On the Whitham Hierarchies: Reductions and Hodograph Solutions*

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Abstract

A general scheme for analyzing reductions of Whitham hierarchies is presented. It is based on a method for determining the S-function by means of a system of first order partial differential equations. Compatibility systems of differential equations characterizing both reductions and hodograph solutions of Whitham hierarchies are obtained. The method is illustrated by exhibiting solutions of integrable models such as the dispersionless Toda equation (heavenly equation) and the generalized Benney system.

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1 Introduction

The study of dispersionless (or quasiclasssical) limits of integrable systems of KdV-type and their applications has been an active subject of research for more than twenty years (see for example [1]-[14]). However, despite of the fact that many important developments on the algebraic and geometric aspects of these systems have been made, the theory of their solution methods seems far from being completed. Indeed, only for a few cases [15]-[17] the dispersionless limit of the inverse scattering method is available and *dispersionless versions* of ordinary direct methods like the $\overline{\partial}$-method are not yet fully developed [18].

In [3]-[4] Kodama and Gibbons provided a direct method for finding solutions of the dispersionless KP (dKP) equation

$$(u_t + 3uu_x)_x = \frac{3}{4}u_{yy}, \tag{1}$$

and its associated dKP hierarchy of nonlinear systems. The main ingredient of their method is the use of reductions of the dKP hierarchy formulated in terms of hydrodynamic-type equations. As a consequence it follows that solutions of the dKP hierarchy turn out to be determined through hodograph equations. Recently, we proposed [19] an alternative direct method for solving the dKP hierarchy from its reductions. It is based on the characterization of reductions and hodograph solutions of the dKP hierarchy by means of certain systems of first-order partial differential equations.

The aim of this paper is to present a generalization of the method of [19] which applies to the Whitham hierachies of dispersionless integrable systems. These hierarchies were introduced by Krichever in [7] and contain many interesting dispersionless models as, for example, the $(2+1)$-dimensional integrable systems

$$\Phi_{xy} + \left(e^{\Phi}\right)_{tt} = 0, \tag{2}$$

known as the dispersionless Toda (dT) equation (*heavenly equation* or Boyer-Finley equation [20]-[21]), and the generalized Benney system [10]

$$\begin{aligned} a_t + (av)_t &= 0, \\ v_t + vv_x + w_x &= 0, \\ w_y + a_x &= 0, \end{aligned} \tag{3}$$

In the next section we review briefly the definition of the Whitham hierarchies (zero genus case) and introduce our main notation conventions.

Section 3 concerns with the method for characterizing reductions and hodograph solutions of the Whitham hierarchies. To this end we take advantage of the same scheme as in [19] to introduce reductions through systems of first-order partial differential equations. The main difference with respect to the procedure used in [19] lies in the more involved construction of the S-function. Like in the study of the dKP hierarchy, we find that the compatibility equations for characterizing diagonal reductions of the Whitham hierarchies are deeply connected with the theory of Combescure transformations of conjugate nets. Finally, Section 4 is devoted to illustrate the method with examples of hodograph solutions of (2) and (3).

2 The Whitham hierarchy

The M-th Whitham hierarchy is related to a family of evolution equations for a set of M functions $z_\alpha = z_\alpha(p, \boldsymbol{t}),\ 1 \leq \alpha \leq M$ depending on a complex variable p and an infinite set of complex time parameters

$$\boldsymbol{t} := \{t_A : A = (\alpha, n) \in \boldsymbol{A}\},$$

where

$$\boldsymbol{A} = \{(\alpha, 0)\}_{\alpha=2}^{M} \bigcup \{(\alpha, n)\}_{\substack{\alpha=1,\dots,M \\ n=1,\dots,\infty}}.$$

It is assumed that a neighborhood $\mathcal{D}$ of ∞ in the extended complex plane of the p variable exists on which each z_α has a simple pole at an associated point $q_\alpha = q_\alpha(\boldsymbol{t})$. In particular, we set $q_1 = \infty$ and assume that z_1 posses the normalized Laurent expansion

$$z_1(p, \boldsymbol{t}) = p + \sum_{n=1}^{\infty} \frac{a_{1,n}(\boldsymbol{t})}{p^n}, \quad p \to \infty. \tag{4}$$

The corresponding expansions for the remaining functions z_α at q_α will be written as

$$z_i(p, \boldsymbol{t}) = \frac{a_{i,-1}(\boldsymbol{t})}{p - q_i(\boldsymbol{t})} + \sum_{n=0}^{\infty} a_{i,n}(\boldsymbol{t})\big(p - q_i(\boldsymbol{t})\big)^n, \quad p \to q_i(\boldsymbol{t}), \quad 2 \leq i \leq M. \tag{5}$$

In order to define the Whitham equations we introduce the system of evolution equations

$$\frac{\partial z_\alpha}{\partial t_A} = \{\Omega_A, z_\alpha\}, \quad 1 \leq \alpha \leq M. \tag{6}$$

Here $\{\cdot,\cdot\}$ is the Poisson bracket

$$\{F_1,F_2\}:=\frac{\partial F_1}{\partial p}\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial x}\frac{\partial F_2}{\partial p},\quad x:=t_{1,1},$$

and the functions $\Omega_A=\Omega_A(p,\boldsymbol{t})$ are defined by

$$\Omega_A=\begin{cases}-\ln(p-q_i(\boldsymbol{t})) & \text{for } A=(i,0),\ 2\leq i\leq M,\\ \\ \big((z_\alpha)^n\big)_+ & \text{for } A=(\alpha,n),\ 1\leq\alpha\leq M,\ n\geq 1,\end{cases}\tag{7}$$

where

$$\big((z_\alpha)^n\big)_+:=P_{(\alpha,+)}(z_\alpha^n),$$

with $P_{(\alpha,+)}$ being the following projectors acting on Laurent series around $p=q_\alpha(\boldsymbol{t})$

$$P_{(1,+)}\big(\sum_{n=-\infty}^{\infty}a_np^n\big)=\sum_{n=0}^{\infty}a_np^n,$$

$$P_{(i,+)}\big(\sum_{n=-\infty}^{\infty}b_n\big(p-q_i(\boldsymbol{t})\big)^n\big)=\sum_{n=1}^{\infty}\frac{b_{-n}}{(p-q_i(\boldsymbol{t}))^n},\quad 2\leq i\leq M.$$

The Whitham hierarchy is the set of equations

$$\frac{\partial\Omega_A}{\partial t_B}-\frac{\partial\Omega_B}{\partial t_A}+\{\Omega_A,\Omega_B\}=0,\quad A,B\in\boldsymbol{A},\tag{8}$$

which describe the compatibility conditions for the system (6). For $M=1$ the Whitham hierarchy becomes the dispersionless Kadomtsev–Petviasvhili (dKP) hierarchy. Some interesting nonlinear models included in the case $M=2$ are, for example,

