

Nonlinear Pseudo-Supersymmetry in the Framework of \mathcal{N} -fold Supersymmetry

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Abstract

We recall the importance of recognizing the different mathematical nature of various concepts relating to \mathcal{PT} -symmetric quantum theories. After clarifying the relation between supersymmetry and pseudo-supersymmetry, we prove generically that nonlinear pseudo-supersymmetry, recently proposed by Sinha and Roy, is just a special case of \mathcal{N} -fold supersymmetry. In particular, we show that all the models constructed by these authors have type A 2-fold supersymmetry. Furthermore, we prove that an arbitrary one-body quantum Hamiltonian which admits two (local) solutions in closed form belongs to type A 2-fold supersymmetry, irrespective of whether or not it is Hermitian, \mathcal{PT} -symmetric, pseudo-Hermitian, and so on.

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I. INTRODUCTION

Since Bender and Boettcher claimed that the reality of the spectrum of the Hamiltonian $H = p^2 + x^2 + ix^3$ is due to the underlying \mathcal{PT} symmetry [1], there have appeared in the literature numerous investigations into the spectral properties of various quantum Hamiltonians with non-real potentials defined on, in general, a complex contour. See, e.g., Refs. [2, 3, 4, 5, 6, 7] and references cited therein for recent developments. The rapid progress in this research field, however, has caused some confusion and several misunderstandings. A typical example is the relation between \mathcal{PT} symmetry and pseudo-Hermiticity. The former concept is meaningful at the operator level, without referring to a vector space on which the operator in question acts; indeed, it can be defined as the invariance of the operator under the formal replacements $x \rightarrow -x$ and $i \rightarrow -i$. On the other hand, the concept of pseudo-Hermiticity, mainly developed by Mostafazadeh in the context of \mathcal{PT} symmetry [8], inevitably needs a Hilbert space on which the Hermitian conjugate is defined. Hence, it makes little sense to discuss, e.g., whether or not \mathcal{PT} symmetry is a special case of pseudo-Hermiticity, without taking into account their different mathematical character.

Recently, we have also found a similar confusion in the paper by Sinha and Roy [9], where the authors claimed to generalize the framework of \mathcal{N} -fold supersymmetry to include pseudo-Hermitian systems. This misunderstanding is apparently inherited from the claim in Ref. [8] that pseudo-supersymmetry is a generalization of (ordinary) supersymmetry.

Considering the current situation in and around this research field, we would like to recall in this paper the importance of recognizing the different mathematical nature of various concepts relating to \mathcal{PT} -symmetric quantum theories. In particular, we focus on the relation among the nonlinear pseudo-supersymmetric models in Ref. [9], the framework of \mathcal{N} -fold supersymmetry, and higher-order Darboux transformations.

The paper is organized as follows. In the next section, we review the definition of various concepts characterizing linear differential operators, which are relevant in \mathcal{PT} -symmetric quantum theories, to avoid ambiguity. Based on these precise definitions, we clarify the exact relation among various types of supersymmetry in Section III. We then proceed to prove that nonlinear pseudo-supersymmetry automatically implies \mathcal{N} -fold supersymmetry in Section IV. To make the relation more transparent, we also show how all the models in Ref. [9] can be explicitly constructed in the framework of \mathcal{N} -fold supersymmetry. These findings clearly suggest that there is an overlooked relation between the higher-order Darboux transformations and \mathcal{N} -fold supersymmetry, which we discuss in Section V. The paper concludes in Section VI with a short discussion of the main results obtained in it and some general remarks on the different mathematical character of the symmetries considered.

II. PSEUDO-HERMITICITY AND \mathcal{PT} SYMMETRY

First of all, we would like to review the definition of \mathcal{PT} symmetry [1] and pseudo-Hermiticity first introduced in Ref. [8]. In this paper, we restrict our discussion to linear operators acting on a linear function space of a single variable, e.g., x , which have generally the following form:

$$L = \sum_{n=0}^{\infty} f_n(x) \frac{d^n}{dx^n}. \quad (1)$$

We first define the *formal* Hermitian conjugate L^f of the operator (1) by

$$L^f = \sum_{n=0}^{\infty} (-1)^n \frac{d^n}{dx^n} f_n^*(x^*), \quad (2)$$

where $*$ denotes complex conjugate. A linear operator L is called *formally Hermitian* if $L^f = L$. Similarly, the transposition L^t of the operator (1) is defined by

$$L^t = \sum_{n=0}^{\infty} (-1)^n \frac{d^n}{dx^n} f_n(x), \quad (3)$$

and L is said to have *transposition symmetry* if $L^t = L$ [10]. We note that the formal Hermitian conjugate and transposition of a product of two linear operators formally satisfies $(L_1 L_2)^f = L_2^f L_1^f$ and $(L_1 L_2)^t = L_2^t L_1^t$, respectively, by the above definition.

