

Four-dimensional naturally reductive pseudo-Riemannian spaces

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Abstract

The classification of 4-dimensional naturally reductive pseudo-Riemannian spaces is given. This classification comprises symmetric spaces, the product of 3-dimensional naturally reductive spaces with the real line and new families of indecomposable manifolds which are studied at the end of the article. The oscillator group is also analyzed from the point of view of this classification.

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1 Introduction

Homogeneous manifolds play a preeminent role in Differential Geometry and have deserved thorough studies and classifications from different perspectives. Among these spaces, naturally reductive manifolds are possibly the simplest class besides the class of Lie groups or symmetric spaces. This is probably due to the fact that they generalize these spaces in a simple way. Classifications of low dimensional naturally reductive Riemannian homogeneous manifolds can be found in classical references. Beyond the trivial result in surfaces, all connected and simply connected 3-dimensional naturally homogeneous spaces are given in [17]: they comprise symmetric spaces together with the Lie groups $SU(2)$, $SL(2, \mathbb{R})$ and the Heisenberg group, endowed with convenient left invariant metrics. The four dimensional case is tackled in [11] where it is proved that under the same topological conditions, a naturally reductive Riemannian 4-manifold necessarily splits as a product of a 3-dimensional naturally reductive manifold and \mathbb{R} . We have to wait for the 5-dimensional case to get new indecomposable naturally reductive manifolds (see [12]).

The study of naturally reductive pseudo-Riemannian spaces also deserves special attention. The classification in the 3-dimensional setting has been recently obtained in [4], [9] where, again, the manifold is either symmetric, $SU(2)$,

$\widetilde{\text{SL}}(2, \mathbb{R})$ or the Heisenberg group with convenient metrics. The four dimensional case has attired much interest in the literature (see for example [2], [15] where the structure of naturally reductive groups are analyzed) probably because of the possible connections of these spaces with plausible relativistic models. The goal of this paper is to provide the complete classification of 4-dimensional naturally reductive pseudo-Riemannian manifolds of $(1, 3)$ or $(2, 2)$ signatures. Surprisingly, the main results (see Theorem 9 and 10) show that, besides the product of a 3-dimensional naturally reductive manifold and \mathbb{R} , there is a family of indecomposable manifolds. This situation has no counterpart in the Riemannian case.

The structure of the article is as follows. We first review the basic concepts and properties of naturally reductive manifolds, specially those connected with the notion of homogeneous structure tensors. We then follow the technique of Kowalski and Vanhecke, although we cannot simply generalize [11] due to the existence of the new families mentioned above. At the end of the article, we explore the geometry of these new manifolds to be sure that they are indecomposable and non-symmetric. Finally, we apply Theorem 9 to the analysis of the 4-dimensional oscillator group, probably the most relevant naturally reductive Lorentzian example in the literature. We give a decomposition of this space which is different to the one of its traditional definition.

2 Preliminaries

2.1 Naturally reductive spaces

Let (M, g) be a reductive pseudo-Riemannian homogeneous manifold of dimension n . This means that $M = G/H$, where G is connected Lie group of isometries acting transitively and effectively on M , H is the isotropy of a point $o \in M$, and the Lie algebra \mathfrak{g} of G admits a decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ such that $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$, where \mathfrak{h} is the Lie algebra of H . The mapping $A \mapsto A_o^* = d/d\varepsilon|_{\varepsilon=0} \exp(\varepsilon A) \cdot o$ defines an isomorphism between \mathfrak{m} and T_oM which, in addition, is used to transfer the metric g to \mathfrak{m} . For convenience, along the article we will denote both the metric in T_oM and in \mathfrak{m} by $\langle \cdot, \cdot \rangle$. The decomposition of \mathfrak{g} is said to be naturally reductive if in addition

$$(1) \quad \langle [X, Y]_{\mathfrak{m}}, Z \rangle + \langle [X, Z]_{\mathfrak{m}}, Y \rangle = 0 \text{ for } X, Y, Z \in \mathfrak{m},$$

where $[\cdot, \cdot]_{\mathfrak{m}}$ is the \mathfrak{m} -part of the bracket (see, *e.g.*, [10, Chapter X, section 3], [14, Chapter 11, Definition 23]). Let $\tilde{\nabla}$ be the canonical connection of the reductive homogenous space $M = G/H$. It is well known that the torsion tensor \tilde{T} and the curvature tensor \tilde{R} of $\tilde{\nabla}$ at the point o read

$$(2) \quad \tilde{T}(X, Y)_o = -[X, Y]_{\mathfrak{m}}, \quad \tilde{R}(X, Y)_o = -[X, Y]_{\mathfrak{h}}, \quad \forall X, Y \in \mathfrak{m}.$$

Recalling that G -invariant tensor fields on M are parallel with respect to the connection $\tilde{\nabla}$, we have

$$(3) \quad \tilde{\nabla}g = \tilde{\nabla}\tilde{R} = \tilde{\nabla}\tilde{T} = 0.$$

Conditions (2) and (3) provide interesting properties. First, the subalgebra $\mathfrak{k} \subset \mathfrak{h}$ generated by all projections $[X, Y]_{\mathfrak{h}} = -\tilde{R}(X, Y)$, $X, Y \in \mathfrak{m}$, belongs to the holonomy algebra and hence its elements $A \in \mathfrak{k}$ act as derivation on the tensor algebra of \mathfrak{m} and

$$A \cdot g = A \cdot \tilde{R} = A \cdot \tilde{T} = 0.$$

Second, the Bianchi identities (see, [10, Chapter III, Theorem 5.3]) become

$$(4) \quad \mathfrak{S}_{X,Y,Z} \tilde{R}(X, Y)Z = \mathfrak{S}_{X,Y,Z} \tilde{T}(\tilde{T}(X, Y), Z),$$

$$(5) \quad \mathfrak{S}_{X,Y,Z} \tilde{R}(\tilde{T}(X, Y), Z) = 0,$$

for all $X, Y, Z \in \mathfrak{m}$, where $\mathfrak{S}_{X,Y,Z}$ denotes the cyclic sum with respect to X, Y, Z .

With both tensors \tilde{T} and \tilde{R} we can recover two important objects. On one hand, the Riemann curvature tensor R defined by the Levi-Civita connection ∇ at T_oM satisfies the formula

$$(6) \quad R(X, Y) = \tilde{R}(X, Y) + [D_X, D_Y] + D_{\tilde{T}(X,Y)},$$

where D_X denotes the difference (1, 1)-tensor $D_X = \nabla_X - \tilde{\nabla}_X$, which from (1) and (2) is

$$(7) \quad D_X Y = -\frac{1}{2} \tilde{T}(X, Y).$$

On the other hand (see [17, Chapter 1, (1.79)]), the brackets of the Lie algebra $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ are defined as

$$(8) \quad \begin{cases} [U, V] = UV - VU, & U, V \in \mathfrak{h}, \\ [U, X] = U(X), & U \in \mathfrak{h}, X \in \mathfrak{m}, \\ [X, Y] = -\tilde{T}(X, Y) + \tilde{R}(X, Y), & X, Y \in \mathfrak{m}. \end{cases}$$

Remark 1 *A homogeneous structure tensor in a pseudo-Riemannian manifold (M, g) is a (1, 1)-tensor D satisfying (3) for the connection $\tilde{\nabla} = \nabla - D$ where ∇ is the Levi-Civita connection. By a classical result of Ambrose and Singer (see [1]) a connected, simply connected and complete manifold is reductive homogeneous if and only if it has a homogenous structure tensor. The set of these tensors are classified in three primitive classes invariant under the action of the orthogonal group of the appropriate signature (see [7]). Tensors D belonging to the class $\mathcal{T}_2 \oplus \mathcal{T}_3$ are those satisfying the property $D_X Y + D_Y X = 0$ and characterize natural reductivity. The tensor in (7) it is obviously in $\mathcal{T}_2 \oplus \mathcal{T}_3$.*

Proposition 2 *If a pseudo-Riemannian manifold (M, g) admits the null tensor as a homogeneous structure tensor, then it is locally symmetric.*

If a naturally reductive homogeneous manifold (M, g) has null intrinsic curvature \tilde{R} , then it is locally symmetric.