1. The dispersionless Toda (dT) equation (*heavenly equation* or Boyer–Finley equation)

$$\Phi_{xy}+(\exp(\Phi))_{tt}=0,\tag{9}$$

which is obtained from (8) by setting $A=(2,0)$, $B=(2,1)$ and

$$\begin{gathered}y:=t_{(2,1)},\ t:=-t_{(2,0)},\\ \Phi:=\ln a_{2,-1},\\ \Omega_{(2,0)}=-\ln(p-q_2),\quad \Omega_{(2,1)}=\frac{a_{2,-1}(\boldsymbol{t})}{p-q_2(\boldsymbol{t})}.\end{gathered}\tag{10}$$

2. The generalized Benney system (generalized gas equation) [10]

$$\begin{aligned} a_t + (av)_x &= 0, \\ v_t + vv_x + w_x &= 0, \\ w_y + a_x &= 0, \end{aligned} \tag{11}$$

can be regarded as a two-dimensional generalization of the equations for one-dimensional gas dynamics. It takes the form (8) by setting $A = (2,1)$, $B = (1,2)$ and

$$\begin{aligned} y := t_{(2,1)},\ t := -\frac{1}{2}t_{(1,2)}, \\ a := a_{2,-1},\quad v := q_2,\quad w := a_{1,1} \\ \Omega_{(2,1)} = \frac{a_{2,-1}(\boldsymbol{t})}{p - q_2(\boldsymbol{t})},\quad \Omega_{(1,2)} = p^2 + 2a_{1,1}. \end{aligned} \tag{12}$$

3 Reductions of the Whitham hierarchy

3.1 The S function

In this paper se shall study *algebraic orbits* of the zero genus Whitham hierarchy defined by [8]

$$z_i = f_i(z), \quad z := z_1, \quad 2 \le i \le M, \tag{13}$$

which are easily checked to be compatible with (6).

Furthermore, it follows from (6) and (8) that

$$\frac{\partial}{\partial t_B}\Omega_A(p(z,\boldsymbol{t}),\boldsymbol{t}) = \frac{\partial}{\partial t_A}\Omega_B(p(z,\boldsymbol{t}),\boldsymbol{t}),$$

and therefore there exists a potential function $S = S(z,\boldsymbol{t})$ satisfying

$$\frac{\partial S(z,\boldsymbol{t})}{\partial t_A} = \Omega_A(p(z,\boldsymbol{t}),\boldsymbol{t}), \quad A \in \boldsymbol{A}. \tag{14}$$

Reciprocally, we can state the following proposition on which our solution method for the Whitham hierarchy will be based

Proposition 1. *Let* $\{z_\alpha(p,\boldsymbol{t})\}_{\alpha=1}^M$ *be a set of functions satisfying a system of time-independent relations* (13) *as well as* (4) *and* (5)*. If a function* $S(z,\boldsymbol{t})$ *verifying* (14) *exists, then the functions* $z_\alpha(p,\boldsymbol{t})$ *provide a solution of the Whitham hierarchy.*

Proof. First we notice that by setting $A=(1,1)$ in (14) it follows

$$p(z,\boldsymbol{t})=\frac{\partial S(z,\boldsymbol{t})}{\partial x},$$

so that

$$\begin{aligned}\frac{\partial p}{\partial t_A}&=\frac{\partial}{\partial x}\frac{\partial S(z,\boldsymbol{t})}{\partial t_A}=\frac{\partial}{\partial x}\Omega_A(p(z,\boldsymbol{t}),\boldsymbol{t})\\&=\frac{\partial \Omega_A}{\partial p}\frac{\partial p}{\partial x}+\frac{\partial \Omega_A}{\partial x}.\end{aligned}$$

Hence, the function $z=z(p,\boldsymbol{t})$ satisfies

$$\begin{aligned}\frac{\partial z}{\partial t_A}&=-\frac{\partial z}{\partial p}\frac{\partial p}{\partial t_A}=-\frac{\partial z}{\partial p}\Big(\frac{\partial \Omega_A}{\partial p}\frac{\partial p}{\partial x}+\frac{\partial \Omega_A}{\partial x}\Big)\\&=\frac{\partial \Omega_A}{\partial p}\frac{\partial z}{\partial x}-\frac{\partial \Omega_A}{\partial x}\frac{\partial z}{\partial p}=\{\Omega_A,z\}.\end{aligned}$$

Therefore, by using (13) we deduce (6). □

3.2 N-reductions of the Whitham hierarchy

We are going to describe a method for finding solutions of the Whitham hierarchy from functions $z=z(p,\boldsymbol{u})$ depending on p and a finite set of variables $\boldsymbol{u}:=(u_1,\dots,u_N)$, such that the inverse function $p=p(z,\boldsymbol{u})$ satisfies a system of equations of the form

$$\frac{\partial p}{\partial u_i}=R_i(p,\boldsymbol{u}),\quad 1\le i\le N, \tag{15}$$

or, equivalently, in terms of $z=z(p,\boldsymbol{u})$

$$\frac{\partial z}{\partial u_i}+R_i(p,\boldsymbol{u})\frac{\partial z}{\partial p}=0,\quad 1\le i\le N. \tag{16}$$

The following conditions for the functions R_i will be assumed

i) The functions R_i are rational functions of p which have singularities only at N simple poles $p_i = p_i(\boldsymbol{u}),\ i = 1, \ldots, N$, and vanish at $p = \infty$. Therefore, they can be expanded as

$$R_i(p, \boldsymbol{u}) = \sum_{j=1}^{N} \frac{r_{ij}(\boldsymbol{u})}{p - p_j(\boldsymbol{u})}. \tag{17}$$

ii) The functions $r_{ij}(\boldsymbol{u}),\ \ p_i(\boldsymbol{u})$ satisfy the compatibility conditions for (16)-(15)

$$\begin{aligned} r_{ik}\frac{\partial p_k}{\partial u_j} - r_{jk}\frac{\partial p_k}{\partial u_i} &= \sum_{l\neq k} \frac{r_{jl}r_{ik} - r_{il}r_{jk}}{p_k - p_l}, \\ \frac{\partial r_{ik}}{\partial u_j} - \frac{\partial r_{jk}}{\partial u_i} &= 2\sum_{l\neq k} \frac{r_{jk}r_{il} - r_{ik}r_{jl}}{(p_k - p_l)^2}, \end{aligned} \tag{18}$$

where $i \neq j$.

