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Crum transformation and Wronskian type solutions for supersymmetric KdV equation

Q.P. Liu¹, M. Mañas²

Departamento de Física Teórica, Universidad Complutense, E28040-Madrid, Spain

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Abstract

Darboux transformation is reconsidered for the supersymmetric KdV system. By iterating the Darboux transformation, a supersymmetric extension of the Crum transformation is obtained for the Manin–Radul SKdV equation, in doing so one gets Wronskian superdeterminant representations for the solutions. Particular examples provide us explicit supersymmetric extensions, super solitons, of the standard soliton of the KdV equation. The KdV soliton appears as the body of the super soliton. © 1997 Elsevier Science B.V.

1. Introduction

Supersymmetric integrable systems have attracted much attention during last decade, and as a consequence a number of results has been established in this field. Thus, several well known integrable systems, such as Kadomtsev–Petviashvili (KP) [10], Korteweg–de Vries (KdV) [10], Sine–Gordon [3], Nonlinear Schrödinger [14], Ablowitz–Kaup–Newell–Segur/Zakharov–Shabat [12] and Harry Dym [7] systems have been embedded into their supersymmetric counterparts. We notice that developing this theory is not only interesting from a mathematical viewpoint, but also may have physical relevance as toy models for more realistic supersymmetric systems and in two-dimensional supersymmetric quantum gravity [1].

Once these supersymmetric integrable models have been constructed one needs to extend the nonsupersymmetric tools to construct solutions to this supersymmetric framework. We are particularly interested in Darboux transformation [11,13], which is very powerful to this end. Indeed, the initial steps along this research line have been taken in [8] and there Darboux and binary Darboux transformations have been constructed for Manin–Radul super KdV (MRSKdV) and Manin–Radul–Mathieu super KdV systems. Furthermore, in [9] vectorial Darboux transformations have been given for the MRSKdV and explicit soliton type solutions were presented.

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In this paper we give a Crum type transformation [4,16] for the MRSKdV system and its Lax operator, a supersymmetric extension of the stationary Schrödinger operator. In doing so, given any solution of the system and wave functions solving the associated linear system, we obtain new solutions represented in terms of superdeterminants of matrices of Wronski type.

The paper is organized as follows. In Section 2 we consider the iteration of the Darboux transformation of [8] obtaining a Crum type transformation and associated Wronskian superdeterminant expressions for the new solutions, we also see how this transformation reduces down to the classical case of KdV equation. In Section 3 we present some simple examples, constructing a family of solutions that can be considered as a super soliton solution because it is a supersymmetrized version of the standard soliton. Indeed, the KdV soliton solution appears as the body of the super soliton, which in turn has its soul exponentially localized.

2. Darboux transformation for MRSKdV: Iteration and reduction

The MRSKdV system was introduced in the context of supersymmetric KP hierarchy [10] and it reads as

$$\partial_t \alpha = \frac{1}{4} \partial (\partial^2 \alpha + 3\alpha D\alpha + 6\alpha u), \quad \partial_t u = \frac{1}{4} \partial (\partial^2 u + 3u^2 + 3\alpha Du), \quad (1)$$

where we use the notation $\partial f := \partial f / \partial x$, $\partial_t f := \partial f / \partial t$, $x, t \in \mathbb{C}$ and D is the super derivation defined by $D := \partial_{\vartheta} + \vartheta \partial$ with ϑ a Grassmann odd variable.

It is known that the following linear system

$$\begin{aligned} (L - \lambda)\psi &:= (\partial^2 + \alpha D + u - \lambda)\psi = 0, \\ (\partial_t - M)\psi &:= [\partial_t - \frac{1}{2}\alpha \partial D - \lambda \partial - \frac{1}{2}u \partial + \frac{1}{4}(\partial \alpha)D + \frac{1}{4}(\partial u)]\psi = 0, \end{aligned} \quad (2)$$

has as its compatibility condition

$$[\partial_t - M, L] = 0,$$

the system (1). Thus, L is the Lax operator for the MRSKdV system, and $L\psi = \lambda\psi$ is a supersymmetric extension of the stationary Schrödinger equation.

