# Higher order generalized Euler characteristics and generating series *

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**Abstract**

For a complex quasi-projective manifold with a finite group action, we define higher order generalized Euler characteristics with values in the Grothendieck ring of complex quasi-projective varieties extended by the rational powers of the class of the affine line. We compute the generating series of generalized Euler characteristics of a fixed order of the Cartesian products of the manifold with the wreath product actions on them.

Let $X$ be a topological space (good enough, say, a quasi-projective variety) with an action of a finite group $G$. For a subgroup $H$ of $G$, let $X^H = \{x \in X : Hx = x\}$ be the fixed point set of $H$. The orbifold Euler characteristic $\chi^{orb}(X, G)$ of the $G$-space $X$ is defined, e.g., in [1], [10]:

$$\chi^{orb}(X,G) = \frac{1}{|G|} \sum_{\substack{(g_0,g_1)\in G\times G:\\ g_0g_1=g_1g_0}} \chi(X^{\langle g_0,g_1\rangle}) = \sum_{[g]\in G_*} \chi(X^{\langle g\rangle}/C_G(g))\,, \qquad (1)$$

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where $G_*$ is the set of conjugacy classes of elements of $G$, $C_G(g) = \{h \in G : h^{-1}gh = g\}$ is the centralizer of $g$, and $\langle g \rangle$ and $\langle g_0, g_1 \rangle$ are the subgroups generated by the corresponding elements.

The higher order Euler characteristics of $(X, G)$ (alongside with some other generalizations) were defined in [3], [13].

**Definition**: The *Euler characteristic* $\chi^{(k)}(X, G)$ *of order* $k$ of the $G$-space $X$ is

$$\chi^{(k)}(X, G) = \frac{1}{|G|} \sum_{\substack{\mathbf{g} \in G^{k+1}: \\ g_i g_j = g_j g_i}} \chi(X^{\langle \mathbf{g} \rangle}) = \sum_{[g] \in G_*} \chi^{(k-1)}(X^{\langle g \rangle}, C_G(g)), \qquad (2)$$

where $\mathbf{g} = (g_0, g_1, \ldots, g_k)$, $\langle \mathbf{g} \rangle$ is the subgroup generated by $g_0, g_1, \ldots, g_k$, and $\chi^{(0)}(X, G)$ is defined as $\chi(X/G)$.

The usual orbifold Euler characteristic $\chi^{orb}(X, G)$ is the Euler characteristic of order 1, $\chi^{(1)}(X, G)$.

The higher order generalized Euler characteristics takes values in the Grothendieck ring of complex quasi-projective varieties extended by the rational powers of the class of the affine line. Let $K_0(\mathrm{Var}_{\mathbb{C}})$ be the Grothendieck ring of complex quasi-projective varieties. This is the abelian group generated by the isomorphism classes $[X]$ of quasi-projective varieties modulo the relation:
— if $Y$ is a Zariski closed subvariety of $X$, then $[X] = [Y] + [X \setminus Y]$.
The multiplication in $K_0(\mathrm{Var}_{\mathbb{C}})$ is defined by the Cartesian product. The class $[X]$ of a variety $X$ is the universal additive invariant of quasi-projective varieties and can be regarded as a generalized Euler characteristic of $X$. Let $\mathbb{L}$ be the class $[\mathbb{A}^1_{\mathbb{C}}]$ of the affine line and let $K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{1/m}]$ be the extension of the Grothendieck ring $K_0(\mathrm{Var}_{\mathbb{C}})$ by all the rational powers of $\mathbb{L}$.