The spatial reflection \mathcal{P} and the time reversal \mathcal{T} of the operator (1) are, respectively, defined by

$$\mathcal{P}L\mathcal{P} = \sum_{n=0}^{\infty} (-1)^n f_n(-x) \frac{d^n}{dx^n}, \quad (4)$$

$$\mathcal{T}L\mathcal{T} = \sum_{n=0}^{\infty} f_n^*(x^*) \frac{d^n}{dx^n}, \quad (5)$$

where we note that $\mathcal{P}^2 = \mathcal{T}^2 = 1$ and $\mathcal{P}\mathcal{T} = \mathcal{T}\mathcal{P}$. A linear operator L is said to have *\mathcal{PT} symmetry* if $\mathcal{P}\mathcal{T}L\mathcal{P}\mathcal{T} = L$ [1].

Let \mathcal{V} be a linear function space, let η be an invertible, formally Hermitian operator on \mathcal{V} , and consider a linear differential operator $L : \mathcal{V} \rightarrow \mathcal{V}$ of the form (1). Then, the *formal pseudo-Hermitian conjugate* $L^\natural : \mathcal{V} \rightarrow \mathcal{V}$ with respect to η is defined by $L^\natural = \eta^{-1} L^f \eta$. A linear operator (1) is called *formally pseudo-Hermitian* if there exists an invertible, formally Hermitian operator η satisfying $L^\natural = L$, or equivalently, $L^f = \eta L \eta^{-1}$. It is evident that formal pseudo-Hermiticity reduces to formal Hermiticity when $\eta = 1$.

The Hermitian conjugate of the linear differential operator (1) acting on a Hilbert space $L^2(S)$ ($S \subset \mathbb{R}$) with the positive definite inner product (ϕ, ψ) defined by

$$(\phi, \psi) = \int_S dx \phi^*(x) \psi(x) \quad (6)$$

is the operator L^\dagger satisfying

$$(\phi, L^\dagger \psi) = (L\phi, \psi), \quad \forall \phi, \psi \in L^2(S), \quad (7)$$

and formally coincides with the formal Hermitian conjugate L^f . Then, the operator L is called *Hermitian (or self-adjoint)* if $L^\dagger = L$ with respect to the inner product (6)¹. It is evident that a Hermitian operator on $L^2(S)$ is always formally Hermitian. Similarly, the operator L is called *pseudo-Hermitian* if there exists an invertible, Hermitian operator η

¹ Note, in particular, that $\mathcal{D}(L) = \mathcal{D}(L^\dagger)$, where \mathcal{D} denotes the domain of the operator.

satisfying $L^\dagger = \eta L \eta^{-1}$ with respect to the inner product (6) [8]. It is also evident that a pseudo-Hermitian operator on $L^2(S)$ is always formally pseudo-Hermitian.

The crucial problems in the construction of pseudo-Hermitian theories are that the eigenvectors of a pseudo-Hermitian operator are not in general orthogonal with respect to the inner product (6), and that ascertaining that these eigenstates span a dense set of the Hilbert space $L^2(S)$ is far from trivial. These facts clearly indicate the difficulty in establishing, e.g., a resolution of the identity and a spectral theorem for pseudo-Hermitian operators in terms of orthogonal spectral projections. Therefore, we should note that many of the results in Refs. [8, 11, 12, 13, 14, 15, 16, 17] including the relation with \mathcal{PT} symmetry, derived from the assumption that there exists a complete set of (bi)orthonormal eigenvectors, cannot be rigorously justified in general, at least at present.