The Darboux transformation for the MRSKdV equation [8], that we will iterate in this paper, is

Proposition 1. Let ψ be a solution of

$$L\psi = \lambda\psi, \quad \partial_t \psi = M\psi, \quad (3)$$

and θ_0 be a particular solution with $\lambda = \lambda_0$. Then, the quantities defined by

$$\begin{aligned} \hat{\psi} &:= (D + \delta_0)\psi, & \delta_0 &:= -\frac{D\theta_0}{\theta_0} \quad (\theta_0: \text{even}) \\ \hat{\alpha} &:= -\alpha - 2\partial\delta_0, & \hat{u} &:= u + (D\alpha) + 2\delta_0(\alpha + \partial\delta_0) \end{aligned}$$

satisfy

$$\hat{L}\hat{\psi} = \lambda\hat{\psi}, \quad \partial_t \hat{\psi} = \hat{M}\hat{\psi},$$

where \hat{L} and \hat{M} are obtained from L and M by replacing α and u with $\hat{\alpha}$ and \hat{u} , respectively.

As a consequence of this Proposition we conclude that \hat{u} and $\hat{\alpha}$ are new solutions of the MRSKdV system (1). We remark that, as usual, the Darboux transformation can be viewed as a gauge transformation:

$$\psi \rightarrow T_0\psi, \quad L \rightarrow \hat{L} = T_0LT_0^{-1}, \quad M \rightarrow \hat{M} = \partial_i T_0 \cdot T_0^{-1} + T_0MT_0^{-1}, \quad T_0 := D + \delta_0.$$

To obtain Crum type transformation, let us start with n solutions $\theta_i, i = 0, \dots, n - 1$, of Eq. (2) with eigenvalues as $\lambda = k_i, i = 0, \dots, n - 1$. To make sense, we choose the θ_i in such way that its index indicates its parity: those with even indices are even and with odd indices are odd variables. We use θ_0 to do our first step transformation and then $\theta_i, i = 1, \dots, n - 1$, are transformed to new solutions $\theta_i[1]$ of the transformed linear equation and θ_0 goes to zero. Next step can be effected by using $\theta_1[1]$ to form a Darboux operator and at this time $\theta_1[1]$ is lost. We can continue this iteration process until all the seeds are mapped to zero. In this way, we have

Proposition 2. Let $\theta_i, i = 0, \dots, n - 1$, be solutions of the linear system (2) with $\lambda = k_i, i = 0, \dots, n - 1$, and parities $p(\theta_i) = (-1)^i$. Then after n iterations of the Darboux transformation of Proposition 1, one obtains a new Lax operator

$$\hat{L} = T_nLT_n^{-1}, \quad T_n = D^n + \sum_{i=0}^{n-1} a_iD^i,$$

where the coefficients a_i of the gauge operator T_n are defined by

$$\left(D^n + \sum_{i=0}^{n-1} a_iD^i \right) \theta_j = 0, \quad j = 0, \dots, n - 1. \tag{4}$$

Proof. With $\theta_i, i = 0, \dots, n - 1$, we may perform n Darboux transformations of Proposition 1 step by step. Indeed, in the first step we transform using θ_0 , obtaining:

$$\begin{aligned} \psi[1] &:= (D + \delta_0)\psi, & \delta_0 &:= -\frac{D\theta_0}{\theta_0}, \\ L[1] &:= (D + \delta_0)L(D + \delta_0)^{-1}, & \text{and } \theta_i[1] &:= (D + \delta_0)\theta_i, \quad i = 1, \dots, n - 1. \end{aligned}$$

Notice that $\theta_0[1] = (D + \delta_0)\theta_0 = 0$.

Using $\theta_1[1]$ to do our next step transformation we get

$$\begin{aligned} \psi[2] &:= (D + \delta_1)\psi[1], & \delta_1 &:= -(\theta_1[1])^{-1}D\theta_1[1], \\ L[2] &:= (D + \delta_1)L[1](D + \delta_1)^{-1}, & \theta_i[2] &:= (D + \delta_1)\theta_i[1], \quad i = 2, \dots, n - 1, \end{aligned}$$

and at this time we have $\theta_1[2] = 0$.

Combining these two steps, we get

$$\psi[2] = (D + \delta_1)(D + \delta_0)\psi, \quad L[2] = (D + \delta_1)(D + \delta_0)L(D + \delta_0)^{-1}(D + \delta_1)^{-1}.$$

Following this way, we iterate the Darboux transformations until all the seeds, $\{\theta_i\}_{i=0}^{n-1}$, are used up. It is clear that our composed gauge operator has the following form

$$T_n = (D + \delta_{n-1}) \cdots (D + \delta_0) = D^n + a_{n-1}D^{n-1} + \cdots + a_0,$$

with the property $T_n(\theta_j) = 0$. This in turn determines the coefficients $a_i, i = 0, \dots, n - 1$. □

The explicit form of the transformed field variables is given by $\hat{L} = T_nLT_n^{-1}$.

