deg p$, for negative α we have the identity

$$\int_{-\infty}^x dx |p(x)|^2 \exp(-x/\alpha) = -\alpha \exp(-x/\alpha) \sum_{m=0}^{\deg(p)} \alpha^{2m} \left| \sum_{j=0}^{\deg(p)-m} \binom{j+m}{m} \alpha^j p^{(j+m)} \right|^2,$$

that can be checked by integrating by parts the integral and using the Leibnitz rule and observing that the result coincides with the expansion of the right hand side term. With this at hand, and the previous proposition for $\dim V = 1$ we deduce an explicit formula for the function ω where its positivity is explicitly shown, and thus the non-singularity property is explicit:

Proposition 6. *Given a heat polynomial p and $k \in \mathbb{C}$, with $\operatorname{Re} k > 0$, the function*

$$\sum_{m=0}^{\deg(p)} \frac{1}{(2 \operatorname{Re} k)^{2m}} \left| \sum_{j=0}^{\deg(p)-m} \binom{j+m}{m} (-1)^j \frac{1}{(2 \operatorname{Re} k)^j} p^{(j+m)}(\tilde{\mathbf{t}}(k)) \right|^2,$$

is a τ -function for the KPI hierarchy that gives rise to a rationally non-singular localized solution of the KPI hierarchy.

In particular, if $p = s_n$ is a Schur polynomial of order n we get the following τ -function:

$$\sum_{m=0}^n \frac{1}{(2 \operatorname{Re} k)^{2m}} \left| \sum_{j=0}^{n-m} \binom{j+m}{m} (-1)^j \frac{1}{(2 \operatorname{Re} k)^j} s_{n-j-m}(\tilde{\mathbf{t}}(k)) \right|^2,$$

for $n = 1, 2$ we recover the 1-lump and the Johnson-Thompson solution.

4.2 KP II hierarchy

For the KP II case the procedure is quite different, now the reality of the B_n and t_n ensures that if b is a solution so is Pb^* , where P is a linear operator, and the same for β and β^*Q . As before we could say that

$$\begin{aligned} b^* &= Pb, \\ \beta &= \beta^*Q, \end{aligned}$$

but this is consistent only when P and Q satisfy $PP^* = QQ^* = \mathbb{I}$. Let us suppose that this is the case and analyze Ω :

$$\begin{aligned} P\partial_n\Omega &= -P \operatorname{res}(\partial^{-1}b \otimes \tilde{B}_n\beta^*\partial^{-1})Q, \\ \partial_n\Omega^*Q &= -P \operatorname{res}(\partial^{-1}b \otimes \tilde{B}_n\beta^*\partial^{-1})Q, \end{aligned}$$

and therefore

$$\partial_n(P\Omega - \Omega^*Q) = 0, \quad n \in \mathbb{N}.$$

Then, if the integration constant Ω_0 satisfies $P\Omega_0 = \Omega_0^*Q$ so does Ω . If we make this choice then we have

$$f_0 - f_0^* = \langle \beta^*, \Omega^{*-1}(\Omega^*Q - P\Omega)\Omega^{-1}b \rangle = 0,$$

so f_0 is real, and so is the new solution. Thus, we conclude that with this choice the KPII hierarchy is preserved by the vectorial Darboux transformation.

Proposition 7. *Let B_n be a solution of the KPII hierarchy, and $b \in V$ be a vectorial solution of*

$$\partial_n b = B_n(b),$$

such that $b^ = Pb$ and $\beta \in V^*$ be solution of the adjoint system*

$$\partial_n \beta = -\tilde{B}_n(\beta),$$

*such that $\beta = \beta^*Q$, where $P, Q \in \text{GL}(V)$ with $PP^* = QQ^* = \mathbb{I}$, and let $\Omega \in \text{GL}(V)$ be a linear operator defined by the compatible equations*

$$\partial_n \Omega = -\text{res}(\partial^{-1}b \otimes \tilde{B}_n \beta \partial^{-1}),$$

with initial condition Ω_0 such that

$$\Omega_0^*Q = P\Omega_0,$$

then the new \hat{B}_n also solves the KPII hierarchy.

