Scaling Self–Similar Formulation of the String Equations of the Hermitian One–Matrix Model

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Abstract

The string equation appearing in the double scaling limit of the Hermitian one–matrix model, which corresponds to a Galilean self–similar condition for the KdV hierarchy, is reformulated as a scaling self–similar condition for the Ur–KdV hierarchy. A non–scaling limit analysis of the one–matrix model has led to the complexified NLS hierarchy and a string equation. We show that this corresponds to the Galilean self–similarity condition for the AKNS hierarchy and also its equivalence to a scaling self–similar condition for the Heisenberg ferromagnet hierarchy.

0. The Hermitian one–matrix model has received much attention in recent years as a non–perturbative formulation of string theory. In [2] the double scaling limit for the even potential case was used to show that the specific heat is a solution of the Korteweg–de Vries (KdV) hierarchy that satisfies an additional constraint, the so called string equation. In [12] it was prove that this corresponds to invariance under Galilean transformations, see also [9]. The model is also relevant for topological gravity and for the Witten–Kontsevich intersection theory of the moduli space of complex curves [18].

In [3] it was performed a non–scaling limit analysis of the the Hermitian one–matrix model with general potential. Now, the specific heat is the second conserved density of the Ablowitz–Kaup–Newell–Segur (AKNS) hierarchy and the string equation, as we shall show, corresponds to invariance under the Galilean transformations

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of the hierarchy. The associated topological field theory is close to the Witten's $\mathbb{C}P^1$ σ–model coupled to topological gravity. Observe that the AKNS hierarchy is a complexfied version of the Non–Linear Schrödinger (NLS) hierarchy.

The aim of this letter is to show that the Hermitian one–matrix model, which corresponds to Galilean self–similarity —L_{-1}–Virasoro constraint— with or without double scaling limit, can be formulated as a L_0–Virasoro constraint, that is, as a scaling invariance condition. In order to do this we need to introduce new integrable hierarchies connected with the previous ones by Miura type transformations.

For the string equation of the double scaling limit of the Hermitian one–matrix model we introduce the fundamental Ur–KdV hierarchy, see [17, 13] and references therein. And, as we shall see, the Galilean self–similarity condition, i.e. the string equation, corresponds in the Ur–KdV hierarchy to scaling self–similarity. For example, the solution corresponding to the Witten–Kontsevich model when only the KdV equation is taken into account, corresponds in the Ur–KdV equation to a linear fractional transformation of a quotient of Airy functions depending on a scaling invariant. Recall that the Airy equation is important in Kontsevich's approach.

We consider as well the string equations of the non-scaling limit of the Hermitian one–matrix model [3]. We not only prove that these are equivalent to the Galilean self–similarity condition for the AKNS hierarchy but also that for the associated Heisenberg ferromagnet hierarchy [5] this corresponds to scaling self–similarity.

In the first section we introduce the Ur–KdV hierarchy. There one can find the proof of the equivalence of the string equation of the double scaling limit of the Hermitian one–matrix model with the scaling self–similar condition for solutions of the Ur–KdV hierarchy. We also illustrate the correspondence with the Galilean self–similar solution of the KdV equation and the appearence of the Airy functions. For the scaling self–similar condition of the KdV hierarchy we found its equivalent in the Ur–KdV hierarchy, giving as example the Adler–Moser rational solutions. We end the section presenting a general formula expressing the solution to the Ur–KdV hierarchy as a quotient of two τ–functions for the KdV hierarchy.

In the next section we analyse the non–scaling limit case. First we prove that the string equation of [3] corresponds to the Galilean self–similarity condition for the AKNS hierarchy. Secondly, we introduce the complexified version of the continuous spin chain with Heisenberg interaction, that is, the Heisenberg ferromagnet hierarchy, showing that the string equation can be recasted as a scaling self–similar condition for this hierarchy. To end we look to the scaling self–similar condition of the AKNS hierarchy and to the corresponding condition in the Heisenberg ferromagnet hierarchy.

