THE $SU(2)$-CHARACTER VARIETIES OF TORUS KNOTS

JAVIER MARTÍNEZ AND VICENTE MUÑOZ

Abstract. Let G be the fundamental group of the complement of the torus knot of type (m,n). We study the relationship between $SU(2)$ and $SL(2,\mathbb{C})$-representations of this group, looking at their characters. Using the description of the character variety of G, $X(G)$, we give a geometric description of $Y(G) \subset X(G)$, the set of characters arising from $SU(2)$-representations.

1. Preliminaries and notation

Given a finitely presented group $G = \langle x_1 \dots x_k | r_1, ..., r_s \rangle$, a $SU(2)$-representation is a homomorphism $\rho : G \to SU(2)$. Every representation is completely determined by the image of the generators, the k-tuple $(A_1, ..., A_k)$ satisfying the relations $r_j(A_1, ..., A_k) = \mathrm{Id}$. It can be shown that the space of all representations, $R_{SU(2)}(G) = \mathrm{Hom}(G, SU(2))$ is an affine algebraic set.

It is natural to declare a certain equivalence relation between these representations: we say that ρ and ρ' are equivalent if there exists $P \in SU(2)$ such that $\rho'(g) = P^{-1}\rho(g)P$ for all $g \in G$.

We want to consider the moduli space of $SU(2)$-representations, the GIT quotient:

$$\mathcal{M}_{SU(2)} = \mathrm{Hom}(G, SU(2))//SU(2).$$

There are also analogous definitions for $SL(2,\mathbb{C})$: we can consider $SL(2,\mathbb{C})$-representations of G, which form a set $R_{SL(2,\mathbb{C})}(G)$, consider $SL(2,\mathbb{C})$-equivalence and construct the associated moduli space:

$$\mathcal{M}_{SL(2,\mathbb{C})} = \mathrm{Hom}(G, SL(2,\mathbb{C}))//SL(2,\mathbb{C}).$$

The natural inclusion $SU(2) \hookrightarrow SL(2,\mathbb{C})$ shows that we can regard every $SU(2)$-representation as a $SL(2,\mathbb{C})$-representation. Moreover, if two representations are $SU(2)$-equivalent, then they are also $SL(2,\mathbb{C})$-equivalent. This leads to a map between moduli spaces:

$$\mathcal{M}_{SU(2)} \xrightarrow{i_*} \mathcal{M}_{SL(2,\mathbb{C})}$$

To every representation $\rho \in R_{SL(2,\mathbb{C})}(G)$ we can associate its character χ_ρ, defined as the map $\chi_\rho : G \to \mathbb{C}$, $\chi_\rho(g) = \mathrm{tr}(\rho(g))$. This defines a map $\chi : R_{SL(2,\mathbb{C})}(G) \to \mathbb{C}^G$,

2010 *Mathematics Subject Classification.* 14D20, 57M25, 57M27.
Key words and phrases. torus knot, character variety, representations.
Partially supported through Spanish MICINN grant MTM2010-17389.

where equivalent representations have the same character. Its image $X_{SL(2,\mathbb{C})}(G) = \chi(R_{SL(2,\mathbb{C})}(G))$ is called the character variety of G.

There is an important relation between the $SL(2,\mathbb{C})$-character variety of G and the moduli space $\mathcal{M}_{SL(2,\mathbb{C})}$. It is seen in [1] that:

- $X_{SL(2,\mathbb{C})}(G)$ can be endowed with the structure of algebraic variety.
- The natural associated map that takes every representation to its character, $\mathcal{M}_{SL(2,\mathbb{C})}(G) \longrightarrow X_{SL(2,\mathbb{C})}(G)$, is bijective. We specify the nature of this correspondence for the case of $SU(2)$-representations in the next section.

We emphasize that $X_{SL(2,\mathbb{C})}(G)$, as a set, consists of characters of $SL(2,\mathbb{C})$-representations. We can also take the set of characters of $SU(2)$-representations, and again we will have a map $X_{SU(2)}(G) \xrightarrow{i^*} X_{SL(2,\mathbb{C})}(G)$.

We focus on the case when G is a torus knot group. Consider the torus of revolution $T^2 \subset S^3$. If we identify it with $\mathbb{R}^2/\mathbb{Z}^2$, the image of the line $y = \frac{m}{n}x$ defines the torus knot of type (m,n), $K_{m,n} \subset S^3$ for coprime m,n. An important invariant of a knot is the fundamental group of its complement in S^3, $G_{m,n} = \pi_1(S^3 - K_{m,n})$. These groups admit the following presentation:

$$G_{m,n} = \langle x, y \mid x^m = y^n \rangle$$

The $SL(2,\mathbb{C})$-character variety of these groups for the case $(m,2)$ was treated in [5]. A complete description for (m,n) coprime was given in [4], and the general case (m,n) was studied using combinatorial tools in [3]. $SU(2)$-character varieties for knot groups were studied in [2]. For the case $(m,2)$, the relation between both character varieties has been recently treated in [6].