When $P = Q = \mathbb{I}$ we have real b , β and Ω . The line soliton solution of the KPII hierarchy appears when the entries of the b and of the β are just exponentials. Another possibility is to take $V = W \oplus W$, even dimensional space and take

$$P = Q = \begin{pmatrix} 0 & \mathbb{I}_W \\ \mathbb{I}_W & 0 \end{pmatrix},$$

with respect to the just written splitting of V . This means that

$$b = \begin{pmatrix} g \\ g^* \end{pmatrix},$$

$$\beta = (\gamma, \gamma^*),$$

where $g(\mathbf{t}) \in W$ and $\gamma(\mathbf{t}) \in W^*$, and

$$\Omega = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{B}^* & \mathcal{A}^* \end{pmatrix},$$

with

$$\partial_n \mathcal{A} = -\text{res}(\partial^{-1} g \otimes \tilde{B}_n \gamma \partial^{-1}),$$

$$\partial_n \mathcal{B} = -\text{res}(\partial^{-1} g \otimes \tilde{B}_n \gamma^* \partial^{-1}).$$

For example, for $W = \mathbb{C}$ we get

$$\det \Omega = |\mathcal{A}|^2 - |\mathcal{B}|^2.$$

Taking $g(\mathbf{t}) = p(\tilde{\mathbf{t}}(k)) \exp(\sum_n k^n t_n)$ and $\gamma(\mathbf{t}) = q(\tilde{\mathbf{t}}(l)) \exp(-\sum_n l^n t_n)$ where p, q are heat polynomials, with degrees $\deg(p), \deg(q)$ and $k, l \in \mathbb{C}$ with $\text{Re } k, \text{Re } l > 0$, we have

$$\mathcal{A} = A + \exp\left(\sum_n (k^n - l^n) t_n\right) \sum_{m=0}^{\deg(p)+\deg(q)} \frac{(-1)^m}{(k-l)^{m+1}} (pq)^{(m)},$$

$$\mathcal{B} = B + \exp\left(\sum_n (k^n - (l^*)^n) t_n\right) \sum_{m=0}^{\deg(p)+\deg(q)} \frac{(-1)^m}{(k-l^*)^{m+1}} (pq^*)^{(m)},$$

where A, B are arbitrary complex constants. If these two constants do not vanish we have a singular solution of mixed rational and exponential character. However, as for KPI, when $A = B = 0$ we have a rational solution

$$\det \Omega(\mathbf{t}) = \exp\left(\sum_n 2 \text{Re}(k^n - l^n) t_n\right) \tau(\mathbf{t}),$$

with the τ -function given by

$$\tau := \left| \sum_{m=0}^{\deg(p)+\deg(q)} \frac{(-1)^m}{(k-l)^{m+1}} (pq)^{(m)} \right|^2 - \left| \sum_{m=0}^{\deg(p)+\deg(q)} \frac{(-1)^m}{(k-l^*)^{m+1}} (pq^*)^{(m)} \right|^2.$$

The singularities lay over algebraic curves in the plane defined by $\mathcal{A} = \mu\mathcal{B}$, with $|\mu| = 1$. For the KP II equation these algebraic curves move in the plane, $x = t_1, y = x_2$, as the time $t = t_3$ evolves. Similar results for the Davey-Stewartson II are given in [7]. Let us remind that the motion of these singularities in y is given by a Calogero-Moser (CM) system while the one in t is a higher order CM system, this constitutes the Krichever theorem, see [32].

5 Reductions to Gel'fand-Dickey I and II

We now discuss these reductions of type I and II in relation with the GD reduction, that is two real reductions of the GD hierarchy, which we denote by GDI and GDII. Observe that the KPI and KP II reductions are compatible with the GD reduction and that for the KdV case we have that the two reductions coincide. The question here is whether there is a vectorial Darboux transformation for these GDI and GDII hierarchies. In order to get such result we just need to combine in an adequate manner the reductions schemes explained above.

First we analyze the GDI hierarchy. On the one hand the n -th GD reduction requests b and β to satisfy the equations $\mathcal{L}(b) = Lb$ and $\tilde{\mathcal{L}}(\beta) = -\beta\Lambda$, where $\mathcal{L} = B_n$ and $L, \Lambda \in L(V)$ are spectral operators. On the other hand the KPI reduction asks b and β to satisfy $\beta = b^\dagger H$, H an arbitrary linear operator in V . The point is to make these two requirements compatible, and this can be done if the operators L, Λ and H are linked by

$$(-1)^n L^\dagger H + H\Lambda = 0,$$

that when H is invertible means that $\Lambda = (-1)^n H^{-1} L^\dagger H$, and so Λ is expressed in terms of L, H . If this is so, then $\mathcal{L}(b) = Lb$ implies $\tilde{\mathcal{L}}(\beta) = (-1)^n (\mathcal{L}(b))^\dagger H = b^\dagger (-1)^n L^\dagger H = -\beta\Lambda$. As we know, the fact that b and $\beta = b^\dagger H$ satisfy the spectral equations ensures, that for adequate initial conditions, the following equation holds

$$(-1)^{n+1} \text{res}(\partial^{-1} b \otimes \mathcal{L}^* b^\dagger \partial^{-1}) H = L\Omega - \Omega\Lambda,$$

we also know, from the KPI reduction, that it is possible to take initials conditions so that

$$\Omega^\dagger H = H^\dagger \Omega,$$

so if it is possible to find an initial condition Ω_0 so that

$$\begin{aligned} (-1)^{n+1} \operatorname{res}(\partial^{-1}b \otimes \mathcal{L}^*b^\dagger \partial^{-1})_0 H &= L\Omega_0 - \Omega_0\Lambda, \\ \Omega_0^\dagger H &= H^\dagger\Omega_0, \end{aligned}$$

we know that the vectorial Darboux transformation preserves the n -th GDI hierarchy.