Proof. If the homogeneous structure tensor D vanishes, then $\nabla R = 0$ and the manifold is locally symmetric. Similarly, if $\tilde{R} = 0$, then H is discrete. The universal covering of M is a Lie group with an invariant metric satisfying $g([X, Y], Z) + g([X, Z], Y) = 0$, which is necessarily symmetric. ■

2.2 Decomposition of manifolds

We now recall the following classical results.

Theorem 3 (de Rham-Wu decomposition) *Let (M, g) be a simply connected and complete pseudo-Riemannian manifold. Then (M, g) can be decomposed as a pseudo-Riemannian product*

$$(M, g) \simeq (M_1, g_1) \times \cdots \times (M_k, g_k)$$

where for each (M_i, g_i) and any $x_i \in M_i$, the tangent space $T_{x_i}M_i$ does not admit a proper non-degenerate subspace, invariant with respect to the holonomy. The decomposition above is unique up to order of the factors. Moreover, the connected components of the identity of the isometry groups satisfy

$$\mathcal{I}^0(M, g) = \mathcal{I}^0(M_1, g_1) \times \cdots \times \mathcal{I}^0(M_k, g_k).$$

The proof of this result can be found, for example, in [18]. As a consequence of this result we have:

Proposition 4 *Let (M, g) be a pseudo-Riemannian manifold and*

$$(M, g) \simeq (M_1, g_1) \times \cdots \times (M_k, g_k)$$

its de Rham-Wu decomposition. Then (M, g) is a naturally reductive homogeneous space if and only if each (M_i, g_i) is a naturally reductive homogeneous space.

Proposition 5 *Let (M, g) be a connected and simply connected naturally reductive homogeneous pseudo-Riemannian manifold and let $o \in M$. Suppose that $T_oM = W \oplus W^\perp$ and that*

$$(9) \quad \tilde{T}(\pi_i X, \pi_i Y) = \pi_i \tilde{T}(X, Y),$$

$$(10) \quad \tilde{R}(\pi_i X, \pi_i Y)\pi_i Z = \pi_i \tilde{R}(X, Y)Z,$$

for all $X, Y, Z \in T_oM$, where $\pi_1: T_oM \rightarrow W$ and $\pi_2: T_oM \rightarrow W^\perp$ are the natural projections. Then M is the pseudo-Riemannian product of two naturally reductive homogeneous pseudo-Riemannian manifolds.

This result is proved for the Riemannian case in [12]. The proof for the case of arbitrary signature is similar, with the only difference that one has to ensure the non degeneracy of the restriction of the metric g to $W \subset T_oM$, in order to apply the de Rham-Wu Theorem. This condition is satisfied as $T_oM = W \oplus W^\perp$. We also have the following result:

Proposition 6 *If $W = \text{span}\{\tilde{T}(X, Y) \mid X, Y \in T_oM\}$ is a proper non-degenerate space then the conditions of Proposition 5 are satisfied and the manifold M is decomposable.*

Proof. From (1) we have $\tilde{T}(X, Y) = 0$ if $Y \in W^\perp$ and therefore (9) is satisfied.

We now check that condition (10) is also satisfied. If $X = \tilde{T}(U, V) \in W$, $Z \in W^\perp$, using (5) with U, V, Z gives $\tilde{R}(X, Z) = 0$. Now, if $X, Y \in W$ and $Z \in W^\perp$ from (4) we get $\tilde{R}(X, Y)Z = 0$ analogously $\tilde{R}(X, Y)Z = 0$ for $X, Y \in W^\perp$, $Z \in W$. Finally, taking (6) into account we get $\langle R(X, Y)Z, U \rangle = \langle \tilde{R}(X, Y)Z, U \rangle$ and $\langle R(X, Y)U, Z \rangle = \langle \tilde{R}(X, Y)U, Z \rangle$ for any $U \in W^\perp$ and from the symmetries of the Riemann curvature tensor we have $\langle \tilde{R}(X, Y)Z, U \rangle = 0$ for any $X, Y, Z \in W$ and hence $\tilde{R}(X, Y)Z \in W$.

If $X, Y, Z \in W^\perp$, $U \in W$, from (6) we have $\langle R(X, Y)Z, U \rangle = \langle \tilde{R}(X, Y)Z, U \rangle$, and $\langle R(Z, U)X, Y \rangle = \langle \tilde{R}(Z, U)X, Y \rangle$. From the symmetries of the Riemann curvature tensor we have $\langle \tilde{R}(X, Y)Z, U \rangle = \langle \tilde{R}(Z, U)X, Y \rangle$ but the right hand side vanishes. Hence $\langle \tilde{R}(X, Y)Z, U \rangle = 0$, $\forall U \in W$ and then $\tilde{R}(X, Y)Z \in W^\perp$. ■

2.3 Normal forms of skew-adjoint operators

Let $(V, \langle \cdot, \cdot \rangle)$ be a metric vector space and let $A: V \rightarrow V$ be a skew-symmetric linear endomorphism, that is an endomorphism satisfying

$$\langle A(u), v \rangle = -\langle u, A(v) \rangle, \quad \forall u, v \in V.$$

If $A(W) \subset W$ for a subspace $W \subset V$ for which the restriction of the metric is non-degenerate, then $A(W^\perp) \subset W^\perp$ and we can decompose $V = W \oplus W^\perp$. In this case, the endomorphism A is said to be reducible. If there is no such an invariant non-degenerate subspace W , we say that A is irreducible.

Proposition 7 *Let $(V, \langle \cdot, \cdot \rangle)$ be a 4-dimensional Lorentzian vector space. For any skew-symmetric endomorphism $A: V \rightarrow V$, there exists an orthonormal basis \mathcal{B} of V with respect to which the matrix of $\langle \cdot, \cdot \rangle$ is $\text{diag}(-1, 1, 1, 1)$ and the matrix of A is one of the following types:*

$$\text{a) } A_1 = \pm \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

b) $A_{\alpha\beta} = \alpha A_2 + \beta A_3$, with $\alpha, \beta \in \mathbb{R}$ and

$$A_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Proof. From [16], every skew-symmetric transformation in a 4-dimensional manifold is reducible. We then have $V = W \oplus W^\perp$ where W is Lorentzian and W^\perp Riemannian. We consider that $A|_W$ is irreducible.

If $\dim W = 1$, then $A|_W = (0)$ and there is an orthonormal basis in W^\perp such that $A|_{W^\perp}$ is a Riemannian skew-symmetric endomorphism. We then get that A is as in the case b) with $\alpha = 0$.

If $\dim W = 2$, then (see [16]) there are orthonormal basis in W and W^\perp for which the matrices of $A|_W$ and $A|_{W^\perp}$ are respectively

$$\begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix}, \alpha \neq 0, \quad \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}, \beta \in \mathbb{R},$$

and we recover the matrices in the case b) with $\alpha \neq 0$.

If $\dim W = 3$, then (see [16]) there is a basis such that $A|_W$ defines a matrix as the top left 3×3 submatrix in a), so that the proof is complete. ■

Proposition 8 *Let $(V, \langle \cdot, \cdot \rangle)$ be a 4-dimensional vector space with a $(2, 2)$ -signature metric. Let $A: V \rightarrow V$ be a skew-symmetric endomorphism. Then we have: If A is reducible, then there is an orthonormal basis \mathcal{B} of V with respect to which the matrix of $\langle \cdot, \cdot \rangle$ is $\text{diag}(-1, -1, 1, 1)$ and the matrix of A is one of the following*

$$\begin{aligned} \text{a1) } A_1 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{pmatrix}, & \text{a2) } A_2 &= \begin{pmatrix} 0 & \alpha & 0 & 0 \\ -\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta \\ 0 & 0 & -\beta & 0 \end{pmatrix}, \beta \neq 0, \\ \text{a3) } A_3 &= \begin{pmatrix} 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \alpha \\ \beta & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \end{pmatrix}, \alpha \neq 0. \end{aligned}$$

If A is irreducible, then there is a basis \mathcal{B} of V with respect to which the matrix of $\langle \cdot, \cdot \rangle$ is

$$\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

and the matrix of A is one of the following

$$\begin{aligned} \text{b1) } B_1 &= \begin{pmatrix} 0 & -\nu & 1 & 0 \\ \nu & 0 & 0 & 1 \\ 0 & 0 & 0 & -\nu \\ 0 & 0 & \nu & 0 \end{pmatrix}, & \text{b2) } B_2 &= \begin{pmatrix} \lambda & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 1 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & -\lambda \end{pmatrix}, \lambda \neq 0, \\ \text{b3) } B_3 &= \begin{pmatrix} \xi & \nu & 0 & 0 \\ -\nu & \xi & 0 & 0 \\ 0 & 0 & -\xi & \nu \\ 0 & 0 & -\nu & -\xi \end{pmatrix}, \xi, \nu \neq 0. \end{aligned}$$