The starting point of the method is a solution $z = z(p, \boldsymbol{u})$ of (15) with a Laurent expansion

$$z(p, \boldsymbol{u}) = p + \sum_{n=1}^{\infty} \frac{a_n(\boldsymbol{u})}{p^n}, \quad p \to \infty, \tag{19}$$

which is assumed to define a univalent analytic function $z : \mathcal{D} \to \mathcal{D}'$ between two neighborhoods $\mathcal{D}$ and $\mathcal{D}'$ of ∞ in the extended complex planes of the variables p and z respectively. The next step is to take $(M - 1)$ different points $z_{0,i} \in \mathcal{D}',\ 2 \leq i \leq M$ and define the functions

$$\begin{aligned} z_1(p, \boldsymbol{u}) &:= z(p, \boldsymbol{u}), \\ z_i(p, \boldsymbol{u}) &:= \frac{1}{z(p, u) - z_{0,i}}, \quad 2 \leq i \leq M. \end{aligned} \tag{20}$$

Obviously, they satisfy the system of equations

$$\frac{\partial z_\alpha}{\partial u_i} + R_i(p, \boldsymbol{u})\frac{\partial z_\alpha}{\partial p} = 0, \quad 1 \leq \alpha \leq M, \tag{21}$$

and admit expansions of the form

$$
\begin{aligned}
z_1(p,\boldsymbol{u}) &= p+\sum_{n=1}^{\infty}\frac{a_{1,n}(\boldsymbol{u})}{p^n}, \quad p\to\infty.\\
z_i(p,\boldsymbol{u}) &= \frac{a_{i,-1}(\boldsymbol{u})}{p-q_i(\boldsymbol{u})}+\sum_{n=0}^{\infty}a_{i,n}(\boldsymbol{u})\big(p-q_i(\boldsymbol{u})\big)^n, \quad p\to q_i(\boldsymbol{u}),
\end{aligned}
\tag{22}
$$

for $2\leq i\leq M$, here

$$
q_i(\boldsymbol{u}):=p(z_{0,i},\boldsymbol{u}). \tag{23}
$$

Observe that introducing the expansions at $p=\infty, q_i$, $i=2,\ldots,M$, of (22) in (21) we get

$$
\begin{cases}
\dfrac{\partial a_{1,1}}{\partial u_i} = -\displaystyle\sum_{j=1}^{N} r_{ij},\\[2ex]
\dfrac{\partial a_{1,2}}{\partial u_i} = -\displaystyle\sum_{j=1}^{N} r_{ij}p_j,\\[2ex]
\dfrac{\partial (a_{1,3}+a_{1,1}^2/2)}{\partial u_i} = -\displaystyle\sum_{j=1}^{N} r_{ij}p_j^2,
\end{cases}
\tag{24}
$$

$$
\begin{cases}
\dfrac{\partial q_\alpha}{\partial u_j} = R_j(q_\alpha),\\[2ex]
\dfrac{\partial \log a_{\alpha,-1}}{\partial u_j} = \dfrac{\mathrm{d}\,R_j}{\mathrm{d}\,p}(q_\alpha)
\end{cases}
\quad \text{for } \alpha=2,\ldots,M
\tag{25}
$$

while the other coefficients $a_{\alpha,n}$ in the expansion of z_α are determined by:

$$
\begin{aligned}
\frac{\partial a_{1,n}}{\partial u_j} &= -R_{j,n}+\sum_{k=1}^{n-2}(n-k)R_{j,k}a_{i,n-k},\\
\frac{\partial a_{i,n}}{\partial u_j} &= \sum_{k=1}^{n+2}\frac{1}{k!}\frac{\mathrm{d}^k\,R_j}{\mathrm{d}\,p^k}(q_i)a_{i,n-k+1}, \quad \text{for } i=2,\ldots,M,
\end{aligned}
$$

with

$$
R_{j,k}=\sum_{i=1}^{N} r_{ji}p_i^{k-1}.
$$

Finally, we introduce the function

$$\begin{aligned} \mathcal{S}(p,\boldsymbol{u},\boldsymbol{t}) &= \mathcal{S}_+(p,\boldsymbol{u},\boldsymbol{t}) + \mathcal{S}_-(p,\boldsymbol{u}),\\ \mathcal{S}_+ &:= \sum_{A\in\boldsymbol{A}} t_A\Omega_A(p,\boldsymbol{u}), \end{aligned} \tag{26}$$

where $\Omega_A(p,\boldsymbol{u})$ are defined by (7) and (22), and $\mathcal{S}_-(p,\boldsymbol{u})$ is an analytic function on $\mathcal{D}$ such that

$$\lim_{p\to\infty} \mathcal{S}_-(p,\boldsymbol{u}) = 0. \tag{27}$$

We can now enounce the following statement

Proposition 2. *If $\mathcal{S}_-(p,\boldsymbol{u})$ satisfies a system of equations*

$$\frac{\partial \mathcal{S}_-}{\partial u_i} + R_i \frac{\partial \mathcal{S}_-}{\partial p} = \sum_k \frac{r_{ik}F_k}{p-p_k}, \quad 1 \leq i \leq N, \tag{28}$$

for a given set of functions $\{F_i = F_i(\boldsymbol{u})\}_{i=1}^N$ verifying the compatibility conditions for (28)

$$r_{ik}\frac{\partial F_k}{\partial u_j} - r_{jk}\frac{\partial F_k}{\partial u_i} = \sum_{l\neq k} \frac{r_{jl}r_{ik} - r_{il}r_{jk}}{(p_k-p_l)^2}(F_k - F_l), \quad i \neq j, \tag{29}$$

and the functions $\{u_i = u_i(\boldsymbol{t})\}_{i=1}^N$ are implicitly determined by means of the hodograph relations

$$\sum_{A\in\boldsymbol{A}} t_A \frac{\partial \Omega_A}{\partial p}(p_i(\boldsymbol{u}),\boldsymbol{u}) + F_i(\boldsymbol{u}) = 0, \quad 1 \leq i \leq N, \tag{30}$$

then

$$S(z,\boldsymbol{t}) := \mathcal{S}\big(p(z,\boldsymbol{u}(\boldsymbol{t})),\boldsymbol{u}(\boldsymbol{t}),\boldsymbol{t}\big), \tag{31}$$

is an S-function for the Whitham hierarchy.

Proof. The proof is based on the following consequence of (26) and (31)

$$\frac{\partial}{\partial t_A}S(z,\boldsymbol{t}) = \Omega_A(p(z,\boldsymbol{u}(\boldsymbol{t})),\boldsymbol{t}) + \sum_{i=1}^N \frac{\partial u_i}{\partial t_A}\Big(\frac{\partial}{\partial u_i}\mathcal{S}(p(z,\boldsymbol{u}),\boldsymbol{u},\boldsymbol{t})\Big)\Big|_{\boldsymbol{u}=\boldsymbol{u}(\boldsymbol{t})}, \tag{32}$$

and our aim is to prove that under the hypothesis of the proposition the functions

$$\frac{\partial}{\partial u_i}\mathcal{S}(p(z,\boldsymbol{u}),\boldsymbol{u},\boldsymbol{t}) = \frac{\partial \mathcal{S}}{\partial p}R_i + \frac{\partial \mathcal{S}}{\partial u_i}. \tag{33}$$

vanish identically, so that $S(z,\boldsymbol{t})$ satisfies (14) and, consequently, the statement will follow at once.