III. SUPERSYMMETRY AND PSEUDO-SUPERSYMMETRY

Before discussing the relation between \mathcal{N} -fold and nonlinear pseudo-supersymmetries, we shall clarify in this section the simplest case, namely, the relation between ordinary and pseudo-supersymmetries. The Poincaré superalgebra in one spacetime dimension is given by

$$[\mathbf{A}^\pm, \mathbf{H}] = 0, \quad \{\mathbf{A}^\pm, \mathbf{A}^\pm\} = 0, \quad \{\mathbf{A}^-, \mathbf{A}^+\} = 2\mathbf{H}. \quad (8)$$

An arbitrary system possessing the dynamical symmetry characterized by the above superalgebra is given by a representation thereof. In particular, a pair of Schrödinger operators H^\pm can be embedded into the following representation:

$$\mathbf{A}^- = \begin{pmatrix} 0 & A^- \\ 0 & 0 \end{pmatrix}, \quad \mathbf{A}^+ = \begin{pmatrix} 0 & 0 \\ A^+ & 0 \end{pmatrix}, \quad \mathbf{H} = \begin{pmatrix} H^+ & 0 \\ 0 & H^- \end{pmatrix} = \frac{1}{2} \begin{pmatrix} A^- A^+ & 0 \\ 0 & A^+ A^- \end{pmatrix}, \quad (9)$$

where the operators A^\pm are given by

$$A^- = \frac{d}{dx} + W(x), \quad A^+ = (A^-)^t = -\frac{d}{dx} + W(x). \quad (10)$$

Arbitrary one-body supersymmetric quantum mechanical systems in the literature are in fact mathematically equivalent to the above system with a specific choice of the function $W(x)$, although various notation, conventions, and terminology have been employed.

The crucial point here is that *the superalgebra (8) always holds for an arbitrary (differentiable) complex function $W(x)$* . Then, if we restrict the function $W(x)$ to be real, A^+ is the formal Hermitian conjugate of A^- , $A^+ = (A^-)^f$, and the operators H^\pm are formally Hermitian, $(H^\pm)^f = H^\pm$. If we further restrict the real function $W(x)$ to be in a special class of real functions, it may be possible to define a Hilbert space $L^2(S)$ ($S \subset \mathbb{R}$) on which H^\pm are (rigorously) Hermitian, $(H^\pm)^f = (H^\pm)^\dagger = H^\pm$. On the other hand, if we restrict $W(x)$ to a class of complex functions such that there exists an invertible, formally Hermitian operator η for which the relation $\mathbf{A}^+ = \eta^{-1}(\mathbf{A}^-)^f \eta$ holds, the operator \mathbf{H} is formally pseudo-Hermitian, $\mathbf{H}^f = \eta \mathbf{H} \eta^{-1}$. A further restriction of the complex function $W(x)$ may enable us to define a Hilbert space $L^2(S)$ on which \mathbf{H} is (rigorously) pseudo-Hermitian [8]. Finally, if the complex function $W(x)$ satisfies $W^*(-x^*) = -W(x)$, the operators H^\pm are \mathcal{PT} -symmetric.

It is thus apparent that Hermitian, \mathcal{PT} -symmetric, or pseudo-Hermitian supersymmetric systems are special cases of *general* supersymmetry, which is characterized by the Poincaré superalgebra (8) in one spacetime dimension, depending on the restrictions one imposes on the function $W(x)$.

IV. \mathcal{N} -FOLD AND NONLINEAR PSEUDO-SUPERSYMMETRY

Next, we shall clarify the relation between \mathcal{N} -fold and nonlinear pseudo-supersymmetry. \mathcal{N} -fold supersymmetry is characterized by a superalgebra of the type

$$[\mathbf{Q}_{\mathcal{N}}^{\pm}, \mathbf{H}_{\mathcal{N}}] = 0, \quad \{\mathbf{Q}_{\mathcal{N}}^{\pm}, \mathbf{Q}_{\mathcal{N}}^{\pm}\} = 0, \quad \{\mathbf{Q}_{\mathcal{N}}^{-}, \mathbf{Q}_{\mathcal{N}}^{+}\} = \Pi_{\mathcal{N}}(\mathbf{H}_{\mathcal{N}}), \quad (11)$$

where $\Pi_{\mathcal{N}}$ is a polynomial of degree \mathcal{N} . The operators $\mathbf{Q}_{\mathcal{N}}^{\pm}$ are called \mathcal{N} -fold supercharges. For a pair of Schrödinger operators $H_{\mathcal{N}}^{\pm}$ and a monic linear differential operator $P_{\mathcal{N}}$ of order \mathcal{N} :

$$P_{\mathcal{N}} = \sum_{k=0}^{\mathcal{N}} w_k(x) \frac{d^k}{dx^k}, \quad (12)$$