Proof. The possible cases are obtained from the classification of irreducible skew-symmetric endomorphisms in spaces with signature $(2, n - 2)$ given in [3] or [13, Theorem 4.1]. ■

3 Classification theorem

Theorem 9 *Let (M, g) be a simply connected naturally reductive Lorentzian manifold of dimension 4. Then M is either symmetric, decomposable or isometric to G/H with*

1. $G = \widetilde{SL(2, \mathbb{R})} \times \mathbb{R}^2$ and H a 1-dimensional subgroup. If the Lie algebra is spanned as $\mathfrak{g} = \text{span}\{Y_1, Y_2, Y_3, T_1, T_2\}$, with non-null brackets $[Y_1, Y_2] = -\lambda Y_3$, $[Y_1, Y_3] = \lambda Y_2$, $[Y_2, Y_3] = Y_1$, then $\mathfrak{h} = \text{span}\{A\}$ and $\mathfrak{m} = \text{span}\{X_1, X_2, X_3, X_4\}$ with

$$\begin{aligned} X_1 &= \frac{1}{\lambda}Y_1 + \left(1 - \frac{1}{\lambda}\right)T_1 + \frac{1}{\lambda}T_2, & X_2 &= \frac{1}{\lambda}Y_1 - \frac{1}{\lambda}T_1 + \left(1 + \frac{1}{\lambda}\right)T_2, \\ X_3 &= Y_2, & X_4 &= Y_3, & A &= \frac{1}{\lambda}Y_1 - \frac{1}{\lambda}T_1 + \frac{1}{\lambda}T_2. \end{aligned}$$

The metric g in M is induced by the metric $\text{diag}(-1, 1, 1, 1)$ in \mathfrak{m} by G invariance.

2. G belonging to the family of simply connected Lie groups with Lie algebra $\mathfrak{g} = \text{span}\{X_1, X_2, X_3, X_4, A, B\}$ and structure constants

$$\begin{aligned} [A, X_1] &= -[A, X_2] = X_3, \\ [B, X_1] &= -[B, X_2] = X_4, \\ [A, X_3] &= [B, X_4] = X_1 + X_2, \\ [X_1, X_3] &= -[X_2, X_3] = -cX_4 + \alpha A + \beta B, \\ [X_1, X_4] &= -[X_2, X_4] = cX_3 + \beta A + \delta B, \\ [X_3, X_4] &= c(X_1 + X_2), \end{aligned}$$

with $c, \alpha, \beta, \delta \in \mathbb{R}$. The Lie subalgebra of H is $\mathfrak{h} = \text{span}\{A, B\}$. The metric g in M is induced by the metric $\text{diag}(-1, 1, 1, 1)$ in the complement $\mathfrak{m} = \text{span}\{X_1, X_2, X_3, X_4\}$ by G invariance.

Theorem 10 *Let (M, g) be a simply connected naturally reductive $(2, 2)$ -signature pseudo-Riemannian manifold of dimension 4. Then M is either symmetric, decomposable or isometric to G/H with*

1. $G = \widetilde{SL(2, \mathbb{R})} \times \mathbb{R}^2$ and H a 1-dimensional subgroup. If the Lie algebra is spanned as $\mathfrak{g} = \text{span}\{Y_1, Y_2, Y_3, T_1, T_2\}$, with non-null brackets $[Y_1, Y_2] = \lambda Y_3$, $[Y_1, Y_3] = \lambda Y_2$, $[Y_2, Y_3] = Y_1$, then $\mathfrak{h} = \text{span}\{A\}$ and $\mathfrak{m} = \text{span}\{X_1, X_2, X_3, X_4\}$ with

$$\begin{aligned} X_1 &= -\frac{1}{\lambda}Y_1 + \left(1 + \frac{1}{\lambda}\right)T_1 + \frac{1}{\lambda}T_2, & X_2 &= Y_2, & X_4 &= Y_3, \\ X_3 &= \frac{1}{\lambda}Y_1 - \frac{1}{\lambda}T_1 + \left(1 - \frac{1}{\lambda}\right)T_2, & A &= \frac{1}{\lambda}Y_1 - \frac{1}{\lambda}T_1 - \frac{1}{\lambda}T_2. \end{aligned}$$

The metric g in M is induced by the metric $\text{diag}(-1, -1, 1, 1)$ in \mathfrak{m} by G invariance.

2. G belonging to the family of simply connected Lie groups with Lie algebra $\mathfrak{g} = \text{span}\{X_1, X_2, X_3, X_4, A, B\}$ and structure constants

$$\begin{aligned} [A, X_2] &= -[A, X_3] = X_4, \\ [B, X_1] &= [A, X_4] = X_2 + X_3, \\ [B, X_2] &= -[B, X_3] = -X_1, \\ [X_1, X_2] &= -[X_1, X_3] = -cX_4 - \beta A - \delta B, \\ [X_2, X_4] &= -[X_3, X_4] = cX_1 - \alpha A + \beta B, \\ [X_1, X_4] &= -c(X_2 + X_3), \end{aligned}$$

with $c, \alpha, \beta, \delta \in \mathbb{R}$. The Lie subalgebra of H is $\mathfrak{h} = \text{span}\{A, B\}$. The metric g in M is induced by the metric $\text{diag}(-1, -1, 1, 1)$ in the complement $\mathfrak{m} = \text{span}\{X_1, X_2, X_3, X_4\}$ by G invariance.

Remark 11 The families of algebras in Theorem 9-2 and Theorem 10-2 include some particular cases where $M = G/H$ is symmetric. That happens when $c = 0$ and in the cases studied in Proposition 12 and 14. In addition, when $\beta = \delta = 0$, the quotient $M = G/H$ is the same as G'/H' , where the Lie algebras are $\mathfrak{g}' = \text{span}\{X_1, X_2, X_3, X_4, A\}$ and $\mathfrak{h}' = \text{span}\{A\}$. In any case, in general, one can check that the structure constants of these families define a solvable 6-dimensional Lie algebra with 5-dimensional non-Abelian nilradical. It thus belongs to the list of all possible algebras with these properties appearing in [5, Table 13], [6, Table 3]. Some computations show that the dependence of \mathfrak{g} on the parameters gives different cases of the aforementioned list.

4 Proof of Theorem 9

Let (X_1, X_2, X_3, X_4) be an orthonormal basis in T_oM such that $\langle X_i, X_j \rangle = \varepsilon_i \delta_{ij}$, with $\varepsilon_1 = -1$, $\varepsilon_i = 1$ for $i = 2, 3, 4$. We write:

$$(11) \quad \tilde{T}(X_i, X_j) = \tilde{T}_{ij}^k X_k, \quad i, j, k = 1, \dots, 4.$$

From (1), we have: $\varepsilon_k \tilde{T}_{ij}^k + \varepsilon_j \tilde{T}_{ik}^j = 0$, $i, j, k = 1, \dots, 4$. Hence $\tilde{T}_{ij}^i = 0$ for $1 \leq i \leq j \leq 4$, and by denoting $\tilde{T}_{12}^3 = a$, $\tilde{T}_{12}^4 = b$, $\tilde{T}_{13}^4 = c$, $\tilde{T}_{23}^4 = d$, we have:

$$(12) \quad \begin{cases} \tilde{T}(X_1, X_2) = aX_3 + bX_4, \\ \tilde{T}(X_1, X_3) = -aX_2 + cX_4, \\ \tilde{T}(X_1, X_4) = -bX_2 - cX_3, \\ \tilde{T}(X_2, X_3) = -aX_1 + dX_4, \\ \tilde{T}(X_2, X_4) = -bX_1 - dX_3, \\ \tilde{T}(X_3, X_4) = -cX_1 + dX_2. \end{cases}$$

We consider the skew-symmetric operator $A = \tilde{R}(X, Y)$ for a choice of $X, Y \in T_oM$. If $A = 0$ for all choices of X, Y , then $\mathfrak{h} = \{0\}$ and M is symmetric (see Proposition 2). We thus assume $\tilde{T} \neq 0$ and there exist $X, Y \in T_oM$ such that $A = \tilde{R}(X, Y) \neq 0$. We use the classification of Proposition 7.