Proposition 8. *Let \mathcal{L} be a solution of the n -th GDI hierarchy and $b(\mathbf{t}) \in V$ a solution of the linear system*

$$\begin{aligned} \mathcal{L}(b) &= Lb, \\ \partial_n b &= B_n(b), \end{aligned}$$

with L a linear operator in V and take linear operators Λ and H subject to

$$(-1)^n L^\dagger H + H\Lambda = 0.$$

Find an invertible operator by integrating the compatible equations

$$\partial_n \Omega = -\operatorname{res}(\partial^{-1}b \otimes \tilde{B}_n b^\dagger \partial^{-1})H,$$

with an initial condition Ω_0 so that

$$\begin{aligned} (-1)^{n+1} \operatorname{res}(\partial^{-1}b \otimes \mathcal{L}^*b^\dagger \partial^{-1})_0 H &= L\Omega_0 - \Omega_0\Lambda, \\ \Omega_0^\dagger H &= H^\dagger\Omega_0, \end{aligned}$$

then the associated vectorial Darboux transformation preserves the n -th GDI hierarchy.

For the GDII hierarchy one proceeds in a similar way. The conditions on P, Q, L and Λ are

$$\begin{aligned} L^*P &= PL, \\ \Lambda^*Q &= Q\Lambda, \end{aligned}$$

that make the conditions $\mathcal{L}(b) = Lb$, $\tilde{\mathcal{L}}(\beta) = -\beta\Lambda$ compatible with $b^* = Pb$, $\beta = \beta^*Q$. So, the conclusion is:

Proposition 9. *Let \mathcal{L} be a solution of the n -th GDII hierarchy, $b(\mathbf{t}) \in V$ a solution of the linear system*

$$\begin{aligned}\mathcal{L}(b) &= Lb, \\ \partial_n b &= B_n(b),\end{aligned}$$

and $\beta(\mathbf{t}) \in V^*$ a solution of

$$\begin{aligned}\tilde{\mathcal{L}}(\beta) &= -\beta\Lambda, \\ \partial_n \beta &= -\tilde{B}_n(b)\end{aligned}$$

with L, Λ linear operators in V and take linear operators P, Q subject to $PP^* = QQ^* = \mathbb{I}$ and

$$\begin{aligned}L^*P &= PL, \\ \Lambda^*Q &= Q\Lambda.\end{aligned}$$

Find an invertible operator by integrating the compatible equations

$$\partial_n \Omega = -\text{res}(\partial^{-1}b \otimes \tilde{B}_n \beta \partial^{-1}),$$

with an initial condition Ω_0 so that

$$\begin{aligned}-\text{res}(\partial^{-1}b \otimes \tilde{\mathcal{L}}\beta \partial^{-1})_0 &= L\Omega_0 - \Omega_0\Lambda, \\ \Omega_0^*Q &= P\Omega_0,\end{aligned}$$

then the associated vectorial Darboux transformation preserves the n -th GDII hierarchy.

While for the 1st GD hierarchy, the KdV hierarchy, both real reductions coincide with the real KdV hierarchy, and so we have two different vectorial Darboux transformations for the same hierarchy, in general the GDI and GDII reductions are different. In particular, the 2nd GD hierarchy is associated with $\mathcal{L} = \partial^3 + u_1\partial + u_0$, and the Boussinesq equation is linked with the t_2 flow given by $B_2 = \partial^2 + u_{2,0}$. The compatibility condition: $\partial_2 \mathcal{L} = [B_2, \mathcal{L}]$ gives $u_{2,0} =: u$, $\partial u_1 = 3/2\partial u$, so that we can take $u_1 = 3/2u + c$, c an arbitrary constant and $\partial(u_0) = 3/4(\partial_2(u) + \partial^2(u))$ and the following equation is satisfied:

$$3\partial_2^2 u + 4c\partial^2(u) + 3\partial^2(u^2) + \partial^4(u) = 0.$$

The Boussinesq I equation takes real u and $x \in \mathbb{R}$ and $t_2 = it$, $t \in \mathbb{R}$, while the Boussinesq II is associated with real u and $x, t_2 = t \in \mathbb{R}$. The Boussinesq I and II equations are

$$\pm 3\partial_t^2 u = 4c\partial^2(u) + 3\partial^2(u^2) + \partial^4(u).$$

with $+$ corresponding to type I and $-$ to type II. Both equations have been considered in the literature, but only Boussinesq I appears to have physical applications, see [1] and references therein.

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