\mathcal{N} -fold supersymmetry can be simply realized by the matrix representation

$$\mathbf{Q}_{\mathcal{N}}^{-} = \begin{pmatrix} 0 & P_{\mathcal{N}} \\ 0 & 0 \end{pmatrix}, \quad \mathbf{Q}_{\mathcal{N}}^{+} = \begin{pmatrix} 0 & 0 \\ P_{\mathcal{N}}^{\dagger} & 0 \end{pmatrix}, \quad \mathbf{H}_{\mathcal{N}} = \begin{pmatrix} H_{\mathcal{N}}^{+} & 0 \\ 0 & H_{\mathcal{N}}^{-} \end{pmatrix}. \quad (13)$$

For a discussion of the general aspects of \mathcal{N} -fold supersymmetry, see, e.g., Refs. [10, 18, 19]. In particular, the system (13) reduces to the ordinary supersymmetric system (9) when $\mathcal{N} = 1$.

From the discussion in the previous section, it should be almost apparent that Hermitian, \mathcal{PT} -symmetric, or pseudo-Hermitian \mathcal{N} -fold supersymmetric systems can all be realized as special cases of (general) \mathcal{N} -fold supersymmetry, depending on the properties of the functions $w_k(q)$ ($k = 0, \dots, \mathcal{N}$) in the component of \mathcal{N} -fold supercharge (12).

We now prove generically that any nonlinear pseudo-supersymmetric system has \mathcal{N} -fold supersymmetry. Indeed, since any nonlinear pseudo-supersymmetric pair of differential operators h_0 and h_N (using the notation of Ref. [9]) satisfies, by definition, intertwining relations with respect to higher-order linear differential operators $A^{(N)}$ and $A^{(N)\dagger}$:

$$A^{(N)}h_0 = h_N A^{(N)}, \quad h_0 A^{(N)\dagger} = A^{(N)\dagger} h_N, \quad (14)$$

the operators h_0 and h_N preserve the finite-dimensional vector spaces $\ker A^{(N)}$ and $\ker A^{(N)\dagger}$, respectively, and thus are weakly quasi-solvable. Applying the theorem on the equivalence between weak quasi-solvability and \mathcal{N} -fold supersymmetry rigorously proved in Ref. [18], and using the fact that the difference between h_0 and h_N is uniquely determined by the given $A^{(N)}$, we immediately conclude that h_0 and h_N must be an \mathcal{N} -fold supersymmetric pair.

To illustrate the above fact more concretely, we shall show in what follows how we can construct the nonlinear pseudo-supersymmetric models in Ref. [9] in the framework of \mathcal{N} -fold supersymmetry with the aid of the algorithm developed by us in Ref. [19]. Our starting point is the two-dimensional linear space

$$\tilde{\mathcal{V}}_2 = \langle 1, z \rangle, \quad (15)$$

and the most general linear second-order differential operator preserving the latter space:

$$\tilde{H}^- = -A(z)\frac{d^2}{dz^2} - B(z)\frac{d}{dz} - C(z), \quad (16)$$

where $A(z)$ is an arbitrary function, and $B(z)$ and $C(z)$ are given by

$$B(z) = b_2 z^2 + b_1 z + b_0, \quad (17)$$

$$C(z) = -b_2 z + c_0, \quad (18)$$

b_i and c_0 being constants. Following the algorithm for constructing an \mathcal{N} -fold supersymmetric system developed in Ref. [19], we easily obtain the components of 2-fold supersymmetry (H^\pm, P_2) as follows:

$$H^\pm = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{4A(z)}\left(\frac{A'(z)}{2} \pm B(z)\right)\left(\frac{3A'(z)}{2} \pm B(z)\right) - \frac{A''(z)}{4} \mp B'(z) - R, \quad (19)$$

$$P_2 = \frac{d^2}{dx^2} - \frac{2B(z)}{\dot{z}}\frac{d}{dx} - \frac{1}{2A(z)}\left(\frac{A'(z)}{2} - B(z)\right)\left(\frac{3A'(z)}{2} + B(z)\right) + \frac{A''(z)}{2} - B'(z), \quad (20)$$

where $R = b_1/2 + c_0$, the dot and the prime denote derivative with respect to x and z , respectively, and the relation between these two variables is determined by

$$\dot{z}^2 = 2A(z). \quad (21)$$

The solvable sector \mathcal{V}_2^- of the Hamiltonian H^- is given by

$$\mathcal{V}_2^- = e^{-\mathcal{W}(z)}\langle 1, z \rangle, \quad (22)$$

with the gauge factor

$$\mathcal{W}(z) = \int \frac{dz}{2A(z)}\left(\frac{A'(z)}{2} - B(z)\right). \quad (23)$$