2. $SU(2)$-character varieties

We recall that $SU(2) \cong S^3$, the isomorphism being given by:

$$\begin{array}{rcl} S^3 \subset \mathbb{C}^2 & \longrightarrow & SU(2) \\ (a,b) & \longrightarrow & \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \end{array}$$

The correspondence is a ring homomorphism if we look at S^3 as the set of unit quaternions. First of all, we want to point out the following fact, which was already true for $SL(2,\mathbb{C})$:

Proposition 1. *The correspondence:*

$$\begin{array}{rcl} \mathcal{M}_{SU(2)}(G) & \longrightarrow & X_{SU(2)}(G) \\ \rho & \longrightarrow & \chi_\rho \end{array}$$

that takes a representation to its character is bijective.

Proof. We follow the steps taken in [1], this time for $SU(2)$. First of all, every matrix A in $SU(2)$ is normal, hence diagonalizable. Since $\det(A) = 1$, the eigenvalues of A are $\{\lambda, \lambda^{-1}\}$ for some $\lambda \in \mathbb{C}^*$. In particular, $\mathrm{tr}(A)$ completely determines the set of eigenvalues $\{\lambda, \lambda^{-1}\}$.

Now, if ρ is a reducible $SU(2)$-representation, there is a common eigenvector e_1 for all $\rho(g)$ and therefore they are all diagonal with respect to the same basis. If ρ' is a second reducible representation such that $\chi_\rho(g) = \chi_{\rho'}(g)$ for all $g \in G$, this means that they share the same eigenvalues for every $g \in G$. After choosing another basis for ρ' such that $\rho'(g)$ is diagonal for all $g \in G$:

$$\rho(g) = \begin{pmatrix} \lambda(g) & 0 \\ 0 & \lambda^{-1}(g) \end{pmatrix} \quad \rho'(g) = \begin{pmatrix} \mu(g) & 0 \\ 0 & \mu^{-1}(g) \end{pmatrix}$$

where either $\lambda(g) = \mu(g)$ or $\lambda(g) = \mu^{-1}(g)$ for every $g \in G$. Interchanging the roles of λ and λ^{-1} if necessary, there is always $g_1 \in G$ such that $\lambda(g_1) = \mu(g_1)$, so there is $g_1 \in G$ such that $\rho(g_1) = \rho'(g_1)$. We also notice that if $\rho(g) = \pm\,\mathrm{Id}$, then $\rho'(g) = \rho(g) = \pm\,\mathrm{Id}$.

We claim that $\rho(g_2) = \rho'(g_2)$ for all $g_2 \in G$. If not, there exists $g_2 \in G$ such that $\rho(g_2) = \rho'(g_2)^{-1} \neq \pm\,\mathrm{Id}$. So $\lambda(g_1) = \mu(g_1)$ and $\lambda(g_2) = \mu^{-1}(g_2)$. On the other hand, we know that $\mathrm{tr}(\rho'(g_1 g_2)) = \mathrm{tr}(\rho(g_1 g_2))$, so:

$$\begin{aligned} \mu(g_1)\mu(g_2) + \mu^{-1}(g_1)\mu^{-1}(g_2) &= \lambda(g_1)\lambda(g_2) + \lambda^{-1}(g_1)\lambda^{-1}(g_2) \\ &= \mu(g_1)\mu^{-1}(g_2) + \mu^{-1}(g_1)\mu(g_2) \end{aligned}$$

Rearranging the terms:

$$\mu(g_2)(\mu(g_1) - \mu^{-1}(g_1)) = \mu^{-1}(g_2)(\mu(g_1) - \mu^{-1}(g_1))$$

which implies that $\mu(g_2) = \pm 1$, so that $\rho(g_2) = \pm\,\mathrm{Id}$, a contradiction. Therefore $\lambda(g) = \mu(g)$ for all $g \in G$. Hence there exists $P \in SU(2)$ such that $\rho(g) = P^{-1}\rho(g)P$ for all $g \in G$, i.e, the representations are equivalent.

For the irreducible case, we point out the following fact: if ρ is a irreducible $SU(2)$-representation and $\rho(g) \neq \pm\,\mathrm{Id}$ for a given $g \in G$, then there exists $h \in G$ such that ρ restricted to the subgroup $H = \langle g, h \rangle$ is again irreducible. To see it, since $\rho(g) \neq \pm\,\mathrm{Id}$, $\rho(g)$ has two eigenspaces L_1, L_2 associated to the pair of different eigenvalues μ_1, μ_2. Since the representation is irreducible, there are elements h_i such that L_i is not invariant under $\rho(h_i)$. We can take $h = h_1$ or $h = h_2$ unless L_1 is invariant under $\rho(h_2)$, or L_2 is invariant under $\rho(h_1)$, in this case we can choose $h = h_1 h_2$.