Lemma 1. The new fields $\hat{\alpha}$ and \hat{u} can be written as

$$\begin{aligned} \hat{\alpha} &= (-1)^n \alpha - 2\partial a_{n-1}, \\ \hat{u} &= u - 2\partial a_{n-2} - a_{n-1}((-1)^n \alpha + \hat{\alpha}) + \frac{1 - (-1)^n}{2} D\alpha. \end{aligned} \tag{5}$$

Next, we solve the linear system (4) and get the explicit solutions in terms of superdeterminants. Since the cases for n even or odd are rather different, we consider them separately.

For $n = 2k$, we denote by

$$\begin{aligned} \mathbf{a}^{(0)} &:= (a_0, a_2, \dots, a_{2k-2}), & \mathbf{a}^{(1)} &:= (a_1, a_3, \dots, a_{2k-1}), \\ \boldsymbol{\theta}^{(0)} &:= (\theta_0, \theta_2, \dots, \theta_{2k-2}), & \boldsymbol{\theta}^{(1)} &:= (\theta_1, \theta_3, \dots, \theta_{2k-1}), \\ \mathbf{b}^{(i)} &:= \partial^k \boldsymbol{\theta}^{(i)}, & W^{(i)} &:= \begin{pmatrix} \boldsymbol{\theta}^{(i)} \\ \partial \boldsymbol{\theta}^{(i)} \\ \vdots \\ \partial^{k-1} \boldsymbol{\theta}^{(i)} \end{pmatrix}, \quad i = 0, 1. \end{aligned}$$

Then, our linear system (4) can be now formulated as

$$(\mathbf{a}^{(0)}, \mathbf{a}^{(1)})\mathcal{W} = -(\mathbf{b}^{(0)}, \mathbf{b}^{(1)}), \tag{6}$$

where

$$\mathcal{W} := \begin{pmatrix} W^{(0)} & W^{(1)} \\ DW^{(0)} & DW^{(1)} \end{pmatrix}.$$

It should be noticed that the supermatrix \mathcal{W} is even and has a Wronski type structure.

We recall some facts about supermatrices [2,5]. Given an even matrix, say $\mathcal{M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, its inverse is

$$\mathcal{M}^{-1} = \begin{pmatrix} (A - BD^{-1}C)^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -D^{-1}C(A - BD^{-1}C)^{-1} & (D - CA^{-1}B)^{-1} \end{pmatrix},$$

and its Berezinian or superdeterminant is

$$\text{sdet } \mathcal{M} = \frac{\det(A - BD^{-1}C)}{\det D} = \frac{\det A}{\det(D - CA^{-1}B)}.$$

Now, we have

Lemma 2. For $n = 2k$, one can write

$$a_{2k-2} = -\frac{\text{sdet } \hat{\mathcal{W}}}{\text{sdet } \mathcal{W}}, \quad a_{2k-1} = D \ln \text{sdet } \mathcal{W},$$

here

$$\hat{\mathcal{W}} = \begin{pmatrix} \hat{W}^{(0)} & \hat{W}^{(1)} \\ DW^{(0)} & DW^{(1)} \end{pmatrix}$$

where $\hat{W}^{(0)}$ and $\hat{W}^{(1)}$ are obtained from the matrices $W^{(0)}$ and $W^{(1)}$ by replacing the last rows with $\mathbf{b}^{(0)}$ and $\mathbf{b}^{(1)}$, respectively.

Proof. Multiplying (6) by \mathcal{W}^{-1} one finds

$$\begin{aligned} \mathbf{a}^{(0)} &= - \left(\mathbf{b}^{(0)} - \mathbf{b}^{(1)}(DW^{(1)})^{-1}DW^{(0)} \right) \left(W^{(0)} - W^{(1)}(DW^{(1)})^{-1}DW^{(0)} \right)^{-1}, \\ \mathbf{a}^{(1)} &= - \left(\mathbf{b}^{(1)} - \mathbf{b}^{(0)}(W^{(0)})^{-1}W^{(1)} \right) \left(DW^{(1)} - (DW^{(0)})(W^{(0)})^{-1}W^{(1)} \right)^{-1}. \end{aligned}$$