Let us first set the arbitrary function $A(z)$ as

$$A(z) = 8(2 - a + z)(2 - a + \sqrt{2 - a + z}), \quad (24)$$

where a is a parameter. From Eq. (21), the change of variable is given by

$$z(x) = (2 - a)(1 - a) - 2(2 - a)\bar{x}^2 + \bar{x}^4, \quad (25)$$

where $\bar{x} = x - x_0$, x_0 being a constant. Applying the formulas (19) and (20), we obtain the following 2-fold supersymmetric system (H^\pm, P_2) :

$$\begin{aligned}
H^- = & -\frac{1}{2} \frac{d^2}{dx^2} + b_2(\dots) + \frac{b_1^2}{32} \bar{x}^2 + \frac{(2 + (1-a)b_1)(6 + (1-a)b_1) + 2b_0b_1}{32\bar{x}^2} \\
& - \frac{48 + 8b_1 - b_1^2}{32(2-a-\bar{x}^2)} + \frac{b_0(4-b_1)}{16\bar{x}^2(2-a-\bar{x}^2)} + \frac{b_0^2}{32\bar{x}^2(2-a-\bar{x}^2)^2} \\
& + \frac{(2-a)(48 + 16b_1 + b_1^2) - 2b_0(8+b_1)}{32(2-a-\bar{x}^2)^2} + \frac{b_1}{16}(4-2b_1+ab_1) - R, \tag{26}
\end{aligned}$$

$$\begin{aligned}
H^+ = & -\frac{1}{2} \frac{d^2}{dx^2} + b_2(\dots) + \frac{b_1^2}{32} \bar{x}^2 + \frac{(2 - (1-a)b_1)(6 - (1-a)b_1) + 2b_0b_1}{32\bar{x}^2} \\
& - \frac{48 - 8b_1 - b_1^2}{32(2-a-\bar{x}^2)} - \frac{b_0(4+b_1)}{16\bar{x}^2(2-a-\bar{x}^2)} + \frac{b_0^2}{32\bar{x}^2(2-a-\bar{x}^2)^2} \\
& + \frac{(2-a)(48 - 16b_1 + b_1^2) + 2b_0(8-b_1)}{32(2-a-\bar{x}^2)^2} - \frac{b_1}{16}(4+2b_1-ab_1) - R, \tag{27}
\end{aligned}$$

$$\begin{aligned}
P_2 = & \frac{d^2}{dx^2} - \frac{1}{2} \left[b_2(\dots) + b_1\bar{x} - \frac{(1-a)b_1}{\bar{x}} + \frac{b_1\bar{x}}{2-a-\bar{x}^2} - \frac{b_0}{\bar{x}(2-a-\bar{x}^2)} \right] \frac{d}{dx} \\
& + b_2(\dots) + \frac{b_1^2}{16} \bar{x}^2 - \frac{(2 + (1-a)b_1)(6 - (1-a)b_1) - 2b_0b_1}{16\bar{x}^2} \\
& + \frac{48 + 4b_1 + b_1^2}{16(2-a-\bar{x}^2)} - \frac{b_0(2+b_1)}{8\bar{x}^2(2-a-\bar{x}^2)} + \frac{b_0^2}{16\bar{x}^2(2-a-\bar{x}^2)^2} \\
& - \frac{(2-a)(48 + 8b_1 - b_1^2) - 2b_0(4-b_1)}{16(2-a-\bar{x}^2)^2} - \frac{b_1}{8}(2+2b_1-ab_1), \tag{28}
\end{aligned}$$

where each $b_2(\dots)$ indicates a term proportional to b_2 , all of which are lengthy and thus will be abbreviated in this paper. We easily see that the Hamiltonians H^\pm are \mathcal{PT} -symmetric, namely, invariant under the formal replacement $x \rightarrow -x$, $i \rightarrow -i$, provided that the parameters are chosen such that $a, b_i, ix_0 \in \mathbb{R}$. Furthermore, one can easily show that the above 2-fold supersymmetric system $(2H^-, 2H^+, P_2^- = P_2, P_2^+ = P_2^t)$ exactly reduces to the second nonlinear pseudo-supersymmetric system (h_0, h_2, A, A^\dagger) in Ref. [9], Section 4.2, if we take the parameters as $b_2 = b_0 = 0$, $b_1 = -4$ and $R = a - 3$, with $a = q\alpha$ and $x_0 = i\epsilon$.