4.1 Case a)

Suppose that a curvature transformation $A = \tilde{R}(X, Y)$ exists such that

$$(13) \quad AX_1 = X_3, \quad AX_2 = -X_3, \quad AX_3 = X_1 + X_2, \quad AX_4 = 0,$$

as in Proposition 7-(a) (for the opposite sign, just consider $\tilde{R}(Y, X)$). By applying $A \cdot \tilde{T} = 0$ to X_i, X_j for $1 \leq i < j \leq 4$, and taking (12) and (13) into account we easily get: $b = 0$ and $c + d = 0$.

In the case, $a \neq 0, c = 0$ (resp. $a \cdot c \neq 0$) if we take $W = \text{span}\{X_1, X_2, X_3\}$ (resp. $W = \text{span}\{X_3, -aX_2 + cX_4, X_1 + X_2\}$) we conclude that M is decomposable by virtue of Proposition 6. We thus consider $a = 0, c \neq 0$. From the Bianchi identities (4), (5) and imposing $A \cdot \tilde{R} = 0$, we obtain:

$$\begin{cases} \tilde{R}(X_1, X_2) = \tilde{R}(X_3, X_4) = 0, \\ \tilde{R}(X_1, X_3) = \tilde{R}_{133}^2 A + \tilde{R}_{143}^2 B, \\ \tilde{R}(X_1, X_4) = \tilde{R}_{143}^2 A + \tilde{R}_{144}^2 B, \\ \tilde{R}(X_2, X_j) = -\tilde{R}(X_1, X_j), \quad j = 3, 4, \end{cases}$$

where

$$B = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}.$$

We have the following possibilities:

- If $(\tilde{R}_{143}^2)^2 \neq \tilde{R}_{133}^2 \tilde{R}_{144}^2$ then $\mathfrak{h} = \text{span}\{\tilde{R}(X, Y) | X, Y \in \mathfrak{m}\} = \text{span}\{A, B\}$. By using (8) we write down the non-vanishing brackets for the Lie algebra $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$:

$$\begin{aligned} [A, X_1] &= -[A, X_2] = X_3, \\ [B, X_1] &= -[B, X_2] = X_4, \\ [A, X_3] &= [B, X_4] = X_1 + X_2, \\ [X_1, X_3] &= -[X_2, X_3] = -cX_4 + \tilde{R}_{133}^2 A + \tilde{R}_{143}^2 B, \\ [X_1, X_4] &= -[X_2, X_4] = cX_3 + \tilde{R}_{143}^2 A + \tilde{R}_{144}^2 B, \\ [X_3, X_4] &= c(X_1 + X_2). \end{aligned}$$

- If $(\tilde{R}_{143}^2)^2 = \tilde{R}_{133}^2 \tilde{R}_{144}^2$ then the dimension of $\mathfrak{h} = \text{span}\{\tilde{R}(X, Y) | X, Y \in \mathfrak{m}\}$ is one. As we are supposing that there exists X, Y such that $\tilde{R}(X, Y) = A$, we have: $\tilde{R}_{144}^2 = \tilde{R}_{143}^2 = 0$. By using (8) we write down the non-vanishing brackets for the Lie algebra $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$:

$$\begin{aligned} [A, X_1] &= -[A, X_2] = X_3, \\ [A, X_3] &= X_1 + X_2, \\ [X_1, X_3] &= -[X_2, X_3] = -cX_4 + \tilde{R}_{133}^2 A, \\ [X_1, X_4] &= -[X_2, X_4] = cX_3, \\ [X_3, X_4] &= c(X_1 + X_2). \end{aligned}$$

This corresponds to the case $\beta = \delta = 0$ of the family in Theorem 9-2.

4.2 Case b)

Suppose that a curvature transformation $A = \tilde{R}(X, Y)$ exists such that

$$(14) \quad A = \alpha A_2 + \beta A_3,$$

as in Proposition 7. If $\alpha\beta \neq 0$, by applying $A \cdot \tilde{T} = 0$ to X_i, X_j for $1 \leq i < j \leq 4$, and taking (12) and (14) into account we easily obtain $\tilde{T} = 0$ and hence M is symmetric. We now assume that $\alpha\beta = 0$ and $A \neq 0$.

In the case $\beta = 0, \alpha \neq 0$, by applying $A_2 \cdot \tilde{T} = 0$ to X_i, X_j for $1 \leq i < j \leq 4$, and taking (12) into account we have $c = d = 0$. Then, if $a \neq 0$ (resp. $a = 0, b \neq 0$) we take $W = \text{span}\{X_1, X_2, X_3 + \frac{b}{a}X_4\}$ (resp. $W = \text{span}\{X_1, X_2, X_4\}$) and we conclude that M is decomposable by virtue of Proposition 6.

We now assume that $\alpha = 0, \beta \neq 0$. By applying $A_3 \cdot \tilde{T} = 0$ to X_i, X_j for $1 \leq i < j \leq 4$, and taking (12) into account we get $a = b = 0$. If $c^2 - d^2 \neq 0$ we take $W = \text{span}\{X_3, X_4, -cX_1 + dX_2\}$ and we conclude that M is decomposable by virtue of Proposition 6.

We thus consider $d = \eta c \neq 0$ with $\eta = \pm 1$. In this case, from straightforward—but rather long—computations, we get

$$\begin{cases} \tilde{R}(X_1, X_3) = \tilde{R}_{133}^1 M + \tilde{R}_{143}^1 N + \tilde{R}_{134}^3 A_3, \\ \tilde{R}(X_1, X_4) = \tilde{R}_{143}^1 M + \tilde{R}_{144}^1 N + \tilde{R}_{144}^3 A_3, \\ \tilde{R}(X_3, X_4) = -\tilde{R}_{134}^3 M - \tilde{R}_{144}^3 N + \tilde{R}_{344}^3 A_3, \\ \tilde{R}(X_1, X_2) = 0, \\ \tilde{R}(X_2, X_i) = \eta \tilde{R}(X_1, X_i), \quad i = 3, 4, \end{cases}$$

where

$$(15) \quad M = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & -\eta & 0 \\ 1 & \eta & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -\eta \\ 0 & 0 & 0 & 0 \\ 1 & \eta & 0 & 0 \end{pmatrix}.$$

By applying $A_3 \cdot \tilde{R} = 0$ to (X_1, X_3, X_3) we obtain $\tilde{R}_{143}^1 = \tilde{R}_{144}^3 = 0$. With the choice (X_1, X_3, X_4) , we get $\tilde{R}_{144}^1 = \tilde{R}_{133}^1$, and with choice (X_1, X_4, X_4) we get $\tilde{R}_{134}^3 = 0$. One can verify that these four relations ($\tilde{R}_{143}^1 = \tilde{R}_{144}^3 = \tilde{R}_{134}^3 = 0, \tilde{R}_{144}^1 = \tilde{R}_{133}^1$) are equivalent to $A_3 \cdot \tilde{R} = 0$. If $\tilde{R}_{133}^1 \neq 0$, then $R(X_1, X_3)$ is a skew-symmetric endomorphism like in Proposition 7-(a) that is already studied in §4.1. Therefore, with $\tilde{R}_{133}^1 = 0$ and $\tilde{R}_{344}^3 \neq 0$, we have $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$, with $\mathfrak{h} = \text{span}\{A_3\}$. By using (8) we obtain that the non-vanishing brackets are

$$\begin{aligned} [A_3, X_3] &= -X_4, & [A_3, X_4] &= X_3, \\ [X_1, X_3] &= -cX_4, & [X_1, X_4] &= cX_3, \\ [X_2, X_3] &= -dX_4, & [X_2, X_4] &= dX_3, \\ [X_3, X_4] &= cX_1 - dX_2 + \tilde{R}_{344}^3 A_3. \end{aligned}$$

Letting

$$T_1 = X_1 - cA_3, \quad T_2 = X_2 - dA_3, \quad Y_1 = cT_1 - dT_2 + \lambda A_3 = cX_1 - dX_2 + \lambda A_3,$$

with $\lambda = \tilde{R}_{344}^3$ then, since $\lambda \neq 0$ and $d = \eta c \neq 0$, a basis of the same Lie algebra is given by $\{Y_1, X_3, X_4, T_1, T_2\}$ and the only non-null brackets are

$$[X_3, Y_1] = \lambda X_4, \quad [Y_1, X_4] = \lambda X_3, \quad [X_3, X_4] = Y_1,$$

that is, $\text{span}\{Y_1, X_3, X_4\} \simeq \mathfrak{sl}(2, \mathbb{R})$, and the Lie algebra \mathfrak{g} is the direct sum of the Abelian Lie algebra $\text{span}\{T_1, T_2\} \simeq \mathbb{R}^2$ and $\mathfrak{sl}(2, \mathbb{R})$. The corresponding simply connected Lie group is thus the direct product $SL(2, \mathbb{R}) \times \mathbb{R}^2$.