The formula for the generating series of the generalized orbifold Euler characteristics of the pairs $(X^n, G_n)$ in [9] uses the (natural) power structure over the Grothendieck ring $K_0(\mathrm{Var}_{\mathbb{C}})$ (and over $K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{1/m}]$) defined in [7]. (See also [8]and [9] for some generalizations of this concept.) This means that for a power series $A(T) \in 1 + t \cdot R[[t]]$ ($R = K_0(\mathrm{Var}_{\mathbb{C}})$ or $K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{1/m}]$) and for an element $m \in R$ there is defined a series $(A(T))^m \in 1 + t \cdot R[[t]]$ so that all the properties of the exponential function hold. For a quasi-projective variety $M$, the series $(1-t)^{-[M]}$ is the Kapranov zeta-function of $M$: $\zeta_{[M]}(t) := (1-t)^{-[M]} = 1 + [M] \cdot t + [\mathrm{Sym}^2 M] \cdot t^2 + [\mathrm{Sym}^3 M] \cdot t^3 + \ldots$, where $\mathrm{Sym}^k M = M^k / S_k$ is the $k$-th symmetric power of the variety $M$. A geometric description of the power structure over the over the Grothendieck ring $K_0(\mathrm{Var}_{\mathbb{C}})$ is given in [7] or [9]. The (natural) power structures over $K_0(\mathrm{Var}_{\mathbb{C}})$ and over $K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{1/m}]$ possess the following properties:

1) $(A(t^s))^m = (A(t))^m\,|_{t\mapsto t^s}$ ;

2) $(A(\mathbb{L}^s t))^m = (A(t))^{\mathbb{L}^s m}$ .

One can define a power structure over the ring $\mathbb{Z}[u_1, \ldots, u_r]$ of polynomials in $r$ variables with integer coefficients in the following way. Let $P(u_1, \ldots, u_r) = \sum\limits_{\underline{k}\in\mathbb{Z}^r_{\geq 0}} p_{\underline{k}}\underline{u}^{\underline{k}} \in \mathbb{Z}[u_1, \ldots, u_r]$, where $\underline{k} = (k_1, \ldots, k_r)$, $\underline{u} = (u_1, \ldots, u_r)$, $\underline{u}^{\underline{k}} = u_1^{k_1}\cdot\ldots\cdot u_r^{k_r}$, $p_{\underline{k}} \in \mathbb{Z}$. Define

$$(1-t)^{-P(u_1,\ldots,u_r)} := \prod_{\underline{k}\in\mathbb{Z}^r_{\geq 0}} (1-\underline{u}^{\underline{k}}t)^{-p_{\underline{k}}},$$

where the power (with an integer exponent $-p_{\underline{k}}$) means the usual one. This gives a $\lambda$-structure on the ring $\mathbb{Z}[u_1, \ldots, u_r]$ and therefore a power structure over it (see, e.g., [9, Proposition 1])

i.e., for polynomials $A_i(\underline{u})$, $i \geq 1$, and $M(\underline{u})$, there is defined a series $(1 + A_1(\underline{u})t + A_2(\underline{u})t^2 + \ldots)^{M(\underline{u})}$ with the coefficients from $\mathbb{Z}[u_1, \ldots, u_r]$.

Let $r = 2$, $u_1 = u$, $u_2 = v$. Let $e : K_0(\mathrm{Var}_{\mathbb{C}}) \to \mathbb{Z}[u, v]$ be the ring homomorphism which sends the class $[X]$ of a quasi-projective variety $X$ to its Hodge–Deligne polynomial $e(X; u, v) = \sum h_X^{ij}(-u)^i(-v)^j$.

**Remark**. Let $R_1$ and $R_2$ be rings with power structures over them. A ring homomorphism $\varphi : R_1 \to R_2$ induces the natural homomorphism $R_1[[t]] \to R_2[[t]]$ (also denoted $\varphi$) by $\varphi\left(\sum a_i\, t^i\right) = \sum \varphi(a_i)\, t^i$. In [9, Proposition 2], it was shown that if a ring homomorphism $\varphi : R_1 \to R_2$ is such that $(1-t)^{-\varphi(m)} = \varphi\left((1-t)^{-m}\right)$ for any $m \in R$, then $\varphi\left((A(t))^m\right) = \left(\varphi\left(A(t)\right)\right)^{\varphi(m)}$ for $A(\underline{t}) \in 1 + tR[[t]]$, $m \in R$.