By construction the functions (33) are analytic on $\mathcal{D}$ up to a set of possible isolated singularities at $\{p_i(\boldsymbol{u}), q_\alpha(\boldsymbol{u})\}$. On the other hand we observe that (28) implies

$$F_i(\boldsymbol{u}) = \frac{\partial \mathcal{S}_-}{\partial p}(p_i(\boldsymbol{u}),\boldsymbol{u}), \tag{34}$$

so that (30) is equivalent to

$$\frac{\partial \mathcal{S}}{\partial p}(p_i(\boldsymbol{u}),\boldsymbol{u}) = 0, \quad 1 \leq i \leq N. \tag{35}$$

As a consequence we deduce that the functions (33) are analytic at $p_i(\boldsymbol{u})$. Hence, their possible singularities reduce to the points $q_\alpha(\boldsymbol{u})$. However, we have

$$\frac{\partial}{\partial u_i}\mathcal{S}(p(z,\boldsymbol{u}),\boldsymbol{u},\boldsymbol{t}) = \sum_{A\in\boldsymbol{A}} t_A \frac{\partial}{\partial u_i}\Omega_A(p(\boldsymbol{u}),\boldsymbol{u}) + \frac{\partial \mathcal{S}_-}{\partial u_i}, \tag{36}$$

and we may rewrite

$$\begin{aligned}
\Omega_{(i,0)} &= \ln\frac{1}{z-z_{0,i}} - P_{(i,-)}\Big(\Omega_{(i,0)} - \frac{1}{z-z_{0,i}}\Big), \quad 2 \leq i \leq M \\
\Omega_{(\alpha,n)} &= z_\alpha^n - P_{(\alpha,-)}\Big(\Omega_{(\alpha,n)} - z_\alpha^n\Big), \quad n \geq 1,
\end{aligned} \tag{37}$$

where $P_{(\alpha,-)} := 1 - P_{(\alpha,+)}$ are the projectors which annihilate the singular terms of Laurent expansions at $p = q_\alpha(\boldsymbol{u})$. Thus, by noticing that the first terms in the right-hand sides of (37) are $\boldsymbol{u}$-independent while the second terms are analytic at $q_\alpha(\boldsymbol{u})$, we conclude that the functions (33) are also analytic at the points $q_\alpha(\boldsymbol{u})$. Hence, these functions are analytic on the whole domain $\mathcal{D}$. Moreover, by taking (27) into account, it follows that there is an expansion of the form

$$\frac{\partial}{\partial u_i}\mathcal{S}(p(z,\boldsymbol{u}),\boldsymbol{u},\boldsymbol{t}) = \sum_{n=1}^{\infty}\frac{s_{i,n}(\boldsymbol{u},\boldsymbol{t})}{p^n}. \tag{38}$$

so that

$$\begin{aligned}\frac{\partial}{\partial u_i}\mathcal{S}(p(z,\boldsymbol{u}),\boldsymbol{u},\boldsymbol{t}) &= P_{(1,-)}\Big(\frac{\partial}{\partial u_i}\mathcal{S}(p(z,\boldsymbol{u}),\boldsymbol{u},\boldsymbol{t})\Big)\\ &= P_{(1,-)}\Big(\frac{\partial \mathcal{S}}{\partial p}R_i\Big)+\frac{\partial \mathcal{S}_-}{\partial u_i}.\end{aligned}$$

Let us now denote by $E = E(p,\boldsymbol{u})$ any entire function of p such that

$$E(p_i(\boldsymbol{u}),\boldsymbol{u}) = F_i(\boldsymbol{u}), \quad i = 1,\dots,N.$$

Then, by taking into account that (30) implies

$$P_{(1,-)}\Big(\Big(\frac{\partial \mathcal{S}_+}{\partial p}+E\Big)R_i\Big) = 0,$$

it follows that

$$\begin{aligned}\frac{\partial}{\partial u_i}\mathcal{S}(p(z,\boldsymbol{u}),\boldsymbol{u},\boldsymbol{t}) &= P_{(1,-)}\Big(\frac{\partial \mathcal{S}_-}{\partial p}R_i - ER_i\Big)+\frac{\partial \mathcal{S}_-}{\partial u_i}\\ &= \frac{\partial \mathcal{S}_-}{\partial p}R_i+\frac{\partial \mathcal{S}_-}{\partial u_i}-\sum_k \frac{r_{ik}F_k}{p-p_k}\\ &= 0.\end{aligned}$$

Hence, the statement follows. □

3.3 Diagonal reductions, symmetric conjugate nets and potentials

In the case of diagonal reductions $r_{ij} = \delta_{ij} r_i$,

$$\frac{\partial p}{\partial u_i} = -\frac{r_i(\boldsymbol{u})}{p-p_i(\boldsymbol{u})}, \tag{39}$$

with $i = 1,\dots,N$, the compatibility conditions (18) and (29) reduce to

$$\begin{aligned}\frac{\partial r_i}{\partial u_j} &= 2\frac{r_i r_j}{(p_j-p_i)^2},\\ \frac{\partial p_i}{\partial u_j} &= \frac{r_j}{p_j-p_i},\\ \frac{\partial F_i}{\partial u_j} &= r_j\frac{F_j-F_i}{(p_j-p_i)^2},\end{aligned} \tag{40}$$

where $i \neq j$. We may extend our observations of [19] by showing that the diagonal reductions of the Whitham hierarchy determine a particular symmetric conjugate net as well as a set of $(M+1)$ Comberscure transformed symmetric conjugate nets. In particular we are going to prove that the coefficients $a_{1,1}, a_{1,2}, a_{1,3}, a_{\alpha,-1}$ and q_α are geometrical potentials associated with these Comberscure transformed nets.