1. The Ur–KdV equation, as named by Wilson [17], is the following non–linear partial differential equation for a complex scalar field z depending on the complex variables t_1, t_3

$$4\partial_3 z = \{z, t_1\}\partial_1 z$$

where

$$\{z, t_1\} := \frac{\partial_1^3 z}{\partial_1 z} - \frac{3}{2}\left(\frac{\partial_1^2 z}{\partial_1 z}\right)^2$$

is the Schwartzian derivative [6]. Here we have use the notation $\partial/\partial t_1 =: \partial_1$ and so on.

This equation is connected to the KdV equation. Given a solution z to the Ur–KdV equation then

$$u = \frac{1}{2}\{z, t_1\} \tag{1}$$

satisfies the KdV equation

$$4\partial_3 u = \partial_3 u + 6u\partial_1 u.$$

The Ur–KdV equation is associated with the Krichever–Novikov equation

$$4\partial_3 z = \{z, t_1\}\partial_1 z + \frac{4z^3 - g_2 z - g_3}{\partial_1 z}$$

which appears in the study of rank 2 and genus 1 solutions to the KP equation, [11]. It is also important in the classification of scalar integrable equations of order 3, [16]. And can be described in terms of certain elliptic homogeneous spaces in analogy to the Landau–Lishfitz equation, [8].

As is well known, the KdV equation has an infinite number of symmetries that preserves the spectral properties of the associated Schrödinger operator

$$\mathcal{L} := \partial_1^2 + u.$$

Therefore, one comes to consider the KdV hierarchy, that can be expressed in terms of the Gel'fand–Dickii potentials $R_n[u]$ (polynomials in $u, \partial_1 u, \partial_1^2 u, \dots$) which are the coefficients of an asymptotic expansion of the resolvent $(\mathcal{L} - \lambda)^{-1}$, [7]. The KdV hierarchy is the following infinite set of compatible equations in the variables $\mathbf{t} := \{t_{2n+1}\}_{n\geq 0}$

$$\partial_{2n+1} u = 4\partial_1 R_{n+1}[u].$$

There is a corresponding Ur–KdV hierarchy

$$\partial_{2n+1} z = 2R_n\left[\frac{1}{2}\{z, t_1\}\right]\partial_1 z, \tag{2}$$

and any of its solutions gives through (1) a solution to the KdV hierarchy. The Ur–KdV hierarchy has a remarkable property, given a solution z any

$$\tilde{z} = \frac{az + b}{cz + d}$$

is also a solution as long as $ab - cd = 1$. Thus, the projective group $PSL(2, \mathbb{C})$ acts on the space of solutions to the hierarchy.

Consider a solution u of the KdV hierarchy, choose two independent functions ψ_1, ψ_2 in the kernel of the Schrödinger operator $\mathcal{L}$ with Wronskian equal to the unity. Hence

$$\mathcal{L}\psi_1 = \mathcal{L}\psi_2 = 0,\ \mathrm{W}(\psi_1, \psi_2) := \psi_1\partial_1\psi_2 - \psi_2\partial_1\psi_1 = 1.$$

In addition we require both ψ_1, ψ_2 to be in the kernel of the evolution operators

$$\mathcal{A}_{2n+1} = \partial_{2n+1} - 2R_n \circ \partial_1 + (\partial_1 R_n). \tag{3}$$

Thus, as one can show

$$z = \frac{\psi_1}{\psi_2}$$

satisfies the Ur–KdV hierarchy and is connected to u through (1). This can be considered as an inversion of (1).

Let us now consider the symmetries defined by translations, scaling and Galilean transformations of the KdV hierarchy. For the infinite set of translational symmetries we define

$$\vartheta(\mathbf{t}) := \mathbf{t} + \boldsymbol{\theta},$$

where

$$\boldsymbol{\theta} := \{\theta_{2n+1}\}_{n\geq 0} \in \mathbb{C}^\infty.$$

If u is a solution to the hierarchy then $\vartheta^* u$ is also a solution. For the scaling symmetry $\mathbf{t} \mapsto \varsigma_\sigma(\mathbf{t})$ we define