Noticing that both

$$\left(W^{(0)} - W^{(1)}(DW^{(1)})^{-1}DW^{(0)} \right) \quad \text{and} \quad \left(DW^{(1)} - (DW^{(0)})(W^{(0)})^{-1}W^{(1)} \right)$$

are even matrices, we use the Cramer's rule to obtain

$$a_{2k-2} = - \frac{\det \left(\hat{W}^{(0)} - \hat{W}^{(1)}(DW^{(1)})^{-1}DW^{(0)} \right)}{\det \left(W^{(0)} - W^{(1)}(DW^{(1)})^{-1}DW^{(0)} \right)} = - \frac{\text{sdet } \hat{W}}{\text{sdet } W}.$$

Similarly, for a_{2k-1} we have the following expression

$$a_{2k-1} = - \frac{\det \left(D\bar{W}^{(1)} - (D\bar{W}^{(0)})(W^{(0)})^{-1}W^{(1)} \right)}{\det \left(DW^{(1)} - (DW^{(0)})(W^{(0)})^{-1}W^{(1)} \right)},$$

where $\bar{W}^{(i)}$ is obtained from $W^{(i)}$ with its last row replaced by its D derivation, that is, $D^{2k-1}\theta^{(i)}$. We will show next that one can write $a_{2k-1} = D \ln \text{sdet } W$.

Introduce the notation

$$V := DW^{(1)} - (DW^{(0)})(W^{(0)})^{-1}W^{(1)},$$

and denote by v_j the j th row of the matrix V . Then, since $(\text{sdet } W)^{-1} = (\det V) / (\det W^{(0)})$, we have

$$D((\text{sdet } W)^{-1}) = \frac{D \det V}{\det W^{(0)}} - \frac{D \ln \det W^{(0)}}{\text{sdet } W} = \frac{D \det V}{\det W^{(0)}} - \frac{\text{Tr}((DW^{(0)})(W^{(0)})^{-1})}{\text{sdet } W}.$$

Notice that

$$\begin{aligned} D \det V &= \sum_{j=1}^k \det \left(V - e_j \otimes (v_j - Dv_j) \right), \\ DV &= \partial W^{(1)} - (\partial W^{(0)})(W^{(0)})^{-1}W^{(1)} + (DW^{(0)})(W^{(0)})^{-1}V, \\ \partial W^{(j)} &= \Lambda W^{(j)} + e_k \otimes b^{(j)} \end{aligned}$$

where $e_i = (\delta_{ij})$ is a column vector with all its entries vanishing except for the i th one, and $\Lambda = (\delta_{i,i+1})$ is the shift matrix.

Hence, we obtain

$$Dv_j = \delta_{j,k} \left(b^{(1)} - b^{(0)}(W^{(0)})^{-1}W^{(1)} \right) + c_j V, \quad j = 1, \dots, k$$

where we denote by c_j the j th row of the matrix $(DW^{(0)})(W^{(0)})^{-1}$.

With the help of above formulae we have

$$\begin{aligned} D \det V &= \det \left(V - e_k \otimes \left[v_k - (b^{(1)} - b^{(0)}(W^{(0)})^{-1}W^{(1)}) \right] \right) + \sum_{j=1}^k \det(1 - e_j \otimes (e_j - c_j)) \det V \\ &= \det \left(D\bar{W}^{(1)} - (D\bar{W}^{(0)})(W^{(0)})^{-1}W^{(1)} \right) + \text{Tr} \left((DW^{(0)})(W^{(0)})^{-1} \right) \det V \end{aligned}$$

so that

$$D((\text{sdet } \mathcal{W})^{-1}) = \frac{\det(D\bar{W}^{(1)} - (D\bar{W}^{(0)})(W^{(0)})^{-1}W^{(1)})}{\det W^{(0)}}$$

which leads to our claimed formula for a_{2k-1} . □

For the $n = 2k + 1$ case, we need to introduce a different set of notations:

$$\begin{aligned} \mathbf{a}^{(1)} &= (a_0, a_2, \dots, a_{2k}), & \mathbf{a}^{(0)} &= (a_1, a_3, \dots, a_{2k-1}), \\ \boldsymbol{\theta}^{(0)} &= (\theta_0, \theta_2, \dots, \theta_{2k}), & \boldsymbol{\theta}^{(1)} &= (\theta_1, \theta_3, \dots, \theta_{2k-1}), \\ \mathbf{b}^{(i)} &= D^{2k+1}\boldsymbol{\theta}^{(j)}, \quad i \neq j, & W^{(i)} &= \begin{pmatrix} \boldsymbol{\theta}^{(i)} \\ \partial\boldsymbol{\theta}^{(i)} \\ \vdots \\ \partial^k\boldsymbol{\theta}^{(i)} \end{pmatrix}, \quad i, j = 0, 1, \end{aligned}$$

then our linear system (4) reads as

$$(\mathbf{a}^{(1)}, \mathbf{a}^{(0)}) \begin{pmatrix} W^{(0)} & W^{(1)} \\ D\bar{W}^{(0)} & D\bar{W}^{(1)} \end{pmatrix} = -(\mathbf{b}^{(1)}, \mathbf{b}^{(0)}),$$

where $\bar{W}^{(i)}$ is the $W^{(i)}$ with the last row removed.