A conjugate net with curvilinear coordinates $\boldsymbol{u}$ can be described in terms of a set of rotation coefficients $\{\beta_{ij}(\boldsymbol{u})\}_{\substack{i,j=1,\dots,N\\ i\neq j}}$ which satisfy the Darboux equations [22]

$$\frac{\partial \beta_{ij}}{\partial u_k} = \beta_{ik}\beta_{kj}$$

for any triple of different labels i, j, k. The associated Lamé coefficients $\{H_i(\boldsymbol{u})\}_{i=1,\dots,N}$ are defined by the solutions of the linear system

$$\frac{\partial H_i}{\partial u_j} = \beta_{ji} H_j.$$

Under a Comberscure transformation a conjugate net transforms into a parallel conjugate net. The rotation coefficients are left invariant but the Lamé coefficients change. The new Lamé coefficients are given by

$$\tilde{H}_i = \sigma_i H_i$$

with

$$\frac{\partial \sigma_i}{\partial u_j} = \beta_{ji}\frac{H_j}{H_i}(\sigma_j - \sigma_i).$$

A conjugate net is said symmetric iff $\beta_{ij} = \beta_{ji}$. Given any pair of parallel symmetric conjugate nets characterized by $\{\beta_{ij}, H_j\}$ and $\{\beta_{ij}, \tilde{H}_j\}$, respectively; then, it follows that locally there exists a potential function ρ so that $\sigma_i H_i^2 = \dfrac{\partial \rho}{\partial u_i}$; to see this just observe that

$$\frac{\partial H_i \tilde{H}_i}{\partial u_j} = \beta_{ij}(H_i\tilde{H}_j + H_j\tilde{H}_i),$$

which is a symmetric expression provided $\beta_{ij} = \beta_{ji}$.

Taking $H_i := \sqrt{r_i}$ and $\beta_{ij} := \dfrac{\sqrt{r_i r_j}}{(p_i - p_j)^2}$, as the first equation on (40) is $\dfrac{\partial H_i}{\partial u_j} = \beta_{ij} H_j$, we can identify H_i and β_{ij} as the Lamé and rotation coefficients, respectively, of a conjugate net.

The functions $\left.\frac{\mathrm{d}\,\Omega_{i,n}}{\mathrm{d}\,p}\right|_{p=q_\alpha}$ determining the hodograph relations are polynomials in

$$p_{i,\alpha} = \begin{cases} p_i, & \text{for } j = 1, \\ \dfrac{1}{p_i - q_\alpha}, & \text{for } \alpha = 2, \dots M; \end{cases}$$

observing that $\beta_{ij} H_j / H_i = \dfrac{r_j}{(p_i - p_j)^2}$ it is easy to see that these coefficients determine a set of M Comberscure transformations. Then, together with the set of Lamé coefficients $\{H_i = \sqrt{r_i}\}_{i=1}^N$ we have the M families of Lamé coefficients

$$\{H_{i,\alpha} := p_{i,\alpha}\sqrt{r_i}\}_{i=1}^N, \quad \text{for } \alpha = 1, \dots, M.$$

It also follows that there is another Comberscure transformed net with Lamé coefficients given by

$$\{h_i := \sqrt{r_i} F_i\}_{i=1}^N.$$

From (24) and (25)we easily find the potentials for $H_i H_{i,j}$ and $H_{i,j}^2$:

$$H_i^2 = -\frac{\partial a_{1,1}}{\partial u_i}, \tag{41}$$

$$H_i H_{i,\alpha} = \begin{cases} -\dfrac{\partial a_{1,2}}{\partial u_i} & \text{for } \alpha = 1 \\ -\dfrac{\partial q_\alpha}{\partial u_i} & \text{for } \alpha = 2, \dots, M, \end{cases} \tag{42}$$

$$H_{i,\alpha}^2 = \begin{cases} -\dfrac{\partial (a_{1,3} + a_{1,1}^2/2)}{\partial u_i} & \text{for } \alpha = 1, \\ -\dfrac{\partial \log a_{\alpha,-1}}{\partial u_i} & \text{for } \alpha = 2, \dots, M. \end{cases} \tag{43}$$

In this way $a_{1,1}, a_{1,2}, a_{1,3}, a_{\alpha,-1}$ and q_α, $\alpha = 2, \dots, M$ acquire a direct geometrical meaning.

Observing that

$$\beta_{ij} = \frac{\sqrt{r_i r_j}}{(p_{i,\alpha} - p_{j,\alpha})^2} p_{i,k\alpha}^2 p_{j,\alpha}^2, \quad \text{for } \alpha = 2, \dots, M,$$

we write our original compatibility conditions as follows

$$\begin{aligned}\frac{\partial r_i}{\partial u_j}&=2\frac{r_ir_j}{(p_{j,\alpha}-p_{i,\alpha})^2}p_{i,\alpha}^2p_{j,\alpha}^2,\\ \frac{\partial p_{i,\alpha}}{\partial u_j}&=\frac{r_j}{p_{j,\alpha}-p_{i,\alpha}}p_{i,\alpha}^2p_{j,\alpha}^2,\\ \frac{\partial F_i}{\partial u_j}&=r_j\frac{F_j-F_i}{(p_{j,\alpha}-p_{i,\alpha})^2}p_{i,\alpha}^2p_{j,\alpha}^2,\end{aligned}\tag{44}$$

for $\alpha=2,\dots,M$. This system determines a particular symmetric conjugate net and two Comberscure transformations of it. Moreover, if we want to recover the original formulation from these $p_{i,\alpha}$ we just need the potential q_α of $p_{i,\alpha}r_i$ and then $r_i, p_i=p_{i,\alpha}^{-1}+q_\alpha$, $\alpha=2,\dots,M$, will fulfill (40).

From (41), (42) and (43) we easily get

$$\frac{\partial^2 a_{1,1}}{\partial u_i\partial u_j}+\beta_{ji}\sqrt{r_i}\sqrt{r_j}=0,$$

$$\begin{cases}-\dfrac{\partial a_{1,2}}{\partial u_i\partial u_j}+\beta_{ji}\sqrt{r_i}\sqrt{r_j}(p_i+p_j)=0\\ \dfrac{\partial^2 q_\alpha}{\partial u_i\partial u_j}+\beta_{ji}\sqrt{r_i}\sqrt{r_j}(p_{i,\alpha}+p_{j,\alpha})=0, \quad \text{for } \alpha=2,\dots,M,\end{cases}$$

$$\begin{cases}\dfrac{\partial^2(a_{1,3}+a_{1,1}^2/2)}{\partial u_i\partial u_j}+2\beta_{ji}\sqrt{r_i}\sqrt{r_j}p_ip_j=0\\ \dfrac{\partial^2 \log a_{\alpha,-1}}{\partial u_i\partial u_j}+2\beta_{ji}\sqrt{r_i}\sqrt{r_j}p_{i,\alpha}p_{j,\alpha}=0 \qquad \text{for } \alpha=2,\dots,M.\end{cases}$$