5 Example: oscillator

One of the most celebrated examples of Lorentzian naturally reductive spaces is the oscillator group. We refer to [8] for notation and definitions. This group is defined as $G = \mathbb{R} \times \mathbb{C} \times \mathbb{R}$ with group structure

$$(p_1, z_1, q_1) \cdot (p_2, z_2, q_2) = (p_1 + p_2 + \frac{1}{2} \text{Im}(\bar{z}_1 e^{iq_1} z_2), z_1 + e^{iq_1} z_2, q_1 + q_2).$$

The corresponding Lie \mathfrak{g} with basis $\mathcal{B} = (P, X, Y, Q)$ has non-vanishing brackets

$$[X, Y] = P, \quad [Q, X] = Y, \quad [Q, Y] = -X.$$

Furthermore, G is endowed with the left invariant metric

$$\begin{pmatrix} \varepsilon & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & \varepsilon \end{pmatrix}$$

with respect to the basis \mathcal{B} and for $-1 < \varepsilon < 1$. Note that g is not, a priori, a product of metrics. The manifold is symmetric if and only if $\varepsilon = 0$. For $\varepsilon \neq 0$, the naturally reductive structure tensors D and the curvature operators \tilde{R} are given in [8]. With respect to the orthonormal basis (P', X, Y, Q') , $P' = (2 - 2\varepsilon)^{-1/2}(P - Q)$, $Q' = (2 + 2\varepsilon)^{-1/2}(P + Q)$, we have $\tilde{R}_{XY}P' = \tilde{R}_{XY}Q' = 0$, $\tilde{R}_{XY}X = -\varepsilon Y$, $\tilde{R}_{XY}Y = 3\varepsilon X$, and $\tilde{T}(X, Y) = -\frac{1}{2}\sqrt{2 - 2\varepsilon}P' - \frac{1}{2}\sqrt{2 + 2\varepsilon}Q'$ which is as in §4.2 with $c^2 - d^2 \neq 0$. Hence G must be the semi-Riemannian product of two naturally reductive spaces with infinitesimal decomposition $T_e G = W \oplus W^\perp$, $W = \text{span}\{X, Y, \tilde{T}(X, Y)\}$. From this, we easily get the splitting $G = M_1 \times M_2$, with

$$M_1 = \{(\lambda, 0, 0, -\varepsilon\lambda) \mid \lambda \in \mathbb{R}\} \simeq \mathbb{R},$$

$$M_2 = \{(p, x, y, 0) \mid p, x, y \in \mathbb{R}\} \simeq \mathbb{R}^3.$$

Using coordinates $(\lambda; p, x, y)$ in $M_1 \times M_2$, one can check that the matrix of g in this system reads

$$\begin{pmatrix} \varepsilon(\varepsilon - 1)(\varepsilon + 1) & 0 & 0 & 0 \\ 0 & \varepsilon & \frac{1}{2}\varepsilon y & -\frac{1}{2}\varepsilon x \\ 0 & \frac{1}{2}\varepsilon y & 1 + \frac{1}{4}\varepsilon y^2 & -\frac{1}{2}\varepsilon xy \\ 0 & -\frac{1}{2}\varepsilon x & -\frac{1}{2}\varepsilon xy & 1 + \frac{1}{4}\varepsilon x^2 \end{pmatrix}$$

which proves that $(G, g_\varepsilon) = (M_1, g_1) \times (M_2, g_2)$, with g_1 Riemannian and g_2 Lorentzian for $-1 < \varepsilon < 0$ and the opposite for $0 < \varepsilon < 1$.

6 Proof of Theorem 10

6.1 Reducible cases

Let (X_1, X_2, X_3, X_4) be a basis in T_oM such that $\langle X_i, X_j \rangle = \varepsilon_i \delta_{ij}$, with $\varepsilon_1 = \varepsilon_2 = -1$, $\varepsilon_3 = \varepsilon_4 = 1$. From (1) we have:

$$0 = \tilde{T}_{ij}^k \varepsilon_k + \tilde{T}_{ik}^j \varepsilon_j, \quad i, j, k = 1, \dots, 4,$$

where \tilde{T}_{ij}^k is introduced in (11). Therefore, $\tilde{T}_{ij}^i = 0$ and $\tilde{T}_{ji}^i = 0$, $1 \leq i \leq j \leq 4$, and by denoting $\tilde{T}_{12}^3 = a$, $\tilde{T}_{12}^4 = b$, $\tilde{T}_{13}^4 = c$, $\tilde{T}_{23}^4 = d$, we get:

$$(16) \quad \begin{cases} \tilde{T}(X_1, X_2) = aX_3 + bX_4, \\ \tilde{T}(X_1, X_3) = aX_2 + cX_4, \\ \tilde{T}(X_1, X_4) = bX_2 - cX_3, \\ \tilde{T}(X_2, X_3) = -aX_1 + dX_4, \\ \tilde{T}(X_2, X_4) = -bX_1 - dX_3, \\ \tilde{T}(X_3, X_4) = -cX_1 - dX_2. \end{cases}$$

As in §9, we assume $\tilde{T} \neq 0$ and there exist $X, Y \in T_oM$ such that $A = \tilde{R}(X, Y) \neq 0$ so that we can apply the classification of Proposition 8.

6.1.1 Case a1)

Suppose that a curvature transformation $A = \tilde{R}(X, Y)$ exists such that

$$(17) \quad AX_1 = 0, \quad AX_2 = X_4, \quad AX_3 = -X_4, \quad AX_4 = X_2 + X_3.$$

By applying $A \cdot \tilde{T} = 0$ to X_i, X_j for $1 \leq i < j \leq 4$, and taking (16) and (17) into account we get $a = 0$, $c = -b$. As we are considering $\tilde{T} \neq 0$ we have $b^2 + d^2 \neq 0$.

In the case, $b = 0$, $d \neq 0$ (resp. $b \cdot d \neq 0$) if we take $W = \text{span}\{X_2, X_3, X_4\}$ (resp. $W = \text{span}\{bX_1 + dX_3, -bX_1 + dX_2, X_4\}$) we conclude that M is decomposable by virtue of Proposition 6.

For the case $b \neq 0$, $d = 0$, (4), (5) we have

$$\begin{cases} \tilde{R}(X_1, X_4) = \tilde{R}(X_2, X_3) = 0, \\ \tilde{R}(X_1, X_3) = -\tilde{R}(X_1, X_2), \\ \tilde{R}(X_2, X_4) = -\tilde{R}(X_3, X_4), \\ \tilde{R}(X_1, X_2) = -\tilde{R}_{134}^3 A_1 - \tilde{R}_{133}^1 B, \\ \tilde{R}(X_3, X_4) = \tilde{R}_{344}^3 A_1 - \tilde{R}_{134}^3 B, \end{cases}$$

with A_1 as in Proposition 8 and

$$B = \begin{pmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We have two possibilities:

- If $(\tilde{R}_{134}^3)^2 \neq -\tilde{R}_{133}^1 \tilde{R}_{344}^3$ then $\mathfrak{h} = \text{span}\{A_1, B\}$ and by using (8) the non-vanishing brackets of the Lie algebra $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ are

$$\begin{aligned} [A_1, X_2] &= -[A_1, X_3] = X_4, \\ [B, X_1] &= [A_1, X_4] = X_2 + X_3, \\ [B, X_2] &= -[B, X_3] = -X_1, \\ [X_1, X_2] &= -[X_1, X_3] = -bX_4 - \tilde{R}_{134}^3 A_1 - \tilde{R}_{133}^1 B, \\ [X_2, X_4] &= -[X_3, X_4] = bX_1 - \tilde{R}_{344}^3 A_1 + \tilde{R}_{134}^3 B, \\ [X_1, X_4] &= -b(X_2 + X_3). \end{aligned}$$