Next, let us choose the function $A(z)$ as

$$A(z) = \frac{1}{1-a} (8 - 4a - 4z + z^2) \left[4 - 2a - 3z + z^2 - (1-z)\sqrt{8 - 4a - 4z + z^2} \right], \tag{29}$$

where the change of variable in this case is given by

$$z(x) = \frac{(2-a)(1-a) - 2(2-a)\bar{x}^2 + \bar{x}^4}{1-a-\bar{x}^2}. \tag{30}$$

Following the same procedure as in the previous case, we obtain a 2-fold supersymmetric system which can be \mathcal{PT} -symmetric and which exactly reduces to the first nonlinear pseudo-supersymmetric system in Ref. [9], Section 4.1, when the parameters take the values $b_2 = b_0 = 0$, $b_1 = -2$ and $R = a - 4$, with $a = q\alpha$ and $x_0 = i\epsilon$.

Similarly, if we take the function $A(z)$ as

$$A(z) = -32(1-p-q)^2 y(1-y) [3 - 4p - 2(3 - 2p - 2q)y]^2, \tag{31}$$

where $y = \frac{1}{2}(1 - i \sinh x)$, the change of variable is given by

$$z(x) = (3 - 4p)(1 - 4p) - 8(3 - 4p)(1 - p - q)y + 8(3 - 2p - 2q)(1 - p - q)y^2. \quad (32)$$

The 2-fold supersymmetric system in this case can be \mathcal{PT} -symmetric with a proper choice of the parameters and completely coincides with the third nonlinear pseudo-supersymmetric model in Ref. [9], Section 5, when $b_2 = b_0 = 0$, $b_1 = 2(1 - p - q)$ and $R = \frac{1}{2}(2 - 2p - 2q + p^2 + 2pq + q^2)$.

Therefore, we have shown that all the nonlinear pseudo-supersymmetric models in Ref. [9] can be constructed in the framework of \mathcal{N} -fold supersymmetry without any difficulty. More precisely, note that the 2-fold supersymmetric system given by (19) and (20) is a realization of type A 2-fold supersymmetry². The previous results thus imply that all the nonlinear pseudo-supersymmetric models constructed in Ref. [9] belong to type A 2-fold supersymmetry.

V. SECOND-ORDER DARBOUX TRANSFORMATION AND TYPE A 2-FOLD SUPERSYMMETRY

We shall now prove the more general fact that an arbitrary one-body quantum Hamiltonian which admits two (local) eigenfunctions in closed form belongs to type A 2-fold supersymmetry, irrespective of whether or not it is Hermitian, \mathcal{PT} -symmetric, pseudo-Hermitian, and so on. Suppose, to this end, that the Hamiltonian H under consideration has two analytic solutions $\psi_i(x)$ and $\psi_j(x)$ with some spectral parameters λ_i and λ_j , respectively:

$$H\psi_i(x) = \lambda_i\psi_i(x), \quad H\psi_j(x) = \lambda_j\psi_j(x). \quad (33)$$

We define two functions $z(x)$ and $\mathcal{W}(z)$ by

$$z(x) = \frac{\psi_j(x)}{\psi_i(x)}, \quad \mathcal{W}(z) = -\ln \psi_i(x). \quad (34)$$

Then, it is evident that the gauged Hamiltonian \tilde{H}^- defined by

$$\tilde{H}^- = e^{\mathcal{W}} H e^{-\mathcal{W}} \quad (35)$$

preserves the vector space

$$\tilde{\mathcal{V}}_2 = \langle 1, z \rangle. \quad (36)$$

Hence, we have a type A 2-fold supersymmetric system (19) and (20) if we follow the procedure described in the previous section, with the specific choices of $z(x)$, $\mathcal{W}(z)$ and \tilde{H}^- given by Eqs. (34) and (35). Therefore, all the models constructed from second-order Darboux transformations with two exact solutions, including those in Refs. [23, 24, 25, 26, 27], belong to type A 2-fold supersymmetry. We note that we have not assumed whether or not the original Hamiltonian H is Hermitian, \mathcal{PT} -symmetric, pseudo-Hermitian, and so on.