Observe that (40) or (44) can be written in terms of two potentials only. For example we can choose these potentials to be q_α and $\log a_{\alpha,-1}$ and use

$$r_i=-\frac{\Big(\dfrac{\partial q_\alpha}{\partial u_i}\Big)^2}{\dfrac{\partial\log a_{\alpha,-1}}{\partial u_i}},\quad p_{i,\alpha}=\frac{\dfrac{\partial\log a_{\alpha,-1}}{\partial u_i}}{\dfrac{\partial q_\alpha}{\partial u_i}}$$

together with

$$\begin{aligned}\beta_{ij}\sqrt{r_i}\sqrt{r_j}&=\frac{r_ir_j}{(p_{i,\alpha}-p_{j,\alpha})^2}p_{i,\alpha}^2p_{j,\alpha}^2\\ &=\frac{\partial\log a_{\alpha,-1}}{\partial u_i}\frac{\partial\log a_{\alpha,-1}}{\partial u_j}\Big(\frac{\partial q_\alpha}{\partial u_i}\frac{\partial q_\alpha}{\partial u_j}\Big)^2\frac{1}{W_{ij}^-(a_{\alpha,-1},q_\alpha)^2}\end{aligned}$$

where

$$W_{ij}^{\pm}(f,g) := \frac{\partial f}{\partial u_i}\frac{\partial g}{\partial u_j} \pm \frac{\partial a}{\partial u_j}\frac{\partial g}{\partial u_j}$$

for $\alpha = 2,\ldots,M$.

$$W_{ij}^{-}(a_{\alpha,-1},q_\alpha)^2\frac{\partial^2 q_\alpha}{\partial u_i\partial u_j} + \frac{\partial \log a_{\alpha,-1}}{\partial u_i}\frac{\partial \log a_{\alpha,-1}}{\partial u_j}\frac{\partial q_\alpha}{\partial u_i}\frac{\partial q_\alpha}{\partial u_j}W_{ij}^{+}(a_{\alpha,-1},q_\alpha) = 0,$$
$$W_{ij}^{-}(a_{\alpha,-1},q_\alpha)^2\frac{\partial^2 \log a_{\alpha,-1}}{\partial u_i\partial u_j} + 2\Big(\frac{\partial \log a_{\alpha,-1}}{\partial u_i}\frac{\partial \log a_{\alpha,-1}}{\partial u_j}\Big)^2\frac{\partial q_\alpha}{\partial u_i}\frac{\partial q_\alpha}{\partial u_j} = 0.$$

For $i,j = 1,\ldots N$ and $i \neq j$.

4 Examples

4.1 Dispersionless Toda equation

In order to find solutions of the dT equation

$$\Phi_{xy} + (\exp(\Phi))_{tt} = 0,$$

we set all t_A equal to zero with the exception of $t_{(2,1)}$ and $t_{(2,0)}$, so that from (10) and by denoting

$$q(\boldsymbol{t}) := q_2(\boldsymbol{t}), \quad \nu(\boldsymbol{t}) := a_{2,-1}(\boldsymbol{t}), \tag{45}$$

we have that

$$\Phi = \ln \nu(\boldsymbol{t}). \tag{46}$$

$N = 1$ reduction Let us first consider reductions $z = z(p,u)$ depending on a single variable u defined by $u = -a_{1,-1}$. Then (15) becomes the Abel's equation

$$\frac{\partial p}{\partial u} = \frac{1}{p - p_1(u)}, \tag{47}$$

and (30) reads

$$\frac{t}{p_1(u) - q(u)} - \frac{y\,\nu(u)}{(p_1(u) - q(u))^2} + x + F(u) = 0, \tag{48}$$

where $q(u)$, $p_1(u)$ and $F(u)$ are arbitrary functions of u. On the other hand,

$$\frac{\partial q(u)}{\partial u} = \frac{1}{q(u) - p_1(u)},$$
$$\frac{\mathrm{d}\ln\nu}{\mathrm{d}\, u} = -\frac{1}{(q(u) - p_1(u))^2}.$$

In this way, we may rewrite (48) as

$$t\sqrt{-\frac{\nu'}{\nu}} - y\nu' - x + F(u) = 0,$$

where $\nu' := \mathrm{d}\,\nu/\,\mathrm{d}\, u$. Therefore, as $p_1(u)$ is an arbitrary function of u we have

$$\begin{aligned} &tT(u) + yY(u) + xX(u) + F(u) = 0,\\ &\Phi = \ln\Big(-\frac{XY}{T^2}\Big), \end{aligned} \tag{49}$$

where $T(u)$, $X(u)$, $Y(u)$ and $F(u)$ are arbitrary functions of u. For example when T, X, Y and F are polynomials of 4th degree we can get explicit examples of solutions. For example, assuming 2nd order polynomials we get

$$u := \gamma\pm\sqrt{\gamma^2-\delta}, \quad \gamma := \frac{1}{2}\frac{X_1x + Y_1y + T_1t + F_1}{X_2x + Y_2y + T_2t + F_2}, \quad \delta := \frac{X_0x + Y_0y + T_0t + F_0}{X_2x + Y_2y + T_2t + F_2},$$

and a solution of dT is

$$\begin{aligned} \Phi =& \ln\Big((X_1 - \gamma X_2)(-\gamma \pm \sqrt{\gamma^2 - \delta}) + X_0 - \delta X_1\Big)\\ &+ \ln\Big((Y_1 - \gamma Y_2)(-\gamma \pm \sqrt{\gamma^2 - \delta}) + Y_0 - \delta Y_1\Big)\\ &- 2\ln\Big(-(T_1 - \gamma T_2)(-\gamma \pm \sqrt{\gamma^2 - \delta}) - T_0 + \delta T_1\Big) \end{aligned}$$

Another example is to take $T = u^3$, $Y = u^2$, $X = u$, $F = 1$ to get the following hodograph relation

$$tu^3 + yu^2 + xu + 1 = 0$$

and the corresponding solution of the dispersionless Toda equation:

$$\Phi = 3\log\Big(\frac{6tf}{12xt - 4y^2 + 8yf - f^2}\Big)$$

where

$$f(x,y,t) := \sqrt[3]{36xyt - 108t^2 - 8y^3 + 12\sqrt{3}t\sqrt{4x^3t - x^2y^2 - 18xyt + 27t^2 + 4y^3}}$$