- If $(\tilde{R}_{134}^3)^2 = -\tilde{R}_{133}^1 \tilde{R}_{344}^3$, the dimension of $\mathfrak{h} = \text{span}\{\tilde{R}(X, Y) | X, Y \in \mathfrak{m}\}$ is one. As we supposed that there exists X, Y such that $\tilde{R}(X, Y) = A_1$, with A_1 as in (17), we have that $\tilde{R}_{133}^1 = \tilde{R}_{134}^3 = 0$. The non-vanishing brackets of the Lie algebra $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ are

$$\begin{aligned} [A_1, X_2] &= -[A_1, X_3] = X_4, \\ [A_1, X_4] &= X_2 + X_3, \\ [X_1, X_2] &= -[X_1, X_3] = -bX_4, \\ [X_2, X_4] &= -[X_3, X_4] = bX_1 - \tilde{R}_{344}^3 A_1, \\ [X_1, X_4] &= -b(X_2 + X_3). \end{aligned}$$

This corresponds to the case $\beta = \delta = 0$ of the family in Theorem 10-2.

6.1.2 Case a2)

Suppose that a curvature transformation $A = \tilde{R}(X, Y)$ exists such that

$$(18) \quad AX_1 = -\alpha X_2, \quad AX_2 = \alpha X_1, \quad AX_3 = -\beta X_4, \quad AX_4 = \beta X_3.$$

If $\alpha \neq 0$, by applying $A \cdot \tilde{T} = 0$ to X_1, X_2 and to X_3, X_4 , and taking (16) and (18) into account we deduce: $a = b = c = d = 0$. Therefore M is symmetric.

For $\alpha = 0$ the condition $A \cdot \tilde{T} = 0$ only gives that $a = b = 0$ in (16). As we are considering $\tilde{T} \neq 0$, we have $c^2 + d^2 \neq 0$. In that case, if we take $W = \text{span}\{cX_1 + dX_2, X_3, X_4\}$ we conclude that M is decomposable by virtue of Proposition 6.

6.1.3 Case a3)

Suppose that a curvature transformation $A = \tilde{R}(X, Y)$ exists such that

$$AX_1 = \beta X_3, \quad AX_2 = \alpha X_4, \quad AX_3 = \beta X_1, \quad AX_4 = \alpha X_2.$$

If $\beta \neq 0$, the condition $A \cdot \tilde{T} = 0$ gives $a = b = c = d = 0$. Therefore M is symmetric.

For $\beta = 0$ the condition $A \cdot \tilde{T} = 0$ now gives $a = c = 0$ in (16). As we are considering $\tilde{T} \neq 0$, we have $b^2 + d^2 \neq 0$.

If $d^2 - b^2 \neq 0$ and we take $W = \text{span}\{X_2, X_4, bX_1 + dX_3\}$ we conclude that M is decomposable by virtue of Proposition 6. We thus assume $d = \eta b$, $\eta = \pm 1$. In this case, from straightforward—but rather long—computations, we get

$$\begin{cases} \tilde{R}(X_1, X_2) = \tilde{R}_{122}^1 M + \tilde{R}_{142}^1 N + \tilde{R}_{124}^2 A_{(\beta=0)}, \\ \tilde{R}(X_1, X_4) = \tilde{R}_{142}^1 M + \tilde{R}_{144}^1 N + \tilde{R}_{144}^2 A_{(\beta=0)}, \\ \tilde{R}(X_2, X_4) = \tilde{R}_{124}^2 M + \tilde{R}_{144}^2 N + \tilde{R}_{244}^2 A_{(\beta=0)}, \\ \tilde{R}(X_1, X_3) = 0, \\ \tilde{R}(X_3, X_j) = -\eta \tilde{R}(X_1, X_j), \quad j = 2, 4, \end{cases}$$

where

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & \eta & 0 \\ 0 & \eta & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \eta \\ 1 & 0 & -\eta & 0 \end{pmatrix}.$$

By applying $A \cdot \tilde{R} = 0$ to (X_1, X_2, X_i) , $i = 1, 2$, we obtain: $\tilde{R}_{142}^1 = \tilde{R}_{144}^2 = 0$, $\tilde{R}_{144}^1 = -\tilde{R}_{122}^1$. With the choice (X_2, X_4, X_4) we get $\tilde{R}_{124}^2 = 0$. One can easily prove that with these four conditions ($\tilde{R}_{142}^1 = \tilde{R}_{144}^2 = \tilde{R}_{124}^2 = 0$, $\tilde{R}_{144}^1 = -\tilde{R}_{122}^1$) are equivalent to the condition $A \cdot \tilde{R} = 0$. Furthermore, from condition $\tilde{R}(X_1, X_4) \cdot \tilde{R} = 0$ we obtain $\tilde{R}_{122}^1 \tilde{R}_{244}^2 = 0$. As we assume that the manifold is non symmetric, either $\tilde{R}_{244}^2 = 0$, $\tilde{R}_{122}^1 \neq 0$ or $\tilde{R}_{244}^2 \neq 0$, $\tilde{R}_{122}^1 = 0$. The former case has been already studied in §6.1.1, for M is a matrix of type a1. Then $\tilde{R}_{244}^2 \neq 0$ and $\mathfrak{h} = \text{span}\{A_3\}$ as in Proposition 8 with $\beta = 0$, $\alpha = 1$. By using (8) the non-vanishing brackets of the Lie algebra $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ are

$$\begin{aligned} [A_3, X_2] &= X_4, & [A_3, X_4] &= X_2, \\ [X_1, X_2] &= -bX_4, & [X_1, X_4] &= -bX_2, \\ [X_2, X_3] &= -dX_4, & [X_3, X_4] &= dX_2, \\ [X_2, X_4] &= bX_1 + dX_3 + \tilde{R}_{244}^2 A_3. \end{aligned}$$

Letting $T_1 = X_1 + bA_3$, $T_2 = X_3 - dA_3$, $Y_1 = bX_1 + dX_3 + \lambda A_3$, with $\lambda = \tilde{R}_{244}^2$, since $\lambda \neq 0$, a basis of the same Lie algebra is given by $(T_1, T_2, Y_1, X_2, X_4)$ and the only non-null brackets are

$$[Y_1, X_2] = \lambda X_4, \quad [Y_1, X_4] = \lambda X_2, \quad [X_2, X_4] = Y_1,$$

that is, Y_1, X_2, X_4 generate $\mathfrak{sl}(2, \mathbb{R})$, and \mathfrak{g} is the direct sum of the 2-dimensional Abelian Lie algebra $\text{span}\{T_1, T_2\}$ and $\mathfrak{sl}(2, \mathbb{R})$. The corresponding simply connected Lie group is thus the direct product $\widetilde{SL(2, \mathbb{R})} \times \mathbb{R}^2$.

6.2 Irreducible cases

For a basis (X_1, X_2, X_3, X_4) of T_oM such that

$$\langle X_2, X_3 \rangle = -\langle X_1, X_4 \rangle = 1 \quad \text{and} \quad \langle X_i, X_j \rangle = 0, \quad \text{otherwise,}$$

condition (1) gives

$$(19) \quad \begin{cases} \tilde{T}(X_1, X_2) = cX_1 - aX_2, \\ \tilde{T}(X_1, X_3) = dX_1 + aX_3, \\ \tilde{T}(X_1, X_4) = dX_2 + cX_3, \\ \tilde{T}(X_2, X_3) = -bX_1 + aX_4, \\ \tilde{T}(X_2, X_4) = -bX_2 + cX_4, \\ \tilde{T}(X_3, X_4) = bX_3 + dX_4. \end{cases}$$

6.2.1 Case b1)

Suppose that a curvature transformation $A = \tilde{R}(X, Y)$ exists such that

$$(20) \quad AX_1 = \nu X_2, \quad AX_2 = -\nu X_1, \quad AX_3 = X_1 + \nu X_4, \quad AX_4 = X_2 - \nu X_3.$$