$$\varsigma_\sigma(\mathbf{t})_{2n+1} := e^{(n+\frac{1}{2})\sigma} t_{2n+1}$$

where $\sigma \in \mathbb{C}$. If u is a solution of the KdV hierarchy then $e^\sigma \varsigma_\sigma^* u$ is a solution as well. The Galilean transformation $\mathbf{t} \mapsto \gamma_b(\mathbf{t})$ is given by

$$\gamma_b(\mathbf{t})_{2n+1} := \sum_{m=0}^{\infty} \binom{n+m+\frac{1}{2}}{m+\frac{1}{2}} b^m t_{2(n+m)+1},$$

where we have used the binomial function that can be expressed in terms of the Euler Γ–function as

$$\binom{a}{b} := \frac{\Gamma(a+1)}{\Gamma(b+1)\Gamma(a-b+1)}.$$

If u is solution of the KdV hierarchy then so is $\gamma_b^* u + b$.

The vector fields generating these symmetries are

$$\partial_{2n+1}, n \geq 0,\ \boldsymbol{\varsigma} = \sum_{n\geq 0}(n+\frac{1}{2})t_{2n+1}\partial_{2n+1},\ \boldsymbol{\gamma} = \sum_{n\geq 1}(n+\frac{1}{2})t_{2n+1}\partial_{2n-1},$$

for translations, scaling and Galilean transformations respectively. Consider the following vector field

$$X := \boldsymbol{\vartheta} + \boldsymbol{\gamma},$$

with

$$\boldsymbol{\vartheta} := \sum_{n\geq 0} \theta_{2n+1}\partial_{2n+1}.$$

Let us denote

$$\mathcal{R} := \sum_{n\geq 0}(n+\frac{1}{2})t_{2n+1}R_n,$$

then, as one can show, a solution u of the KdV hierarchy is self–similar under the vector field X, i.e. u remains invariant under the symmetry transformation generated by X, if and only if it satisfies the string equation

$$\sum_{n\geq 0}\theta_{2n+1}R_{n+1} + \mathcal{R} = \frac{c}{4} \tag{4}$$

for some $c \in \mathbb{C}$. Notice that the self–similar solutions under the vector field X can be understood as Galilean self–similar solutions once we shift the times by $t_{2n+1} \mapsto t_{2n+1} + (n+1/2)^{-1}\theta_{2n-1}$ where $n \geq 1$. A shift in t_1 changes the constant c. Therefore, any Galilean self–similar solution of the KdV hierarchy is of the form $u(t_1 - c, t_3, \dots)$ where u is a solution of the string equation of the double scaling limit of the Hermitian one–matrix model with even potentials [2, 12, 9]

$$\mathcal{R} = 0.$$

In the Ur–KdV hierarchy there is no Galilean local symmetry, only the scaling transformation and the translations are local symmetries. If z is a solution of the Ur–KdV hierarchy then $\vartheta^* z$ and $\varsigma_\sigma^* z$ are solutions as well. Now, we consider the vector field

$$Y := \widetilde{\boldsymbol{\vartheta}} + \boldsymbol{\varsigma},$$

with

$$\widetilde{\boldsymbol{\vartheta}} := -\frac{c}{2}\partial_1 + \sum_{n\geq 1}\theta_{2n-1}\partial_{2n+1}.$$

Using Eqs.(2,3) one can show

LEMMA 1 *The following relations holds*

$$\begin{aligned} Y\psi_i + \frac{\psi_i}{4} &= 2(\mathcal{R} + \sum_{n\geq 0}\theta_{2n+1}R_{n+1} - \frac{c}{4})\partial_1\psi_i - \psi_i\partial_1(\mathcal{R} + \sum_{n\geq 0}\theta_{2n+1}R_{n+1} - \frac{c}{4}), \quad (5)\\ Yz &= 2(\mathcal{R} + \sum_{n\geq 0}\theta_{2n+1}R_{n+1} - \frac{c}{4})\partial_1 z. \quad (6)\end{aligned}$$

From where one deduces

PROPOSITION 1 *A solution u to the KdV hierarchy satisfies the string equation*

$$\mathcal{R} + \sum_{n\geq 0}\theta_{2n+1}R_{n+1} = \frac{c}{4}$$

only if the corresponding solution z to the Ur–KdV hierarchy satisfies

$$Yz = 0.$$

And if $Yz = 0$ then the corresponding solution u to the KdV hierarchy either satisfies the string equation or $u = 0$.