We introduce two block matrices

$$\mathcal{W} = \begin{pmatrix} W^{(0)} & W^{(1)} \\ D\bar{W}^{(0)} & D\bar{W}^{(1)} \end{pmatrix}, \quad \check{\mathcal{W}} = \begin{pmatrix} W^{(0)} & W^{(1)} \\ D\check{W}^{(0)} & D\check{W}^{(1)} \end{pmatrix},$$

where $\check{W}^{(i)}$ is the matrix $W^{(i)}$ with its second to last row removed.

Lemma 3. For $n = 2k + 1$, one has

$$a_{2k-1} = -\frac{\text{sdet } \mathcal{W}}{\text{sdet } \check{\mathcal{W}}}, \quad a_{2k} = -\frac{\det(\hat{W}^{(0)} - \hat{W}^{(1)}(D\bar{W}^{(1)})^{-1}(D\bar{W}^{(0)}))}{\det(W^{(0)} - W^{(1)}(D\bar{W}^{(1)})^{-1}D\bar{W}^{(0)})},$$

where $\hat{W}^{(i)}$ is the matrix $W^{(i)}$ with its last row replaced by the vector $\mathbf{b}^{(j)}$, $i, j = 0, 1, i \neq j$.

Proof. This lemma follows easily from Cramer’s rule. □

Now, from the last three lemmas one can easily arrive at the main result of this paper:

Theorem 1. Let α, u be a seed solution of (1) and $\{\theta_j\}_{j=0}^{n-1}$ be a set of n solutions of the associated linear system (2), such that the parity is $p(\theta_j) = (-1)^j$. Then:

- (i) If $n = 2k$, we have new solutions $\hat{\alpha}, \hat{u}$ of (1) given by

$$\begin{aligned} \hat{\alpha} &= \alpha - 2D^3 \ln \text{sdet } \mathcal{W}, \\ \hat{u} &= u + 2\partial \left(\frac{\text{sdet } \hat{\mathcal{W}}}{\text{sdet } \mathcal{W}} \right) + (\alpha + \hat{\alpha})D \ln \text{sdet } \mathcal{W}. \end{aligned}$$

(ii) If $n = 2k + 1$, we have new solutions $\hat{\alpha}$, \hat{u} of (1) given by

$$\hat{\alpha} = \alpha + 2\partial \left(\frac{\det(\hat{W}^{(0)} - \hat{W}^{(1)}(D\hat{W}^{(1)})^{-1}D\hat{W}^{(0)})}{\det(W^{(0)} - W^{(1)}(DW^{(1)})^{-1}DW^{(0)})} \right),$$

$$\hat{u} = u + 2\partial \left(\frac{\text{sdet } \mathcal{W}}{\text{sdet } \check{\mathcal{W}}} \right) + D\alpha - (\alpha - \hat{\alpha}) \left(\frac{\det(\hat{W}^{(0)} - \hat{W}^{(1)}(D\hat{W}^{(1)})^{-1}D\hat{W}^{(0)})}{\det(W^{(0)} - W^{(1)}(DW^{(1)})^{-1}DW^{(0)})} \right).$$

Some remarks are in order here

- (a) The proof of Lemma 2 given above is inspired by the one in [15].
- (b) In the $n = 2k + 1$ case, it is not possible to write a_{2k} in terms of superdeterminants, this is so because the block structure of the matrix is not preserved after using Cramer's rule.
- (c) The elegant representation for a_{2k-1} obtained in the even case is lost in the odd case.