If $\nu \neq 0$, by applying $A \cdot \tilde{T} = 0$ to X_i, X_j , $1 \leq i < j \leq 4$, and taking (19) and (20) into account we get $a = b = c = d = 0$, hence $\tilde{T} = 0$ and therefore M is symmetric. If $\nu = 0$, we just get $a = c = 0$. As we are considering $\tilde{T} \neq 0$, at least one of b and d is different to 0. We can assume that $b \neq 0$ (if $b = 0$, the new basis $(-X_2, X_1, -X_4, X_3)$ preserves the metric and the expression of A but switches b to d). In this case from the Bianchi identities (4), (5) and imposing $A \cdot \tilde{R} = 0$, we obtain:

$$\begin{cases} \tilde{R}(X_1, X_2) = 0, \\ \tilde{R}(X_1, X_i) = -\frac{d}{b}\tilde{R}(X_2, X_i), \quad i = 3, 4, \\ \tilde{R}(X_2, X_3) = -\frac{d}{b}\tilde{R}(X_2, X_4), \\ \tilde{R}(X_2, X_4) = \tilde{R}_{244}^3 B + \tilde{R}_{344}^3 B_1, \\ \tilde{R}(X_3, X_4) = \tilde{R}_{344}^3 B + \tilde{R}_{344}^2 B_1, \end{cases}$$

where B_1 is as in Proposition 8 with $\nu = 0$ and

$$(21) \quad B = \begin{pmatrix} -\frac{d}{b} & 1 & 0 & 0 \\ -\frac{d^2}{b^2} & \frac{d}{b} & 0 & 0 \\ 0 & 0 & -\frac{d}{b} & 1 \\ 0 & 0 & -\frac{d^2}{b^2} & \frac{d}{b} \end{pmatrix}.$$

One can check that both $B \pm B_1$ are reducible matrix equivalent to A_1 in Proposition 8 (note that $\text{span}\{\mp X_2 + X_3 + \frac{d}{b}X_4\}$ is an invariant and non-degenerate subspace). This means that $\mathfrak{h} = \text{span}\{B, B_1\}$ is also generated by the two reducible endomorphisms $B' = B + B_1$ and $B'' = B - B_2$. This case has been already studied in 6.1.1. This implies that \mathfrak{g} and \mathfrak{h} must be as in case 6.1.1 above.

6.2.2 Case b2)

Suppose that a curvature transformation $A = \tilde{R}(X, Y)$ exists such that

$$(22) \quad AX_1 = \lambda X_1, \quad AX_2 = -\lambda X_2, \quad AX_3 = X_1 + \lambda X_3, \quad AX_4 = X_2 - \lambda X_4,$$

with $\lambda \neq 0$. By applying $A \cdot \tilde{T} = 0$ to X_1, X_2 and to X_3, X_4 , and taking (19) and (22) into account we deduce: $a = b = c = d = 0$, hence $\tilde{T} = 0$ and therefore M is symmetric.

6.2.3 Case b3)

Suppose that a curvature transformation $A = \tilde{R}(X, Y)$ exists such that

$$(23) \quad \begin{cases} AX_1 = \xi X_2 + \nu X_4, & AX_2 = \xi X_1 + \nu X_3, \\ AX_3 = -\nu X_3 + \xi X_4, & AX_4 = -\nu X_1 + \xi X_3, \end{cases}$$

with $\xi \cdot \nu \neq 0$. By applying $A \cdot \tilde{T} = 0$ to X_i, X_j for $1 \leq i < j \leq 4$, and taking (19) and (23) into account we obtain: $a = b = c = d = 0$ and therefore M is symmetric.

7 Study of the new manifolds

In this section we analyze the geometry of the manifolds given in Theorems 9 and 10. We prove that in the generic case they are not symmetric nor decomposable. For that purpose, we need the computation of the covariant derivative of the curvature tensor and the holonomy of the Levi-Civita connection. With respect to the former, from (3) and (7) we have

$$(24) \quad (\nabla_X R)(Y, Z)W = -\frac{1}{2}\tilde{T}(X, R(Y, Z)W) + \frac{1}{2}R(\tilde{T}(X, Y), Z)W \\ + \frac{1}{2}R(Y, \tilde{T}(X, Z))W + \frac{1}{2}R(Y, Z)\tilde{T}(X, W).$$

For the latter, we recall that (see [10, X, Corollary 4.5]) the holonomy algebra of a reductive homogeneous manifold $M = G/H$, with H being the isotropy of $o \in M$, is the smallest subalgebra $\mathfrak{hol} \subset \mathfrak{so}(\mathfrak{m}, g_o)$ containing the $R(X, Y)_o$, $X, Y \in \mathfrak{m}$, such that $[\Lambda_{\mathfrak{m}}(X), \mathfrak{hol}] \subset \mathfrak{hol}$, for all $X \in \mathfrak{m}$, where $\Lambda_{\mathfrak{m}}(X): \mathfrak{m} \rightarrow \mathfrak{m}$ is $\Lambda_{\mathfrak{m}}(X)(Y) = \frac{1}{2}[X, Y]_{\mathfrak{m}}$. Note that if the holonomy algebra does not possess any proper non-degenerate invariant subspace, the manifold must be indecomposable. We then have the following results.

Proposition 12 *The Lorentzian manifold G/H in Theorem 9-2, with $c \neq 0$ and $\alpha\delta - \beta^2 \neq 0$, is flat if and only if $\beta = 0$ and $\alpha = \delta = \frac{1}{4}c^2$. Otherwise it is indecomposable. Furthermore, it is symmetric if and only if $\beta = 0$ and $\alpha = \delta$.*

The subalgebra $\text{span}\{X_1, X_2, X_3, X_4, A\}$ for $\beta = \delta = 0$ which corresponds to 4.1, $\beta^2 = \alpha\delta$, is also non-symmetric and indecomposable.

Proof. Taking (6) and (7) into account we get

$$\begin{cases} R(X_1, X_2) = R(X_3, X_4) = 0, & R(X_1, X_3) = (\alpha - \frac{1}{4}c^2)A + \beta B, \\ R(X_1, X_4) = \beta A + (\delta - \frac{1}{4}c^2)B, & R(X_2, X_j) = -R(X_1, X_j), \quad j = 3, 4, \end{cases}$$

and we get the condition about flatness. For the covariant derivative of the curvature, from (24) we have $(\nabla_{X_1} R)(X_1, X_3)X_1 = c\beta X_3 + \frac{1}{2}c(\delta - \alpha)X_4$,

which only vanishes for $\beta = 0$ and $\alpha = \delta$. In that case, it is easy to see that $(\nabla_{X_i} R)(X_j, X_k)X_l = 0$, for all i, j, k, l and M is locally symmetric.

If $(\alpha - \frac{1}{4}c^2)(\delta - \frac{1}{4}c^2) - \beta^2 \neq 0$, in view of the expressions of $R(X_1, X_3)$ and $R(X_1, X_4)$ above, condition $[\Lambda_m(X), \mathfrak{hol}] \subset \mathfrak{hol}$ gives $\mathfrak{hol} = \text{span}\{A, B\}$. These matrices do not have any common invariant non-degenerate subspace and therefore M is irreducible.

If $(\alpha - \frac{1}{4}c^2)(\delta - \frac{1}{4}c^2) - \beta^2 = 0$, then $R(X, Y)_o$ is generated by a single element, for instance $(\alpha - \frac{1}{4}c^2)A + \beta B$. On the other hand, one can check that $[\Lambda_m(X_1), A] = -\frac{c}{2}B$, $[\Lambda_m(X_1), B] = \frac{c}{2}A$. Then condition $[\Lambda_m(X_1), \mathfrak{hol}] \subset \mathfrak{hol}$ gives again $\mathfrak{hol} = \text{span}\{A, B\}$ unless $\alpha - \frac{1}{4}c^2 = \delta - \frac{1}{4}c^2 = \beta = 0$.

The study of the subalgebra $\text{span}\{X_1, X_2, X_3, X_4, A\}$ for $\beta = \delta = 0$ is done similarly. ■

Proposition 13 *The Lorentzian manifold $(SL(2, \mathbb{R}) \times \mathbb{R}^2)/\mathbb{R}$ in Theorem 9-1 defined infinitesimally by the Lie brackets*

$$\begin{aligned} [A, X_3] &= -X_4, & [A, X_4] &= X_3, \\ [X_1, X_3] &= -cX_4, & [X_1, X_4] &= cX_3, \\ [X_2, X_3] &= -\eta cX_4, & [X_2, X_4] &= \eta cX_3, \\ [X_3, X_4] &= cX_1 - \eta cX_2 + \alpha A, \end{aligned}$$

as in §4.2 with $c \neq 0$, $\eta = \pm 1$, $\alpha \neq 0$, is not flat, non-symmetric and indecomposable.