Proof: If the string equation is satisfied then Eq.(5) imply that

$$Y\psi_i + \frac{\psi_i}{4} = 0,$$

thus

$$Yz = Y(\frac{\psi_1}{\psi_2}) = 0.$$

This proves the only if part. Now, if $Yz = 0$ then Eq.(6) gives that either the string equation holds or $\partial_1 z = 0$, so that $z = z_0 \in \mathbb{C}$ and $u = 0$.□

We arrive to the conclusion that given a non–constant self–similar solution z of the Ur–KdV hierarchy under the action of the vector field Y —generating scaling transformations in shifted coordinates— then the associated solution of the KdV hierarchy by means of (1) is self–similar under the action of the vector field X and viceversa.

As an example we consider the Galilean self–similar solution of the KdV equation

$$u = -\frac{2t_1}{3t_3}.$$

The corresponding scaling self–similar solution of the Ur–KdV equation is

$$z(t_1, t_3) = \frac{a\mathrm{Ai}(\zeta) + b\mathrm{Bi}(\zeta)}{c\mathrm{Ai}(\zeta) + d\mathrm{Bi}(\zeta)},\ ad - bc = 1$$

being Ai and Bi the standard Airy functions [14] and

$$\zeta^3 := \frac{2}{3}\frac{t_1^3}{t_3}.$$

The solution u when extended to all the t_{2n+1} and once the shift $t_3 \mapsto t_3 + 3/2$ is performed is the one considered by Kac and Schwarz [10] and also the one associated to the Witten–Kontsevich model for the intersection theory of the moduli space of complex curves [18]. Observe the appearance of the Airy functions, essential in the Kontsevich approach, in the Ur–KdV context.

To end this section we shall give the string equation for the Ur–KdV hierarchy corresponding to the scaling self–similar condition to the KdV equation, i.e. to 2D–stable gravity [4]. One can easily show that the following relation holds

$$e^{\sigma}\varsigma_{\sigma}^{*}u = \frac{1}{2}\{\varsigma_{\sigma}^{*}z, t_1\}.$$

Thus, if we want u to be scaling self–similar, we arrive to the condition

$$\{\varsigma_\sigma^* z, t_1\} = \{z, t_1\}$$

so that

$$\varsigma_\sigma^* z = \frac{(\cosh(\sigma\rho) + A_0 \frac{\sinh(\sigma\rho)}{\rho})z + A_+ \frac{\sinh(\sigma\rho)}{\rho}}{(A_- \frac{\sinh(\sigma\rho)}{\rho})z + \cosh(\sigma\rho) - A_0 \frac{\sinh(\sigma\rho)}{\rho}},$$

with $A = A_+ E + A_0 H + A_- F \in \mathfrak{sl}(2, \mathbb{C})$ and $\rho = -\det A$. Hence, the scaling self–similarity condition for the KdV hierarchy can be recasted as scaling self–similarity of the Ur–KdV hierarchy modulo the global $PSL(2, \mathbb{C})$–gauge invariance.

PROPOSITION 2 *A solution u of the KdV hierarchy is scaling self–similar if and only if the corresponding solution z to the Ur–KdV hierarchy satisfies the string equation*

$$\varsigma z = -A_- z^2 + 2A_0 z + A_+.$$

for some $A_+, A_0, A_- \in \mathbb{C}$.

When $A = 0$ we recover the Galilean case already exposed. Another example is $A = H/4$, then $\varsigma_\sigma^* \partial_1 z = \partial_1 z$. The function

$$w = \frac{1}{2} \ln(\partial_1 z),$$

is a solution of the potential modified KdV hierarchy self–similar under scaling transformations. As was shown in [15] this corresponds to the double scaling limit of the symmetric unitary one–matrix model with no boundary terms. We see that this sector of the double scaling limit of the one–matrix model can be encoded with the Hermitian one with the aid of the Ur–KdV hierarchy.