² In this respect, we recall the important fact that type A \mathcal{N} -fold supersymmetry with $\mathcal{N} = 2$ is special due to the lack of the condition $d^5 A(z)/dz^5 = 0$ [20, 21]. As a consequence, type A 2-fold supersymmetric models are more general than the $\mathfrak{sl}(2)$ Lie-algebraic quasi-solvable models in Ref. [22].

In fact, with this procedure we can obtain all the nonlinear pseudo-supersymmetric models in Ref. [9]. Another point worth mentioning is that the gauged Hamiltonian (35) must be diagonal in the basis (36) because of the assumption (33) and the choice (34). It follows that the function $B(z)$ calculated from $A(z) = \dot{z}^2/2$ and $\mathcal{W}(z)$ via the relation (23) must be proportional to z , which results in $b_2 = b_0 = 0$ in Eq. (17). This is the underlying reason why the nonlinear pseudo-supersymmetric models of Ref. [9] always emerge when $b_2 = b_0 = 0$ in our previous arguments. This observation also indicates that the framework of Darboux transformations of order \mathcal{N} based on \mathcal{N} eigenfunctions is in general more restrictive than the framework of \mathcal{N} -fold supersymmetry, for arbitrary integer $\mathcal{N} > 2$.

VI. CONCLUDING REMARKS

One of the most important lessons drawn from the above results is the recognition of the different characters of symmetries. The realization of \mathcal{N} -fold supersymmetry, including ordinary one, in terms of linear differential operators is essentially *local*, in the sense that it is solely characterized by pointwise relations through a superalgebra. That is exactly the reason why a couple of significant aspects of \mathcal{N} -fold supersymmetry has an intimate relation with other local concepts such as quasi-solvability [18] and transposition symmetry [10]. It was shown [28] that the relation among these local concepts is also crucial in another realization of \mathcal{N} -fold supersymmetry for von Roos operators [29]. Higher-order Darboux transformations also make sense at the local level. On the other hand, the concepts of Hermiticity, pseudo-Hermiticity, and so on are *global*, in the sense that they make sense rigorously only when they are formulated in a Hilbert space which encodes global properties such as the domain of operators, boundary conditions, and so on.

For a given Hilbert space, any (pseudo-)Hermitian operator defined on it inevitably has a particular form. That is, any (pseudo-)Hermitian linear differential operator defined on $L^2(S)$ ($S \subset \mathbb{R}$) must be formally (pseudo-)Hermitian. Hence, we can discuss whether or not \mathcal{N} -fold supersymmetric linear differential operators can be in addition formally Hermitian, \mathcal{PT} -symmetric, or formally pseudo-Hermitian at the local level without referring to a Hilbert space. If it is the case, the system can possess both of these characteristic features. For instance, a system which is \mathcal{N} -fold supersymmetric and formally Hermitian as well is weakly quasi-solvable and, if there is a self-adjoint extension on a suitable Hilbert space $L^2(S)$, its eigenvalues are all real. What the authors of Ref. [9] have achieved is exactly that they constructed a few 2-fold supersymmetric Schrödinger operators which are \mathcal{PT} -symmetric as well. Needless to say, this does not mean that they generalized the framework of \mathcal{N} -fold supersymmetry.

Regarding the relation between \mathcal{PT} symmetry and pseudo-Hermiticity, on the other hand, much more care must be exercised. This is because an eigenvalue problem of a \mathcal{PT} -symmetric operator is often defined on a complex contour rather than on the real line. Due to this fact, a \mathcal{PT} -symmetric linear differential operator which is formally pseudo-Hermitian as well need not share the properties of pseudo-Hermitian operators (provided that they are rigorously justified) when the eigenvalue problem is set for it. In this respect, there was an attempt to map \mathcal{PT} -symmetric eigenvalue problems on a complex contour to those on the real line [30]. However, the method in Ref. [30] needs the knowledge that the \mathcal{PT} symmetry of the system is unbroken, and thus would hardly apply in the general situation where we cannot know *a priori* whether or not \mathcal{PT} symmetry is dynamically broken.

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