For a group generated by two elements, $H = \langle g, h \rangle$, the reducibility of a representation is completely determined by $\chi_\rho([g, h])$. It can be seen in the following chain of equivalences:

$$\begin{aligned}
\rho|_H \text{ is reducible} &\Leftrightarrow \rho(g), \rho(h) \text{ share a common eigenvector} \\
&\Leftrightarrow \rho(g), \rho(h) \text{ are simultaneously diagonalizable} \\
&\Leftrightarrow [\rho(g), \rho(h)] = \mathrm{Id} \\
&\Leftrightarrow \mathrm{tr}[\rho(g), \rho(h)] = 2 \\
&\Leftrightarrow \chi_\rho([g,h]) = 2
\end{aligned}$$

Let ρ, ρ' be two $SU(2)$-representations such that $\chi_\rho = \chi_{\rho'}$. By the previous observation, there are $g, h \in G$ such that $\rho|_{\langle g,h\rangle}$ is irreducible, i.e, $\chi_\rho([g,h]) \neq 2$. It follows that, since $\chi_\rho = \chi_{\rho'}$, $\chi_{\rho'}([g,h]) \neq 2$, so $\rho'|_{\langle g,h\rangle}$ is irreducible too. Varying ρ, ρ' in their equivalence classes, we can assume that there are basis B, B' such that:

$$\rho(h) = \rho'(h) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

The matrices $\rho(g), \rho'(g)$ will not be triangular, by irreducibility, and conjugating again by diagonal unitary matrices, we can assume that:

$$\rho(g) = \begin{pmatrix} a & -b \\ b & \bar{a} \end{pmatrix}, \qquad \rho'(g) = \begin{pmatrix} a' & -b' \\ b' & \bar{a}' \end{pmatrix}$$

for $a, a' \in \mathbb{C}$, $b, b' \in \mathbb{R}^+$. Notice that $b, b' \neq 0$ since $\rho|_{\langle g,h\rangle}$ is irreducible. More in general, for any $\alpha \in G$:

$$\rho(\alpha) = \begin{pmatrix} x & -\bar{y} \\ y & \bar{x} \end{pmatrix}, \qquad \rho'(\alpha) = \begin{pmatrix} x' & -\bar{y}' \\ y' & \bar{x}' \end{pmatrix}$$

Now, the equations $\chi_\rho(\alpha) = \chi_{\rho'}(\alpha)$, $\chi_\rho(h\alpha) = \chi_{\rho'}(h\alpha)$ imply that:

$$\begin{aligned}
x + \bar{x} &= x' + \bar{x}' \\
\lambda x + \lambda^{-1}\bar{x} &= \lambda x' + \lambda^{-1}\bar{x}'
\end{aligned}$$

and since $\lambda \neq \pm 1$, we get that $x = x'$.

Substituting $\alpha = g$, we get that $a = a'$ and since $\det(\rho(g)) = \det(\rho'(g)) = 1$, $b = b'$, so $\rho(g) = \rho'(g)$.

Substituting again α for $g\alpha$, we arrive at the equation $ax - by = ax - by'$, which implies that $y = y'$ and finally that $\rho(\alpha) = \rho'(\alpha)$: we have proved that the representations ρ and ρ', after $SU(2)$-conjugation, are the same, i.e, they are equivalent. □

Corollary 2. *We have a commutative diagram:*

$$\begin{array}{ccc}
\mathcal{M}_{SU(2)}(G) & \xrightarrow{1:1} & X_{SU(2)}(G) \\
\downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i_*} \\
\mathcal{M}_{SL(2,\mathbb{C})}(G) & \xrightarrow{1:1} & X_{SL(2,\mathbb{C})}(G)
\end{array}$$

The previous corollary shows that we can equivalently study the relationship between $SU(2)$ and $SL(2,\mathbb{C})$-representations of G from the point of view of their characters or from the point of view of their representations. Looking at the diagram, we also deduce that:

Corollary 3. *The natural inclusion $i_*: \mathcal{M}_{SU(2)}(G) \longrightarrow \mathcal{M}_{SL(2,\mathbb{C})}(G)$ is injective.*

3. $SU(2)$-character varieties of torus knots

We focus now on the specific case of the torus knot $G_{m,n}$ of coprime type (m,n). Henceforth, we will often denote $X_{SL(2,\mathbb{C})} = X_{SL(2,\mathbb{C})}(G)$ and omit the group in our notation. In this case:

$$R_{SL(2,\mathbb{C})}(G) = \{(A,B) \in SL(2,\mathbb{C}) \mid A^m = B^n\}$$

and:

$$R_{SU(2)}(G) = \{(A,B) \in SU(2) \mid A^m = B^n\}$$

We have a decomposition of $X_{SL(2,\mathbb{C})}$:

$$X_{SL(2,\mathbb{C})} = X_{red} \cup X_{irr}$$

where X_{red} is the subset of characters of reducible representations and X_{irr} is the subset of characters of irreducible representations. Inside $X_{SL(2,\mathbb{C})}$ we have $i_*(X_{SU(2)})$, i.e, the set of characters of $SU(2)$-representations. For simplicity, we will denote $Y = i_*(X_{SU(2)})$. Again, Y decomposes in $Y_{red} \cup Y_{irr}$.

Reducible representations.