Reduction to KdV Eq. (1) reduces to the KdV equation when $\alpha = 0$. Hence, it is natural to ask whether our iterated Darboux transformation reduces down to a Darboux transformation of the KdV equation. Actually, that aim is achieved when we take, $n = 2k$, $a_{2i-1} = 0$ with $\alpha = 0$. Under this condition the linear system (6) breaks into:

$$a^{(0)}W^{(0)} = -b^{(0)}, \quad a^{(0)}W^{(1)} = -b^{(1)}. \tag{7}$$

Thus, while the first equation leads to the well known formula for the KdV system [11], the second one is a constraint for θ_i , $i = 0, \dots, n - 1$. This constraint can be easily solved with the choice

$$\theta_{2i}(x, t, \vartheta) = \theta_{2i}(x, t), \quad \theta_{2i-1}(x, t, \vartheta) = \vartheta\theta_{2i}(x, t)$$

so that $W^{(1)} = \vartheta W^{(0)}$ and $b^{(1)} = \vartheta b^{(0)}$, therefore the equation $a^{(0)}W^{(1)} = -b^{(1)}$ holds whenever $a^{(0)}W^{(0)} = -b^{(0)}$ does.

3. Super solitons

In this section we present some explicit examples obtained by dressing the zero background $u = \alpha = 0$ for $n = 2$. The general solutions of the linear system (4) are:

$$\theta_0 = c_+ \exp(\eta) + c_- \exp(-\eta), \quad \theta_1 = \gamma_+ \exp(\eta) + \gamma_- \exp(-\eta),$$

where $\eta = kx + k^3t$, $k \in \mathbb{C}$ and c_{\pm} are even and γ_{\pm} are odd and

$$c_{\pm}(\vartheta) = c_{\pm}^{(0)} + \vartheta c_{\pm}^{(1)}, \quad \gamma_{\pm}(\vartheta) = \gamma_{\pm}^{(1)} + \vartheta \gamma_{\pm}^{(0)},$$

here the superfix indicates the parity.

In order to simplify the final expressions we take $c_{\pm}^{(0)}, \gamma_{\pm}^{(0)} \in \mathbb{C}$. One can show that

$$\text{sdet } \mathcal{W} = \frac{c_+^{(0)} \exp(\eta) + c_-^{(0)} \exp(-\eta)}{\gamma_+^{(0)} \exp(\eta) + \gamma_-^{(0)} \exp(-\eta)} + \frac{(c_+^{(1)} \exp(\eta) + c_-^{(1)} \exp(-\eta)) (\gamma_+^{(1)} \exp(\eta) + \gamma_-^{(1)} \exp(-\eta))}{(\gamma_+^{(0)} \exp(\eta) + \gamma_-^{(0)} \exp(-\eta))^2}$$

$$+ \frac{2\vartheta k (c_+^{(0)}\gamma_-^{(1)} - c_-^{(0)}\gamma_+^{(1)})}{(\gamma_+^{(0)}\exp(\eta) + \gamma_-^{(0)}\exp(-\eta))^2} - \frac{4\vartheta k\gamma_-^{(1)}\gamma_+^{(1)} (c_+^{(1)}\exp(\eta) + c_-^{(1)}\exp(-\eta))}{(\gamma_+^{(0)}\exp(\eta) + \gamma_-^{(0)}\exp(-\eta))^3}.$$

Hence, we have

$$a_0 = f - k(\gamma_-^{(1)}\gamma_+^{(1)} + \vartheta(\gamma_+^{(1)}\gamma_-^{(0)} - \gamma_-^{(1)}\gamma_+^{(0)}))g - \vartheta k(c_+^{(0)}\gamma_-^{(1)} - c_-^{(0)}\gamma_+^{(1)})fg,$$

$$a_1 = (k(c_+^{(0)}\gamma_-^{(1)} - c_-^{(0)}\gamma_+^{(1)}) + \vartheta(c_+^{(0)}\gamma_-^{(0)} - c_-^{(0)}\gamma_+^{(0)}))g,$$

where

$$f := -k \left(\frac{c_+^{(0)}\exp(\eta) - c_-^{(0)}\exp(-\eta)}{c_+^{(0)}\exp(\eta) + c_-^{(0)}\exp(-\eta)} \right),$$

$$g := \frac{2}{(c_+^{(0)}\exp(\eta) + c_-^{(0)}\exp(-\eta))(\gamma_+^{(0)}\exp(\eta) + \gamma_-^{(0)}\exp(-\eta))}.$$

Our solution is

$$\hat{\alpha} = -2\partial a_1, \quad \hat{u} = -2\partial a_0.$$

Notice that our solution can be understood as a super soliton which has the KdV soliton, $-2\partial f$, as its body, and that the choice $c_+^{(0)} = \gamma_+^{(0)}$ and $c_-^{(0)} = \gamma_-^{(0)}$ gives the solution found in [6].

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