There are two natural homomorphism from the Grothendieck ring $K_0(\mathrm{Var}_{\mathbb{C}})$ to the ring $\mathbb{Z}$ of integers and to the ring $\mathbb{Z}[u, v]$ of polynomials in two variables: the Euler characteristic (with compact support) $\chi : K_0(\mathrm{Var}_{\mathbb{C}}) \to \mathbb{Z}$ and the Hodge–Deligne polynomial. Both possesses the following well known identities:

(1) the formula of I.G. Macdonald [12]:

$$\chi(1 + [X]t + [\mathrm{Sym}^2 X]t^2 + [\mathrm{Sym}^3 X]t^3 + \ldots) = (1-t)^{-\chi(X)},$$

(2) and the corresponding formula for the Hodge–Deligne polynomial (see [4, Proposition 1.2]):

$$e(1+[X]t+[\mathrm{Sym}^2X]t^2+\ldots) = (1-T)^{-e(X;u,v)} = \prod_{p,q}\left(\frac{1}{1-u^pv^qt}\right)^{e^{p,q}(X)}.$$

These properties and the previous remark imply that the corresponding homomorphisms respect the power structures over the corresponding rings: $K_0(\mathrm{Var}_{\mathbb{C}})$ and $\mathbb{Z}[u,v]$ respectively, see [8].

A generalization of the orbifold Euler characteristic to the orbifold (or stringly) Hodge numbers and the orbifold Hodge–Deligne polynomial (for an action of a finite group $G$ on a non-singular quasi-projective variety $X$) was defined in [5], [15], [2].

Let $X$ be a smooth quasi-projective variety of dimension $d$ with an (algebraic) action of the group $G$. For $g \in G$, the centralizer $C_G(g)$ of $g$ acts on the manifold $X^{\langle g\rangle}$ of fixed points of the element $g$. Suppose that its action on the set of connected components of $X^{\langle g\rangle}$ has $N_g$ orbits, and let $X_1^{\langle g\rangle}$, $X_2^{\langle g\rangle}$, ..., $X_{N_g}^{\langle g\rangle}$ be the unions of the components of each of the orbits. At a point $x \in X_{\alpha_g}^{\langle g\rangle}$, $1 \le \alpha_g \le N_g$, the differential $dg$ of the map $g$ is an automorphism of finite order of the tangent space $T_xX$. Its action on $T_xX$ can be represented by a diagonal matrix $\mathrm{diag}(\exp(2\pi i\theta_1),\ldots,\exp(2\pi i\theta_d))$ with $0 \le \theta_j < 1$ for $j = 1,2,\ldots,d$ ($\theta_j$ are rational numbers). The *shift number* $F_{\alpha_g}^g$ associated with $X_{\alpha_g}^{\langle g\rangle}$ is $F_{\alpha_g}^{\langle g\rangle} = \sum_{j=1}^d \theta_j \in \mathbb{Q}$. (It was introduced in [15].)

**Definition**: The *generalized orbifold Euler characteristic* of the pair $(X,G)$ (see [9]) is

$$[X,G] = \sum_{[g]\in G_*}\sum_{\alpha_g=1}^{N_g}[X_{\alpha_g}^{\langle g\rangle}/C_G(g)]\cdot\mathbb{L}^{F_{\alpha_g}^g} \in K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{1/m}]\,. \tag{3}$$

Since the Euler characteristic and the Hodge–Deligne polynomial are additvie invariants they factor through $K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{1/m}]$ and the Euler characteristic morphisms sends $[X,G]$ to the orbifold Euler characteristic $\chi^{orb}(X,G)$. The Hodge–Deligne polynomial morphism sends it to the orbifold Hodge–Deligne polynomial from [2], [14].