Proposition 4. *There is an isomorphism $Y_{red} \cong [-2,2] \subset \mathbb{R}$*

Proof. We will use, from now on, the explicit description of $X_{SL(2,\mathbb{C})}$ given in [4]. There is an isomorphism $X_{red} \cong \mathbb{C}$ given by:

$$\left(A = \begin{pmatrix} t^n & 0 \\ 0 & t^{-n} \end{pmatrix}, B = \begin{pmatrix} t^m & 0 \\ 0 & t^{-m} \end{pmatrix} \longrightarrow s = t + t^{-1} \in \mathbb{C} \right)$$

This is because given a reducible $SL(2,\mathbb{C})$-representation ρ, we can consider the associated split representation $\rho = \rho' + \rho''$, for which in a certain basis takes the form:

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, B = \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix}$$

and the equality $A^m = B^n$ implies that $\lambda = t^n, \mu = t^m$ for a unique $t \in \mathbb{C}$ (here we use that m,n are coprime). Now, since $A, B \in SU(2)$, t must satisfy that $|t|^2 = 1$, i.e, $t \in S^1 \subset \mathbb{C}$. We have to also take account of the change of order of the basis elements and therefore $t \sim \frac{1}{t}$. So the parameter space is isomorphic to $[-2,2]$ (under the correspondence $t \in S^1 \longrightarrow s = t + t^{-1} = 2\,\mathrm{Re}(t) \in [-2,2]$). $\square$

To explicitly describe when a pair (A, B) is reducible, we follow [4, 2.2]. First of all, A and B are diagonalizable (recall that $A, B \in SU(2)$), so we can rule out the Jordan type case since it is not possible. So:

Proposition 5. *In any of the cases:*

- $A^m = B^n \neq \pm\,\mathrm{Id}$
- $A = \pm\,\mathrm{Id}$ *or* $B = \pm\,\mathrm{Id}$

the pair (A, B) *is reducible.*

Proof. Let us deal with the first case, when $A^m = B^n \neq \pm\,\mathrm{Id}$. A is diagonalizable with respect to a basis $\{e_1, e_2\}$, and takes the form $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$. Then:

$$B^n = A^m = \begin{pmatrix} \lambda^m & 0 \\ 0 & \lambda^{-m} \end{pmatrix}$$

so B is diagonal in the same basis and the pair is reducible. For the second case, if $A = \alpha\,\mathrm{Id}$, where $\alpha = \pm 1$, then any basis diagonalizing B diagonalizes A, hence the pair is reducible. The case $B = \alpha\,\mathrm{Id}$ follows in the same way. □

Irreducible representations.

Now we look at the irreducible set of representations, since we want to study Y_{irr}. Let $(A, B) \in R_{SU(2)}(G)$ be an irreducible pair. Both are diagonalizable, and using Proposition 5, they must satisfy that $A^m = B^n = \pm\,\mathrm{Id}$, $A, B \neq \pm\,\mathrm{Id}$. The eigenvalues $\lambda, \lambda^{-1} \neq \pm 1$ of A satisfy $\lambda^m = \pm 1$, the eigenvalues μ, μ^{-1} of B satisfy $\mu^n = \pm 1$ and $\lambda^m = \mu^n$.

We can associate to A a basis $\{e_1, e_2\}$ under which it diagonalizes, and the same for B, obtaining another basis $\{f_1, f_2\}$. The eigenvalues λ, μ and the eigenvectors e_i, f_i completely determine the representation (A, B). We are interested in $i_*(\mathcal{M}_{SU(2)})$, $SL(2, \mathbb{C})$-equivalence classes of such pairs (A, B), and these are fully described by the projective invariant of the four points $\{e_1, e_2, f_1, f_2\}$, the cross ratio:

$$[e_1, e_2, f_1, f_2] \in \mathbb{P}^1 - \{0, 1, \infty\}$$

(we may assume that the four eigenvectors are different since the representation is irreducible, see [4] for details).

Since both $A, B \in SU(2)$, we know that $e_1 \perp e_2$ and $\|e_1\| = \|e_2\| = 1$, so shifting the vectors by a suitable rotation $C \in SU(2)$, we can assume that $e_1 = [1 : 0], e_2 = [0 : 1]$, and therefore $f_1 = [a : b], f_2 = [-\bar{b} : \bar{a}]$, since they are orthogonal too. So the pair (A, B) inside $X_{SL(2,\mathbb{C})}$ is determined by λ, μ satisfying the conditions above and the projective cross ratio:

$$r = \Big[e_1, e_2, f_1, f_2\Big] = \Big[0, \infty, \frac{b}{a}, -\frac{\bar{a}}{\bar{b}}\Big] = \frac{b\bar{b}}{-a\bar{a}} = \frac{b\bar{b}}{b\bar{b} - 1} = \frac{t}{t-1}$$

where we have used that $a\bar{a} + b\bar{b} = 1$ and $t = |b|^2, b \in (0,1)$. We also get that r is real and $r \in (-\infty, 0)$.

The converse is also true: if the triple (λ, μ, r), satisfies that $\lambda^m = \mu^n = \pm 1$, $\lambda, \mu \neq \pm 1$ and $r \in (-\infty, 0)$, then $(A,B) \in i_*(\mathcal{M}_{SU(2)})$. To see this, r determines uniquely $t = |b|^2$ since $r(t)$ is invertible for $t \in (0,1)$. Once $|b|$ is fixed, we get that $|a|$ is fixed too, using $|a|^2 = 1 - |b|^2$. We can choose any $(a,b) \in S^1 \times S^1$ and we conclude that (A,B) is $SL(2,\mathbb{C})$-equivalent to a $SU(2)$ representation. To be more precise, it is equivalent to the representation with eigenvalues λ, μ and eigenvectors $[1:0], [0:1], [a:b], [-\bar{b}, \bar{a}]$.