Proof. Taking (6) and (7) into account we get

$$\begin{cases} R(X_1, X_2) = 0, & R(X_1, X_3) = -\frac{1}{4}c^2M, \\ R(X_1, X_4) = -\frac{1}{4}c^2N, & R(X_2, X_j) = \eta R(X_1, X_j), j = 3, 4, \\ R(X_3, X_4) = \alpha A, \end{cases}$$

where M, N are defined in (15). Moreover, from (24), we have

$$(\nabla_{X_3} R)(X_1, X_3)X_3 = -\frac{1}{2}c\alpha X_4,$$

so that the manifold is not locally symmetric.

In view of the expressions of $R(X_i, X_j)$ above, condition $[\Lambda_m(X), \mathfrak{hol}] \subset \mathfrak{hol}$ gives $\mathfrak{hol} = \text{span}\{M, N, A\}$. It is easy to see that these matrices do not share any common invariant non-degenerate subspace. ■

Proposition 14 *The $(2, 2)$ -signature manifold G/H in Theorem 10-2 with $b \neq 0$, $\alpha^2 \neq -\beta\delta$, in theorem 10 is flat if and only if $\alpha = 0$ and $\beta = \delta = \frac{1}{4}b^2$. Otherwise it is indecomposable. Furthermore, it is symmetric if and only if $\alpha = 0$ and $\beta = \delta$.*

The subalgebra $\text{span}\{X_1, X_2, X_3, X_4, A_1\}$ for $\beta = \alpha = 0$ which corresponds to 6.1.1, $\alpha^2 = -\beta\delta$, is also non-symmetric and indecomposable.

Proof. Taking (6) and (7) into account we get

$$\begin{cases} R(X_1, X_2) = -\alpha A_1 - (\beta - \frac{1}{4}b^2)B, & R(X_2, X_4) = -(\delta - \frac{1}{4}b^2)A + \alpha B, \\ R(X_2, X_3) = R(X_1, X_4) = 0, & R(X_3, X_j) = -R(X_2, X_j), \quad j = 1, 4, \end{cases}$$

where B is defined in (21) and we get the condition about flatness. From (24), we have $(\nabla_{X_2} R)(X_3, X_1)X_2 = -b\alpha X_1 + \frac{1}{2}b(\beta - \delta)X_4$, which only vanishes for $\alpha = 0$ and $\beta = \delta$. In that case, it is easy to see that all $(\nabla_{X_i} R)(X_j, X_k)X_l = 0$, for all i, j, k, l and M is locally symmetric.

If $(\beta - \frac{1}{4}b^2)(\delta - \frac{1}{4}b^2) - \alpha^2 \neq 0$, in view of the expressions of $R(X_1, X_2)$ and $R(X_2, X_4)$ above, condition $[\Lambda_m(X_2), \mathfrak{hol}] \subset \mathfrak{hol}$ gives $\mathfrak{hol} = \text{span}\{A, B\}$. These matrices do not have any common invariant non-degenerate subspace and therefore M is irreducible. If $(\beta - \frac{1}{4}b^2)(\delta - \frac{1}{4}b^2) - \alpha^2 = 0$, then $R(X, Y)_o$ is generated by a single element, for instance $\alpha A_1 + (\beta - \frac{1}{4}b^2)B$. On the other hand, one can check that $[\Lambda_m(X_2), A] = -\frac{b}{2}B$, $[\Lambda_m(X_2), B] = -\frac{b}{2}A$. Then condition $[\Lambda_m(X_2), \mathfrak{hol}] \subset \mathfrak{hol}$ gives $\mathfrak{hol} = \text{span}\{A, B\}$ unless $\beta - \frac{1}{4}b^2 = \pm\alpha$. Otherwise, $\mathfrak{hol} = \text{span}\{A_1 \pm B\}$ which do not have invariant non-degenerate subspaces.

The study of the subalgebra $\text{span}\{X_1, X_2, X_3, X_4, A_1\}$ for $\beta = \alpha = 0$ is done similarly. ■

Proposition 15 *The (2, 2)-signature manifold $(SL(2, \mathbb{R}) \times \mathbb{R}^2)/\mathbb{R}$ in Theorem 10-1 defined infinitesimally by the Lie brackets*

$$\begin{aligned} [A, X_2] &= X_4, & [A, X_4] &= X_2, \\ [X_1, X_2] &= -bX_4, & [X_1, X_4] &= -bX_2, \\ [X_2, X_3] &= -\eta bX_4, & [X_3, X_4] &= \eta bX_3, \\ [X_3, X_4] &= b(X_1 + \eta X_3) + \alpha A, \end{aligned}$$

as in §6.1.3 with $b \neq 0$, $\eta = \pm 1$, $\alpha \neq 0$, in is non-symmetric and indecomposable.

We do not include the proof as it is similar to the one of Proposition 13.

References

- [1] W. Ambrose, I.M. Singer, *On homogeneous Riemannian manifolds*, Duke Math. J. **25** (1958), 647–669.
- [2] V. del Barco, G.P. Ovando, F. Vittone, *Naturally reductive pseudo-Riemannian Lie groups in low dimensions*, preprint arXiv:1211.0884.
- [3] Ch. Boubel, *Sur l'holonomie des variétés pseudo-riemanniennes*, PhD thesis, Institut Élie Cartan (Nancy), 2000.
- [4] G. Calvaruso, R.A. Marinosci, *Homogeneous geodesics of non-unimodular Lorentzian Lie groups and naturally reductive Lorentzian spaces in dimension three*, Adv. Geom. **8** (2008), no. 4, 473–489.
- [5] R. Campoamor-Stursberg, *Some Remarks Concerning the Invariants of Rank One Solvable Real Lie Algebras*, Algebra Colloq., **12** (2005), no. 3, 497–518.
- [6] M. Freibert, F. Schulte-Hengesbach, *Half-flat structures on indecomposable Lie groups*, Transform. Groups **17** (2012), no. 3, 657–689.

- [7] P.M. Gadea, J.A. Oubiña, *Reductive homogeneous pseudo-Riemannian manifolds*, Monatsh. Math. **124** (1997), no. 1, 1734.
- [8] P.M. Gadea, J.A. Oubiña, *Homogeneous Lorentzian structures on the oscillator groups*, Arch. Math. **73** (1999), no. 4, 311–320.
- [9] N. Huaoari, W. Batat, N. Rahmani and S. Rahmani, *Three-Dimensional naturally reductive homogeneous Lorentzian manifolds*, Mediterr. J. Math. **5** (2008), 113–131.
- [10] S. Kobayashi, K. Nomizu, *Foundations of Differential Geometry, Volume II*, John Wiley & Sons, Inc., N.Y., 1969.
- [11] O. Kowalski and L. Vanhecke, *Four dimensional naturally reductive homogeneous spaces*, Rend. Sem. Mat. Univ. Politec. Torino 1983 (1984), Special Issue, 223–232.
- [12] O. Kowalski and L. Vanhecke, *Classification of five-dimensional naturally reductive spaces*, Math. Proc. Cambridge Phil. Soc. **97** (1985), 445–463.
- [13] F. Leitner, *Imaginary Killing spinors in Lorentzian geometry*, J. Math. Phys. **44**, (2003), 4795–4807.
- [14] B. O’Neill, *Semi-Riemannian geometry with applications to Relativity*, Academic Press, New York, London, 1983.
- [15] G. Ovando, *Naturally reductive pseudo-Riemannian spaces*, J. Geom. Phys. **61** (2011), no. 1, 157–171.
- [16] D.A. Singer, D.H. Steinberg, *Normal forms in Lorentzian spaces*, Nova J. Algebra Geom. **3** (1994), no. 1, 1–9.
- [17] F. Tricerri, L. Vanhecke, *Homogeneous structures on Riemannian manifolds*, *London Mathematical Society Lecture Note Series*, **83**, Cambridge Univ. Press (1983).
- [18] H. Wu, *On the de Rham decomposition theorem*, Illinois J. Math. **8**, no. 2 (1964), 291–311.

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