Let $G^n = G\times\ldots\times G$ be the Cartesian power of the group $G$. The symmetric group $S_n$ acts on $G^n$ by permutation of the factors: $s(g_1,\ldots,g_n) =$

$(g_{s^{-1}(1)}, \ldots, g_{s^{-1}(n)})$. The *wreath product* $G_n = G \wr S_n$ is the semidirect product of the groups $G^n$ and $S_n$ defined by the described action. Namely the multiplication in the group $G_n$ is given by the formula $(\mathbf{g}, s)(\mathbf{h}, t) = (\mathbf{g} \cdot s(\mathbf{h}), st)$, where $\mathbf{g}$, $\mathbf{h} \in G^n$, $s$, $t \in S_n$. The group $G^n$ is a normal subgroup of the group $G_n$ via the identification of $\mathbf{g} \in G^n$ with $(\mathbf{g}, 1) \in G_n$. For a variety $X$ with a $G$-action, there is the corresponding action of the group $G_n$ on the Cartesian power $X^n$ given by the formula

$$((g_1, \ldots, g_n), s)(x_1, \ldots, x_n) = \left(g_1 x_{s^{-1}(1)}, \ldots, g_n x_{s^{-1}(n)}\right),$$

where $x_1, \ldots, x_n \in X$, $g_1, \ldots, g_n \in G$, $s \in S_n$. One can see that the quotient $X^n/G_n$ is naturally isomorphic to the space $\mathrm{Sym}^n(X/G) = (X/G)^n/S_n$. In particular, in the Grothendieck ring of complex quasi-projective varieties one has $[X^n/G_n] = [(X/G)^n/S_n] = [\mathrm{Sym}^n(X/G)]$.

A formula for the generating series of the $k$-th order Euler characteristics of the pairs $(X^n, G_n)$ in terms of the $k$-th order Euler characteristics of the $G$-space $X$ was given in [13] (see also [3]).

The generating series of the orbifold Hodge–Deligne polynomials $e(X^n, G_n; u, v)$ of the pairs $(X^n, G_n)$ was computed in [14].

A reformulation of the result of [14] in terms of the generalized orbifold Euler characteristic with values in $K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{1/m}]$ was given in [9]. Using properties of the power structure one has ([9, Theorem 4]):

$$\sum_{n \geq 0} [X^n, G_n] t^n = \left(\prod_{r=1}^{\infty} (1 - \mathbb{L}^{(r-1)d/2} t^r)\right)^{-[X,G]}. \tag{4}$$

Here we define higher order generalized Euler characteristics of a pair $(X, G)$ (with $X$ non-singular) and give a formula for the generating series of the $k$-th order generalized Euler characteristic of the pairs $(X^n, G_n)$.

Before giving the definition of the higher order generalized Euler characteristic of a pair $(X, G)$ we discuss some versions of the definition (3) and of the equation (4).

For a $G$-variety $X$ (not necessarily non-singular) its *inertia stack* (or rather *class*) $I(X, G)$ is defined by

$$I(X, G) := \sum_{[g] \in G_*} [X^g / C_G(g)] \tag{5}$$

(see e.g. [11], [6]). One can see that it is an analogue of the generalized orbifold Euler characteristic (3) without the shift factor $\mathbb{L}^{F^{\langle g\rangle}_{\alpha_g}}$. This inspires the following version of the definition (3).

**Definition**: For a rational number $\varphi_1$, let

$$[X,G]_{\varphi_1} := \sum_{[g]\in G_*}\sum_{\alpha_g=1}^{N_g}[X^{\langle g\rangle}_{\alpha_g}/C_G(g)]\cdot \mathbb{L}^{\varphi_1 F^{\langle g\rangle}_{\alpha_g}} \in K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{1/m}]\,. \qquad (6)$$

That is the Zaslow shift $F^{\langle g\rangle}_{\alpha_g}$ is multiplied by $\varphi_1$. For $\varphi_1 = 1$ one gets the generalized Euler characteristic $[X,G]$ from (3), for $\varphi_1 = 0$ one gets the inertia class $I(X,G)$. The arguments from [9] easily give the following version of Equation (4).