$N \geq 2$ **diagonal reductions** Let us consider now reductions $z = z(p, \boldsymbol{u})$ involving $N > 1$ variables $\boldsymbol{u} := (u_1, \ldots, u_N)$ associated with a system of equations (16) (or (15)). Consequently, the functions $r_{ij}(\boldsymbol{u})$, $p_i(\boldsymbol{u})$ are assumed to satisfy the compatibility conditions (29). In this case we get the following system of equations for determining $q(\boldsymbol{u})$ and $\nu(\boldsymbol{u})$

$$\begin{aligned} \frac{\partial q}{\partial u_i} &= R_i(q, \boldsymbol{u}), \\ \frac{\partial \ln\nu}{\partial u_i} &= \frac{\partial R_i}{\partial p}(q(\boldsymbol{u}), \boldsymbol{u}), \end{aligned} \tag{50}$$

where $i = 1, \ldots, N$. Thus, given a set of functions $\{F_i(\boldsymbol{u})\}_{i=1}^N$ satisfying (29), the hodograph relations (30) read

$$\frac{t}{p_i(\boldsymbol{u}) - q(\boldsymbol{u})} - \frac{y\,\exp\Big(\int^{\boldsymbol{u}} \sum_{j=1}^N \frac{\partial R_j}{\partial p}(q(\boldsymbol{u}), \boldsymbol{u})\,\mathrm{d}\,u_j\Big)}{(p_i(\boldsymbol{u}) - q(\boldsymbol{u}))^2} + x + F_i = 0,$$

for $i = 1, \ldots, N$, where now q solves

$$\frac{\partial q}{\partial u_i} = \frac{r_i}{p_i - q},$$

If we define

$$P_i := \frac{1}{p_i - q}$$

the above equation reads

$$tP_i - yP_i^2\exp\Big(-\int^{\boldsymbol{u}} \sum_{j=1}^N r_j P_j^2\,\mathrm{d}\,u_j\Big) + x + F_i = 0.$$

For $N = 2$ a simple solution we can take

$$r_1 = -r_2 = \frac{1}{8}(u_1 - u_2), p_1 = \frac{1}{4}(3u_1 + u_2),\ p_2 = \frac{1}{4}(u_1 + 3u_2),$$
$$F_1 = -F_2 = \frac{c}{u_2 - u_1}, \tag{51}$$

where c is an arbitrary complex constant. In this case we can get the explicit solution $z(p, \boldsymbol{u})$ of (16) satisfying (19). It is given by

$$z = p + \frac{(u_1 - u_2)^2}{16p - 8(u_1 + u_2)}. \tag{52}$$

Thus, from (23) we can set

$$q(\boldsymbol{u}) = -\frac{1}{2}\sqrt{(u_1 + z_0)(u_2 + z_0)} + \frac{1}{4}(u_1 + u_2 - 2z_0), \tag{53}$$

so that by denoting

$$U_i := \sqrt{u_i + z_0}, \quad i = 1, 2,$$

the hodograph relations become

$$x + \frac{4y}{(U_1U_2)^2} - \frac{c}{(U_1 - U_2)^2} = 0,$$
$$4t + \Big(2x - \frac{c}{(U_1 - U_2)^2}\Big)(U_1 + U_2)^2 - c = 0, \tag{54}$$

and we have

$$\Phi = \ln \frac{(U_1 + U_2)^2}{U_1U_2}. \tag{55}$$

The system (54) reduces to a quartic equation as we shall show. We first write the system (54) in terms of

$$u_\pm := (U_1 \pm U_2)^2,$$

as follows

$$x + \frac{64y}{(u_+ - u_-)^2} - \frac{c}{u_-} = 0,$$

$$4t + \Big(2x - \frac{c}{u_-}\Big)u_+ - c = 0.$$

By eliminating $u_+ = (c-4t)(2xu_- - c)^{-1}u_-$ we get

$$x^3u_-^4 + (-3c+4t)x^2u_-^3 + (4t^2 - 8ct + 3c^2 + 64xy)xu_-^2 + (-4t^2 + 4ct - 64xy - c^2)cu_- + 16c^2y = 0,$$

and the associated solution (55) of the dT equation equation is then given by

$$\Phi = \log\Big(\frac{8t-2c}{2t-c+xu_-}\Big). \tag{56}$$

4.2 Generalized gas equation

We consider now solutions of the generalized gas equation

$$\begin{aligned} a_t + (av)_x &= 0,\\ v_t + vv_x + w_x &= 0,\\ w_y + a_x &= 0. \end{aligned} \tag{57}$$

We set all time variables t_A equal to zero except for $t_{(2,1)}$ and $t_{(1,2)}$ and use the notation conventions (45). Then, from (12) it follows that the dependent variables are given by

$$a = \nu(\boldsymbol{t}),\ v = q(\boldsymbol{t}),\ w = a_{1,1}(\boldsymbol{t}).$$

$N=1$ reductions Reductions $z = z(p,u)$ depending on a single variable u, defined by $u = -a_{1,-1}$, lead to the Abel's equation (47) and to a hodograph relation (30) of the form

$$-t\,p_1 - \frac{y\,\nu(u)}{(p_1(u) - q(u))^2} + x + F(u) = 0, \tag{58}$$

where $q(u)$, $p_1(u)$ and $F(u)$ are arbitrary functions of u. We may rewrite (58) as

$$t\Big(\frac{1}{P(u)} - \int^u P(u)\,\mathrm{d}\,u\Big) - yP^2\exp\Big(-\int^u P^2\,\mathrm{d}\,u\Big) + x + F(u) = 0, \tag{59}$$

where $P := \partial_u q(u)$ is an arbitrary function of u and

$$a = \exp\Big(-\int^u P^2\,\mathrm{d}\,u\Big),\quad v = \int^u P\,\mathrm{d}\,u,\quad w = -u. \tag{60}$$

An equivalent form of (57) is

$$\begin{aligned} a_t + (av)_x &= 0, \\ (v_t + vv_x)_y - a_{xx} &= 0. \end{aligned} \tag{61}$$

To prove this fact just consider a solution (a, v) of (61), then integrating the second equation with respect to the y variable we conclude the existence of a function $f(x, t)$ such that

$$v_t + vv_x - \int_{y_0}^{y} a_{xx} \,\mathrm{d}\, y + f_x(x, t) = 0.$$

Then, a solution of (57) is given by (a, v, w) with

$$w(x, y, t) := f(x, t) - \int_{y_0}^{y} a_x(x, y, t) \,\mathrm{d}\, y.$$

Now we will show two reformulations of the previous $N = 1$ technique providing us with explicit solutions to (61).