As an illustration let us consider the rational solutions of the KdV hierarchy that vanishes when $t_1 \to \pm\infty$, [1]. These are self–similar solutions under scaling transformations. The rational solution u_n is characterized by $u_n(t_1, 0, 0, \cdots) = n(n+1)/t_1^2$ where $n \in \mathbb{N} \cup \{0\}$. One has the expression $u_n = 2\partial_1^2 \ln \Theta_n$, where Θ_n is a polynomial in $\mathbf{t}$ of degree $n(n+1)/2$ ($\deg t_{2n+1} = 2n+1$) and can be considered as a theta function for the rational curve $\mu^2 = \lambda^{2n+1}$, they are τ–functions. For the corresponding Schrödinger operator $\mathcal{L}_n = \partial_1^2 + u_n$ one has the kernel $\mathrm{Ker}\mathcal{L}_n = \mathbb{C}\{\Theta_{n+1}/\Theta_n, \Theta_{n-1}/\Theta_n\}$, so that

$$z = \frac{a\Theta_{n+1} + b\Theta_{n-1}}{c\Theta_{n+1} + d\Theta_{n-1}}$$

is a solution of the Ur–KdV hierarchy. For example $z_n := \Theta_{n+1}/\Theta_{n-1}$ satisfies $\varsigma_\sigma^* z_n = e^{(n+1/2)\sigma} z_n$.

The possibility of expressing z as a quotient of τ–functions for the KdV hierarchy is true in general and not only for the rational case considered above. Let $u_0 =$

$2\partial_1^2 \ln \tau_0$ be a solution of the KdV hierarchy. Consider the expressions $u_0 = \partial_1 v_+ - v_+^2 = -(\partial_1 v_- + v_-^2)$ where v_+ and v_- are solutions to the modified KdV hierarchy. Then u_0 is a Bäcklund transformation of $u_- = \partial_1 v_- - v_-^2 = 2\partial_1^2 \ln \tau_-$, solution of the KdV hierarchy, and generates the solution $u_+ = -(\partial_1 v_+ + v_+^2) = 2\partial_1^2 \ln \tau_+$. Then, the corresponding solution of the Ur–KdV hierarchy is of the form $z = (a\tau_+ + b\tau_-)/(c\tau_+ + d\tau_-)$. Connected with this see [13].

2. In this section we shall show that the string equation found for the Hermitian one–matrix model in [3] is the Galilean self–similarity condition for the AKNS hierarchy and then we shall prove that this is equivalent to the scaling self–similarity condition for the associated Heisenberg ferromagnet hierarchy.

The AKNS hierarchy for p, q, functions depending on $\mathbf{t} = \{t_n\}_{n\geq 0}$ is the following collection of compatible equations

$$\begin{cases} \partial_n p = 2p_{n+1}, \\ \partial_n q = -2q_{n+1}, \end{cases} \tag{7}$$

where $n \geq 0$, $\partial_n := \partial/\partial t_n$ and p_n, q_n and h_n are defined recursively by the relations

$$\begin{aligned} p_n &= \frac{1}{2}\partial_1 p_{n-1} + p h_{n-1}, \\ q_n &= -\frac{1}{2}\partial_1 q_{n-1} + q h_{n-1}, \\ \partial_1 h_n &= p q_n - q p_n, \quad n \geq 1 \end{aligned}$$

with the initial data $p_0 = q_0 = 0$, $h_0 = 1$.

The $n = 0$ flow is usually not considered in the standard AKNS hierarchy, but its inclusion will prove convenient. The equations for that flow are $\partial_0 p = 2p, \partial_0 q = -2q$, which means that $p(t_0, t_1, \dots) = \exp(2t_0)\tilde{p}(t_1, \dots), q(t_0, t_1, \dots) = \exp(-2t_0)\tilde{q}(t_1, \dots)$. The functions $(\tilde{p}, \tilde{q})$ satisfy the standard AKNS hierarchy, and this t_0–flow reflects the fact that given a solution $(\tilde{p}, \tilde{q})$ to the standard AKNS hierarchy $(n > 0)$ then $(e^c\tilde{p}, e^{-c}\tilde{q})$ is a solution as well for any $c \in \mathbb{C}$. The $n = 2$ flow is

$$\begin{cases} 2\partial_2 p = \partial_1^2 p - 2p^2 q, \\ 2\partial_2 q = -\partial_1^2 q + 2pq^2. \end{cases}$$