Finally, we have to take account of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ action given by the permutation of the eigenvalues:

- Permuting e_1, e_2 takes (λ, μ, r) to $(\lambda^{-1}, \mu, r^{-1})$
- Permuting f_1, f_2 takes (λ, μ, r) to $(\lambda, \mu^{-1}, r^{-1})$

Since $\lambda^m = \mu^n = \pm 1$, we get that:

$$\lambda = e^{\pi i k/m}, \qquad \mu = e^{\pi i k'/n}, \tag{1}$$

where since $\lambda \sim \lambda^{-1}, \mu \sim \mu^{-1}$ and $\lambda \neq \pm 1, \mu \neq \pm 1$, we can restrict to the case when $0 < k < m, 0 < k' < n$. We also notice that $\lambda^m = \mu^n$ implies that $k \equiv k' \pmod 2$. So the irreducible part is made of $(m-1)(n-1)/2$ intervals.

We have just proved:

Proposition 6.

$$Y_{irr} \cong \{(\lambda, \mu, r) : \lambda^m = \mu^n = \pm 1; \lambda, \mu \neq \pm 1; r \in (-\infty, 0)\}/\mathbb{Z}_2 \times \mathbb{Z}_2$$

This real algebraic variety consists of $\frac{(m-1)(n-1)}{2}$ *open intervals.*

To describe the closure of the irreducible orbits, we have to consider the case when $e_1 = f_1$, since this is what happens in the limit (the situation is analogous when $e_2 = f_2$). In this situation $r = 0$, and the representation is equivalent to a reducible representation. Taking into account Lemma 4, it corresponds to a certain $t \in S^1$ such that $\lambda = t^n$, $\mu = t^m$. We have another limit case $r = -\infty$, if we allow $e_1 = f_2$. The representation is again reducible and corresponds to another $t' \in S^1$ such that $\lambda = (t')^n$, $\mu^{-1} = (t')^m$.

Remark 7. *The explicit description of the set of* $SU(2)$*-representations allows us to give an alternative proof of Corollary 3, which stated that the inclusion* $i_* : \mathcal{M}_{SU(2)} \to \mathcal{M}_{SL(2,\mathbb{C})}$ *is injective.*

Let us see this. Suppose that (A,B) *and* (A',B') *are two* $SU(2)$*-representations which are mapped to the same point in* $\mathcal{M}_{SL(2,\mathbb{C})}$*, i.e, which are* $SL(2,\mathbb{C})$*-equivalent. If we denote by* u_1, u_2, u_3, u_4 *the set of eigenvectors of* (A,B) *and by* v_1, v_2, v_3, v_4 *the* *set of eigenvectors of (A', B'), we know that:*

$$[u_1, u_2, u_3, u_4] = [v_1, v_2, v_3, v_4] = r \in (-\infty, 0)$$

Since their cross ratio is the same, we know that there exists $P \in SL(2, \mathbb{C})$ that takes the set u_i to v_i. Moreover, since P takes the unitary basis u_1, u_2 to the unitary basis v_1, v_2, we get that $P \in SU(2)$, and therefore both representations are $SU(2)$-equivalent.

Topological description. We finally describe Y topologically. We refer to [4] for a geometric description of $X_{SL(2,\mathbb{C})}$.

Using proposition 6, Y_{irr} is a collection of real intervals (parametrized by $r \in (-\infty, 0)$) for a finite number of (λ, μ) that satisfy the required conditions. By our last observation, the limit cases when $r = 0, \infty$ (i.e, points in the closure of Y_{irr}) correspond to the points where Y_{irr} intersects Y_{red}.

As we saw before, each interval has two points in its closure: these are $t_0 \in S^1$ such that $t_0^n = \lambda$, $t_0^m = \mu$ ($r = 0$) and $t_1 \in S^1$ corresponding to $t_1^n = \lambda$, $t_1^m = \mu^{-1}$ ($r = -\infty$). The conditions on λ, μ force that $t_0 \neq t_1$ so that we get different intersection points with Y_{red}.

Y is topologically a closed interval (Y_{red}) with $(m-1)(n-1)/2$ open intervals (Y_{irr}) attached at $(m-1)(n-1)$ different points (without any intersections among them). The interval $Y_{red} = [-2, 2]$ sits inside $X_{red} \cong \mathbb{C}$ and every real interval in Y_{irr} is inside the corresponding complex line in X_{irr}.

The situation is described in the following two pictures:

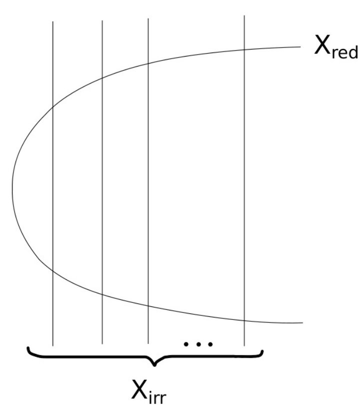

FIGURE 1. Picture of $X_{SL(2,\mathbb{C})}$, defined over $\mathbb{C}$. The drawn lines are curves isomorphic to $\mathbb{C}$.