1. If we parametrize in terms of $a(u) = \exp(-\int^u P^2(u) \,\mathrm{d}\, u)$ assuming that a is a solution of the following ODE

$$\frac{\mathrm{d}\, a}{\mathrm{d}\, u} = -\frac{1}{af'(a)^2}$$

for a given function $f = f(a)$ $(f'(a) = \dfrac{\mathrm{d}\, f}{\mathrm{d}\, a})$ from $\log a = -\int^u P^2(u) \,\mathrm{d}\, u$ we have $1/P := -af'(a)$ and we get the following hodograph relation

$$(af'(a) + f(a))af'(a)^2 t + y - (x + F(a))af'(a)^2 = 0. \tag{62}$$

Then, given two arbitrary functions f and F, and $a(x, y, t)$ a solution with of (62) then $a, v = f(a)$ is a solution of (61)

For example, if $f = Aa + B$ and $F = -(Ca^3 + Da^2 + Ea + G)$, with A, B, C, D, E and G arbitrary constants we get the hodograph relation

$$A^2Ca^4 + A^2Da^3 + A^2(2At + E)a^2 + A^2(Bt - Ax + AG)a + y = 0$$

and the solution of (61) is a, v with vv given by

$$v = Aa + B.$$

If we take $f = a + 1$ and $F = a^3$ we get the following solution

$$a = \alpha - \frac{3(t-x) - 4t^2}{9\alpha} - \frac{2t}{3},$$
$$v = a + 1$$

with

$$\alpha := \Big(12t(t-x) - 18y - \frac{32}{3}t^3 + 2\big(12t^3 - 36xt^2 - 12t^4 + 36tx^2 + 24xt^3 - 12x^3 - 12x^2t^2 - 108t^2y + 108txy + 81y^2 + 96yt^3\big)^{1/2}\Big)^{1/3}.$$

Another simple example appears when we take $f(a) = \log a$, the solution to the hodograph equation (for $F = 0$) is

$$a(x,y,t) = \frac{t}{y} W\Big(\frac{y}{t}\exp(x/t - 1)\Big)$$

and

$$v(x,y,t) = W\Big(\frac{y}{t}\exp(x/t - 1)\Big) - 1 + \frac{x}{t}$$

where W is the Lambert function defined by

$$W(z)\exp(W(z)) = z.$$

2. Alternatively, we can parametrize in terms of $v = \int^u P \,\mathrm{d}\,u$ where v is subject to the ODE $\dfrac{\mathrm{d}\,v}{\mathrm{d}\,u} = -g'(v)/g(v)$. Then, we get from (59) the following hodograph relation

$$-t(g(v) + vg'(v))g(v) - yg'(v)^3 + g(v)g'(v) + F(v) + g(v)g'(v) = 0.$$

This equation is gotten by takin into account that $g(v) = f^{-1}(v)$, is the inverse function of f. Thus, given two arbitrary functions g, F and a solution $v(x,y,t)$ to this hodograph relation we get a solution a, v with a given by

$$a = g(v),$$

of (61). In particular if $g := A\mu^2 + B\mu + C$ and $F = D\mu + E$ we get the following hodograph relation

$$\begin{aligned}&-3A^2t\mu^4 + (2A^2x - 8A^3y - 5ABt + AD)\mu^3\\ &+ (3ABx - 12A^2By - 2(2AC + B^2)t + AE + BD)\mu^2\\ &+ ((2AC + B^2)x - 6AB^2y - 3BCt + BE + CD)\mu\\ &+ BCx - B^3y - C^2t + EC = 0\end{aligned}$$

$N \geq 2$ diagonal reductions Reductions $z = z(p, \boldsymbol{u})$ involving $N > 1$ variables $\boldsymbol{u} := (u_1, \ldots, u_N)$ can be analyzed by the same scheme as in the case of the dT equation. They are associated with a system of equations (16) (or (15)), where the functions $r_{ij}(\boldsymbol{u})$, $p_i(\boldsymbol{u})$ are assumed to verify the compatibility conditions (29). The functions $q(\boldsymbol{u})$ and $\nu(\boldsymbol{u})$ are determined by solving the system (50). Thus, given a set of functions $\{F_i(\boldsymbol{u})\}_{i=1}^N$ satisfying (29), the hodograph relations (30) read

$$-t\, p_i(\boldsymbol{u}) - \frac{y\, \exp\Big(\int^{\boldsymbol{u}} \sum_{j=1}^N \frac{\partial R_j}{\partial p}(q(\boldsymbol{u}), \boldsymbol{u})\, \mathrm{d}\, u_j\Big)}{(p_i(\boldsymbol{u}) - q(\boldsymbol{u}))^2} + x + F_i = 0, \tag{63}$$

where $1 \leq i \leq N$. The dependent variables of the Generalized Benney system are then given by

$$\begin{gathered} a = \exp\Big(\int^{\boldsymbol{u}} \sum_{j=1}^N \frac{\partial R_j}{\partial p}(q(\boldsymbol{u}), \boldsymbol{u})\, \mathrm{d}\, u_j\Big), \quad v = q(\boldsymbol{u}), \\ w = \int^{\boldsymbol{u}} \sum_{i,j=1}^N \mathrm{Res}(R_i(p, \boldsymbol{u}), p_j(\boldsymbol{u}))\, \mathrm{d}\, u_j. \end{gathered} \tag{64}$$

In the particular case of the $N = 2$ reduction of diagonal type defined by

$$\begin{gathered} r_1 = -r_2 = \frac{1}{8}(u_1 - u_2), \\ p_1 = \frac{1}{4}(3u_1 + u_2),\ p_2 = \frac{1}{4}(u_1 + 3u_2), \end{gathered} \tag{65}$$

the function $q(\boldsymbol{u})$ is given by (53), so that by denoting

$$U_i := \sqrt{u_i + z_0}, \quad i = 1, 2,$$

we have

$$\begin{gathered} q(\boldsymbol{u}) = -\frac{1}{2}U_1 U_2 + \frac{1}{4}(U_1^2 + U_2^2 - 4z_0), \\ p_1(\boldsymbol{u}) = \frac{1}{4}(3U_1^2 + U_2^2 - 4z_0), p_2(\boldsymbol{u}) = \frac{1}{4}(U_1^2 + 3U_2^2 - 4z_0). \end{gathered} \tag{66}$$

The hodograph relations (63) reduce to

$$-\frac{4y}{U_1^3U_2}-\frac{1}{4}(3U_1^2+U_2^2-4z_0)t+x+F_1=0,$$
$$-\frac{4y}{U_2^3U_1}-\frac{1}{4}(3U_2^2+U_1^2-4z_0)t+x+F_2=0, \tag{67}$$

and (64) implies

$$a=\frac{(U_1+U_2)^2}{U_1U_2},\ v=\frac{1}{4}(U_1-U_2)^2-z_0,\ w=\frac{1}{16}(U_1^2-U_2^2)^2. \tag{68}$$

In particular, for $F_1=F_2=0$ one finds the following explicit solution

$$\begin{aligned} a &= \frac{2}{3}\frac{x+z_0t}{(t^2y)^{1/3}}+2\\ v &= \frac{x}{3t}-\left(\frac{y}{t}\right)^{1/3}-\frac{2}{3}z_0,\\ w &= \frac{(x+z_0t)^2}{9t^2}-\left(\frac{y}{t}\right)^{2/3}. \end{aligned} \tag{69}$$

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