Notice that the real reduction $q = \mp p^*$ and $t_n \mapsto i t_n$ produces the NLS$^\pm$ hierarchy for which the t_2–flow is $2i\partial_2 p = -\partial_1^2 p \pm 2|p|^2 p$, the NLS$^\pm$ equation.

Let us now describe the local symmetries of the AKNS hierarchy. First we have the shifts in the time variables. Let ϑ be

$$\vartheta(\mathbf{t}) := \mathbf{t} + \boldsymbol{\theta},$$

the action of translations, where

$$\boldsymbol{\theta} := \{\theta_n\}_{n\geq 0} \in \mathbb{C}^\infty,$$

are the shifts of the time variables. If (p, q) is a solution to the AKNS hierarchy then so is $(\vartheta^* p, \vartheta^* q)$.

The Galilean transformation $\mathbf{t} \mapsto \gamma_b(\mathbf{t})$ is given by

$$\gamma_b(\mathbf{t})_n := \sum_{m \geq 0} \binom{n+m}{m} b^m t_{n+m}$$

where $b \in \mathbb{C}$. The scaling transformation $\mathbf{t} \mapsto \varsigma_\sigma(\mathbf{t})$ is represented by the relations

$$\varsigma_\sigma(\mathbf{t})_n := e^{n\sigma} t_n$$

where $\sigma \in \mathbb{C}$. If (p, q) is a solution of the AKNS hierarchy then so are $(\gamma_b^* p, \gamma_b^* q)$ and $(e^\sigma \varsigma_\sigma^* p, e^\sigma \varsigma_\sigma^* q)$.

Notice that for the corresponding solutions $(\tilde{p}, \tilde{q})$ of the standard AKNS hierarchy the Galilean action is $(\exp(2t(a))\gamma_a^* \tilde{p}, (\exp(-2t(a))\gamma_a^* \tilde{q})$, the exponential factors are a result of the flow in t_0 induced by the Galilean transformation. The related fundamental vector fields, infinitesimal generators of the action of translation, Galilean and scaling transformations are given by

$$\partial_n,\ n \geq 0,\ \boldsymbol{\gamma} = \sum_{n \geq 0} (n+1) t_{n+1} \partial_n,\ \boldsymbol{\varsigma} = \sum_{n \geq 1} n t_n \partial_n,$$

respectively.

Consider the vector field

$$X := \boldsymbol{\gamma} + \boldsymbol{\vartheta},$$

with

$$\boldsymbol{\vartheta} := \sum_{n \geq 0} \theta_n \partial_n.$$

Then, (p, q) is a self–similar solution under the action of the vector field X if

$$Xp = Xq = 0.$$

These are precisely the string equations appearing in the non–scaling limit analysis performed in [3] for the Hermitian one–matrix model with arbitrary potential, being the specific heat the first non–trivial conserved density $2h_2 = -pq$ of the AKNS hierarchy and $p = \exp(s)$ and $q = -u \exp(-s)$ with $u = R$ and $S = \partial_1 s$. The corresponding topological field theory is very close to the Witten's $\mathbb{C}P^1$–sigma model coupled with topological gravity, see [3].

Let us introduce the complexified version of the continuous one–dimensional Heisenberg spin chain or simply the Heisenberg ferromagnet equation. The rôle of the spin field is played by a vector field S depending on t_1, t_2 with $S(t_1, t_2) \in \mathfrak{sl}(2, \mathbb{C})$ such that $-\det S = 1$. The Heisenberg ferromagnet equation is

$$4\partial_2 S = [S, \partial_1^2 S].$$

As is well known, see [5] and references therein, this equation is equivalent to the AKNS t_2–flow. We can write

$$S = \mathrm{Ad}aH$$

where $\{E, H, F\}$ is the standard Weyl basis for $\mathfrak{sl}(2, \mathbb{C})$. Then, the solutions of the equation

$$\partial_1 a \cdot a^{-1} + \mathrm{Ad}a(pE + qF) = 0 \tag{8}$$

provides solutions to the AKNS t_2–flow.