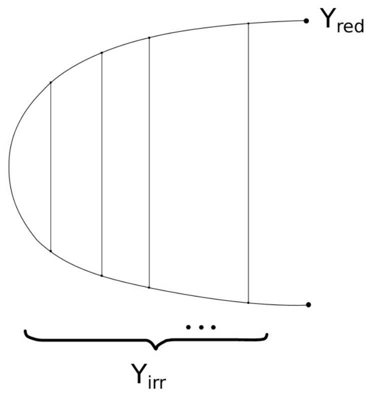

FIGURE 2. Picture of $Y \subset X_{SL(2,\mathbb{C})}$, defined over $\mathbb{R}$. The picture displays the set of real segments which form Y_{irr}.

4. Noncoprime case

If $\gcd(m,n) = d > 1$, then $G_{m,n}$ does no longer represent a torus knot, since these are only defined in the coprime case. However, the group $G_{m,n} = \langle x, y \mid x^n = y^m \rangle$ still makes sense and we can study the representations of this group into $SL(2,\mathbb{C})$ and $SU(2)$ using the method described above. We will denote by a, b the integers that satisfy:

$$\begin{aligned} m &= a\,d, \\ n &= b\,d. \end{aligned}$$

As we did before, we focus on $Y = i_*(X_{SU(2)})$, the set of characters of $SU(2)$-representations.

Reducible representations. First of all, we describe what happens in the $SL(2,\mathbb{C})$ case:

Proposition 8. *There is an isomorphism:*

$$X_{red} \cong \bigsqcup_{i=0}^{\lfloor d/2 \rfloor} X^i_{red}$$

where:

- $X^i_{red} \cong \mathbb{C}^*$ *for* $0 < i < \frac{d}{2}$.
- $X^i_{red} \cong \mathbb{C}$ *for* $i = 0$ *and* $i = \frac{d}{2}$ *if* d *is even*

Proof. As it is shown in [4], an element in X_{red} can be regarded as the character of a split representation, $\rho = \rho' \oplus \rho'^{-1}$. There is a basis such that:

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad B = \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix},$$

where $A^m = B^n$ implies that $\lambda^m = \mu^n$. We deduce that $(\lambda^a)^d = (\mu^b)^d$, so that (λ, μ) belong to one of the components:

$$X^i_{red} = \{(\lambda, \mu) | \lambda^a = \xi^i \mu^b\} = \{(\lambda, \mu) | \lambda^a \mu^{-b} = \xi^i\},$$

where ξ is a primitive d-th root of unity. These components are disjoint, and each one of them is parametrized by $\mathbb{C}^*$. To see this, let us fix a component, X^i_{red}, and let α be a b-th root of ξ^i. Then:

$$\begin{aligned} X^i_{red} &= \{(\lambda, \mu) | \lambda^a = \xi^i \mu^b\} \\ &= \{(\lambda, \mu) | \lambda^a = \alpha^b \mu^b\} \\ &= \{(\lambda, \nu) | \lambda^a = \nu^b\} \cong \mathbb{C}^* . \end{aligned}$$

In other words, for each $(\lambda, \mu) \in X^i_{red}$ there is a unique $t \in \mathbb{C}^*$ such that $t^b = \lambda$, $t^a = \alpha\mu$. However, we have to take account of the action given by permuting the two vectors in the basis, which corresponds to the change $(\lambda, \mu) \sim (\lambda^{-1}, \mu^{-1})$. In our decomposition, if $(\lambda, \mu) \in X^i_{red}$, then $(\lambda^{-1}, \mu^{-1}) \in X^{-i}_{red}$. So $t \in X^i_{red}$ is equivalent to $1/t \in X^{-i}_{red}$.

For $0 \leq i \leq d-1$, we have two possibilities. If $i \not\equiv -i \pmod d$, then X^i_{red} and X^{-i}_{red} get identified. If $i \equiv -i \pmod d$, then $t \sim t^{-1} \in X^i_{red} \cong \mathbb{C}$, and thus $X^i_{red}/_{\sim} \cong \mathbb{C}^*/_{a \sim a^{-1}} \cong \mathbb{C}$.

When d is even, there are two $i \in \mathbb{Z}/d\mathbb{Z}$ such that $i \equiv -i \pmod d$, so we get two copies of $\mathbb{C}$ in Y_{red}. When d is odd we get just one, since there is only one solution ($i \equiv 0$). The remaining copies of X^i_{red} get identified pairwise: $X^i_{red} \sim X^{-i}_{red}$. □

Now, for the case of $SU(2)$-representations, we have:

Proposition 9. *There is an isomorphism:*

$$Y_{red} \cong \bigsqcup_{i=0}^{\lfloor \frac{d}{2} \rfloor} Y^i_{red}$$

where:

- $Y^i_{red} \cong S^1$ *for* $0 < i < \frac{d}{2}$
- $Y^i_{red} \cong [-2, 2]$ *for* $i = 0$, $i = \frac{d}{2}$ *if* d *is even*

Proof. If (A, B) is a reducible $SU(2)$-representation, both are diagonalizable with respect to a certain basis and therefore:

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \quad B = \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix}$$

The equality $A^m = B^n$ gives us that $\lambda^m = \mu^n$. So the pair (λ, μ) belongs to a certain component X^i_{red}. Since it is a $SU(2)$-representation, the eigenvalues λ and μ satisfy that $|\lambda| = |\mu| = 1$. This implies that $(\lambda, \mu) \in S^1 \subset \mathbb{C}^* \cong X^i_{red}$: we define $Y^i_{red} := S^1 \subset X^i_{red}$.