The Heisenberg ferromagnet hierarchy, constructed similarly to the AKNS hierarchy, for the spin field S is the following set of compatible equations

$$\partial_n S = [S, S_{n+1}]$$

with the recurrence relations

$$\partial_1 S_n = [S, S_{n+1}].$$

For example $S_0 = 0, S_1 = S, S_2 = [S, \partial_1 S]/4$. The t_0 flow is trivial $\partial_0 S = 0$.

If $Q_n = p_n E + h_n H + q_n F$ then a satisfies the evolution equations

$$\partial_n a \cdot a^{-1} + \mathrm{Ad}aQ_n = 0, \tag{9}$$

notice that $\mathrm{Ad}aQ_n = S_{n+1}$.

Observe that the Heisenberg ferromagnet hierarchy is invariant under the adjoint action of $SL(2, \mathbb{C})$. Given a spin field S then $\mathrm{Ad}a_0 S$, with $a_0 \in SL(2, \mathbb{C})$, is also a solution. Thus, we have an action of $PSL(2, \mathbb{C}) = SL(2, \mathbb{C})/\{\mathrm{id}, -\mathrm{id}\}$ in the space of solutions. In fact, when one considers $\mathbb{C}^3 \cong \mathfrak{sl}(2, \mathbb{C})$ and the Cartan–Killing bilinear form $(X, Y) = 1/2\mathrm{Tr}XY$, the action above can be understood as an $SO(3, \mathbb{C}) \cong PSL(2, \mathbb{C})$ action. This is the complex version of the well known isotropy property of the SO_3–Heisenberg ferromagnet ($PSU_2 \cong SO_3$). The local symmetries of the Heisenberg ferromagnet are the translations and the scaling transformation.

A solution S to this hierarchy gives through (8) a solution (p, q) to the AKNS hierarchy. We introduce the vector field

$$Y := \boldsymbol{\varsigma} + \widetilde{\boldsymbol{\vartheta}},$$

where

$$\widetilde{\boldsymbol{\vartheta}} := \sum_{n \geq 0} \theta_n \partial_{n+1}.$$

From Eqs.(7,9) and

$$\partial_n S = [\partial_n a \cdot a^{-1}, S]$$

one deduces

LEMMA 2 *The following equation holds*

$$YS = \mathrm{Ad}a(Xp\,E + Xq\,F),$$

being S, p, q connected by (8).

from where it follows

PROPOSITION 3 *The solution (p, q) of the AKNS hierarchy satisfies the string equations*

$$Xp = Xq = 0$$

if and only if the corresponding spin field S satisfies

$$YS = 0.$$

Now, we consider the scaling self–similar condition in the AKNS hierarchy and its representation in terms of the Heisenberg ferromagnet hierarchy. From (8) one easily gets the relation

$$(\varsigma_\sigma^* a)^{-1} \cdot (\partial_1 \varsigma_\sigma^* a) + e^\sigma (\varsigma_\sigma^* p\,E + \varsigma_\sigma^* q\,F) = 0.$$

So that the solutions (p, q) of the AKNS hierarchy are scaling self–similar if and only if

$$\varsigma_\sigma^* a = \pm \exp(\sigma A) \cdot a,$$

with $A \in \mathfrak{sl}(2, \mathbb{C})$, or iff

$$\varsigma_\sigma^* S = \mathrm{Ad} \exp(\sigma A)\, S.$$

PROPOSITION 4 *The solution (p, q) of the AKNS hierarchy is self–similar under scaling transformations if and only if the corresponding solution to the Heisenberg ferromagnet hierarchy satisfies the string equation*

$$(\boldsymbol{\varsigma} - \mathrm{adA})S = 0,$$

for some $A \in \mathfrak{sl}(2, \mathbb{C})$.

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