We have to take into account the equivalence relation in X_{red} given by the permutation of the eigenvectors. If $i \not\equiv -i \pmod d$, then $Y^i_{red} \cong Y^{-i}_{red}$. If $i \equiv -i \pmod d$, then $Y^i_{red} \cong S^1/_{a\sim a^{-1}} \cong [-2, 2]$. This gives the desired result. □

Irreducible representations. We start by describing what happens in the $SU(2)$ case.

Proposition 10. *We have an isomorphism*

$$Y_{irr} \cong \{(\lambda, \mu, r) : \lambda^m = \mu^n = \pm 1; \lambda, \mu \neq \pm 1, r \in (-\infty, 0)\}/\mathbb{Z}_2 \times \mathbb{Z}_2 .$$

This real algebraic variety consists of:

- $\frac{(m-1)(n-1)+1}{2}$ *open intervals if* m, n *are both even,*
- $\frac{(m-1)(n-1)}{2}$ *open intervals in any other case.*

Proof. By Proposition 5, a representation (A, B) is reducible unless $A^m = B^n = \pm \mathrm{Id}$, $A, B \neq \pm \mathrm{Id}$. So the set of irreducible representations can be described using the same tools as before: the set of equivalence classes of irreducible representations is a collection of intervals $r \in (-\infty, 0)$ parametrized by pairs (k, k') satisfying:

$$0 < k < m, \quad 0 < k' < n, \quad k \equiv k' \pmod 2. \tag{2}$$

We compute the number of such pairs, separating in three different cases according to the parity of m and n:

Suppose m, n are both even. If $k \equiv k' \equiv 0 \pmod 2$, then $k \in \{2, 4, \ldots, m-2\}$, $k' \in \{2, 4, \ldots n-2\}$, so there are $\frac{(m-2)(n-2)}{4}$ such pairs. If $k \equiv k' \equiv 1 \pmod 2$, $k \in \{1, 3, \ldots m-1\}$, $k' \in \{1, 3, \ldots, n-1\}$, we have $\frac{mn}{4}$ pairs. The sum is $\frac{(m-2)(n-2)}{4} + \frac{mn}{4} = \frac{(m-1)(n-1)+1}{4}$.

Suppose m is even and n is odd (the case m odd and n even is similar). Then if $k \equiv k' \equiv 0 \pmod 2$, $k \in \{2, 4, \ldots, m-2\}$, $k' \in \{2, 4, \ldots n-1\}$, we get $\frac{(m-2)(n-1)}{4}$ such pairs. If $k \equiv k' \equiv 1 \pmod 2$, $k \in \{1, 3, \ldots m-1\}$, $k' \in \{1, 3, \ldots, n-2\}$, and there are $\frac{m(n-1)}{4}$ such pairs. We get in total $\frac{m(n-1)}{4} + \frac{(m-2)(n-1)}{4} = \frac{(m-1)(n-1)}{2}$.

Finally, suppose both m, n odd. If $k \equiv k' \equiv 0 \pmod 2$, $k \in \{2, 4, \ldots, m-1\}$, $k' \in \{2, 4, \ldots n-1\}$, and we get $\frac{(m-1)(n-1)}{4}$ such pairs. If $k \equiv k' \equiv 1 \pmod 2$, $k \in \{1, 3, \ldots m-2\}$, $k' \in \{1, 3, \ldots, n-2\}$, there are $\frac{(m-1)(n-1)}{4}$ such pairs. We get $\frac{(m-1)(n-1)}{2}$ pairs in total.

We have obtained a decomposition:

$$Y_{irr} = \bigsqcup_{k,k'} Y^{(k,k')}_{irr}$$

where every $Y^{(k,k')}_{irr}$ is an open interval isomorphic to $(-\infty, 0)$. □

For the case of $SL(2, \mathbb{C})$ representations, we have the following:

Proposition 11. *The component $X_{irr} \subset X_{SL(2,\mathbb{C})}$ is described as*

$$X_{irr} = \bigsqcup_{k,k'} X_{irr}^{(k,k')}$$

where k, k' satisfy (2), and $X_{irr}^{(k,k')} = \mathbb{P}^1 - \{0, 1, \infty\}$. This complex algebraic variety consists of $\frac{(m-1)(n-1)+1}{2}$ components if m, n are both even, of $\frac{(m-1)(n-1)}{2}$ components if one of m, n is odd. Moreover $Y_{irr}^{(k,k')} = (-\infty, 0) \subset X_{irr}^{(k,k')}$ in the natural way.

The limit cases $r = 0$, $r = -\infty$ correspond to the closure of the irreducible components, and these points are exactly where $\overline{Y}_{irr}$ intersects Y_{red}. The triples $(\lambda, \mu, 0)$, $(\lambda, \mu, -\infty)$ correspond to the reducible representations with eigenvalues (λ, μ) and (λ, μ^{-1}). Since $\lambda, \mu \neq \pm 1$, we get two different intersection points. Note that the pattern of intersections for $\overline{X}_{irr}$ and X_{red} is the same, but the components are complex algebraic varieties now.

To understand the way the closure of the components of Y_{irr} intersect Y_{red}, we have the following:

Proposition 12. *The closure of $Y_{irr}^{(k,k')}$ is a closed interval that joins $Y_{red}^{i_0}$ with $Y_{red}^{i_i}$, where:*

$$i_0 = \frac{k - k'}{2}, \quad i_i = \frac{k + k'}{2} \pmod{d}.$$

Proof. Set $D = 2d\, ab$, and consider ω a primitive D-th root of unity. Then $\xi := \omega^{D/d} = \omega^{2ab}$ is a primitive d-th root of unity. The irreducible component $Y_{irr}^{(k,k')}$ is the interval (λ, μ, r), $r \in (-\infty, 0)$, where

$$\lambda = (\omega^b)^k, \quad \mu = (\omega^a)^{k'},$$

and k, k' are subject to the conditions (2), see equation (1). The points in the closure of $Y_{irr}^{(k,k')}$ correspond to the reducible representations with eigenvalues (λ, μ) and (λ, μ^{-1}). Clearly $(\lambda, \mu) \in X_{red}^{i_0}$, since

$$\lambda^a \mu^{-b} = \omega^{kab} \omega^{-k'ab} = \omega^{\frac{k-k'}{2} 2ab} = \omega^{i_0 2ab} = \xi^{i_0},$$

and $(\lambda, \mu^{-1}) \in X_{red}^{i_1}$, since

$$\lambda^a \mu^b = \omega^{kab} \omega^{k'ab} = \xi^{i_1}.$$

□

Proposition 12 gives a clear rule to depict $Y = Y_{irr} \cup Y_{red}$ for every pair (m, n). Acutally, Y is a collection of intervals attached on their endpoints to Y_{red}, which consists of several disjoint copies of S^1 and $[-2, 2]$. Note that the pattern of intersections for the irreducible components of $X_{SL(2,\mathbb{C})} = X_{irr} \cup X_{red}$ is the same as that of Y.

When m, n are coprime, we recover our previous pictures.

Corollary 13. *For any two different components $Y_{red}^{i_0}, Y_{red}^{i_1} \subset Y_{red}$, there is a pair (k,k') such that $\overline{Y}_{irr}^{(k,k')}$ joins them.*

In particular, Y is a connected topological space.

Proof. We can assume $0 \leq i_0 < i_i \leq \frac{d}{2}$. Then $0 < k = d + i_0 - i_1 < d \leq m$ and $0 < k' = d - i_0 - i_1 < d \leq n$ both satisfy that $k \equiv k' \pmod 2$ and $\frac{k-k'}{2} = i_0$, $\frac{k+k'}{2} = i_1$. □

Remark 14. *It can be checked that there is no component $\overline{Y}_{irr}^{(k,k')}$ which joins $Y_{red}^{i_0}$ to itself when $m = n$, or when one of m, n divides the other, and we are dealing with $i_0 = 0$ or $i_0 = d/2$ (the latter only if d is even).*

Actually, such component would correspond to a pair (k,k') such that $\frac{k-k'}{2} \equiv \pm i_0 \pmod d$ and $\frac{k+k'}{2} \equiv \pm i_0 \pmod d$. Accounting for all possibilities of signs, we have either $k \equiv \pm 2i_0$, $k' \equiv 0 \pmod d$, or $k \equiv 0$, $k' \equiv \pm 2i_0 \pmod d$. This has solutions unless $m > n = d$, $i_0 = 0, d/2$; $n > m = d$, $i_0 = 0, d/2$; or $m = n = d$, any i_0.

References

[1] M. Culler and P.B. Shalen, *Varieties of group representations and splittings of 3-manifolds*, Ann. of Math. (2) **117** (1983), no. 1, 109-146.

[2] E.P. Klassen, *Representations of knot groups in $SU(2)$*, Trans. Amer. Math. Soc **326** (1991), no. 2, 795-828.

[3] J. Martín-Morales and A.M. Oller-Marcén, *Combinatorial aspects of the character variety of a family of one-relator groups*, Topology Appl., **156** (2009), no. 14, 2376-2389.

[4] V. Muñoz, *The $SL(2,\mathbb{C})$-character varieties of torus knots*, Rev. Mat. Complut., **22** (2009), no.2, 489-497.

[5] A. M. Oller-Marcén, *The $SL(2,\mathbb{C})$ character variety of a class of torus knots*, Extracta Math., **23** (2008), no. 2, 163-172.

[6] A. M. Oller-Marcén, *$SU(2)$ and $SL(2,\mathbb{C})$ representations of a class of torus knots*, Extracta Math., 2012, to appear.

Facultad de Matemáticas, Universidad Complutense de Madrid, Plaza de Ciencias 3, 28040 Madrid, Spain

E-mail address: javiermartinez@mat.ucm.es

Facultad de Matemáticas, Universidad Complutense de Madrid, Plaza de Ciencias 3, 28040 Madrid, Spain

E-mail address: vicente.munoz@mat.ucm.es