**Proposition 1**

$$\sum_{n\geq 0}[X^n, G_n]_{\varphi_1} t^n = \left(\prod_{r=1}^{\infty}(1-\mathbb{L}^{\varphi_1(r-1)d/2}t^r)\right)^{-[X,G]}. \qquad (7)$$

Thus multiplication of Zaslow's shift by a number (at least by 1 or 0) makes sense. For the corresponding definition of the higher order generalized Euler characteristic one can use factors $\varphi_k$ depending on the order of the Euler characteristic.

Let $X$ be a non-singular $d$-dimensional quasi-projective variety with a $G$ action and let $\underline{\varphi} = (\varphi_1, \varphi_2, \ldots)$ be a fixed sequence of rational numbers. We use the notations introduced before (3).

**Definition**: The *generalized orbifold Euler characteristic of order* $k$ of the pair $(X,G)$ is

$$[X,G]^k_{\underline{\varphi}} := \sum_{[g]\in G_*}\sum_{\alpha_g=1}^{N_g}[X^{\langle g\rangle}_{\alpha_g}, C_G(g)]^{k-1}_{\underline{\varphi}}\cdot \mathbb{L}^{\varphi_k F^{\langle g\rangle}_{\alpha_g}} \in K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{1/m}]\,, \qquad (8)$$

where $[X,G]^1_{\underline{\varphi}} := [X,G]_{\varphi_1}$ is the (modified) generalized orbifold Euler characteristic given by (6).

**Remark**. The definition (2) (as well as (1)) contains two equivalent versions. One can say that here we formulate an analogues of the second one. A formula analogous to the first one (with the factor $\frac{1}{|G|}$ in front) cannot work directly, at least without tensoring the ring $K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{1/m}]$ by the field $\mathbb{Q}$ of rational numbers. Moreover, it seems that there is no analogue of Theorem 1 in terms of the power structure. This gives the hint that a definition of this sort makes small geometric sense (if any).

Taking the Euler characteristic, one gets $\chi([X,G]^k_{\underline{\varphi}}) = \chi^{(k)}(X,G)$.

To prove the formula for the generating series of $[X^n, G_n]^k_{\underline{\varphi}}$, we will use some technical statements.

**Lemma 1**

$$[X' \times X'', G' \times G'']^k_{\underline{\varphi}} = [X', G']^k_{\underline{\varphi}} \times [X'', G'']^k_{\underline{\varphi}}\,. \tag{9}$$

The proof is obvious.

Let $X_1$ and $X_2$ be two $G$-manifolds and let $X_1^m \times X_2^{n-m}$ be embedded into $(X_1 \coprod X_2)^n$ in the natural way: a pair of elements $(x_{1,1}, \ldots, x_{1,m}) \in X_1^m$ and $(x_{2,1}, \ldots, x_{2,n-m}) \in X_2^{n-m}$ is identified with $(x_{1,1}, \ldots, x_{1,m}, x_{2,1}, \ldots, x_{2,n-m}) \in (X_1 \coprod X_2)^n$. Let $\mathrm{Sym}^n(X_1^m \times X_2^{n-m})$ be the orbit of $X_1^m \times X_2^{n-m}$ under the $S_n$-action on $(X_1 \coprod X_2)^n$. The wreath product $G_n$ acts on $\mathrm{Sym}^n(X_1^m \times X_2^{n-m})$.

**Lemma 2**

$$[\mathrm{Sym}^n(X_1^m \times X_2^{n-m}), G_n]^k_{\underline{\varphi}} = [X_1^m, G_m]^k_{\underline{\varphi}} \times [X_2^{n-m}, G_{n-m}]^k_{\underline{\varphi}}\,. \tag{10}$$

**Proof**. An element $(\mathbf{g}, s) \in G_n$ has fixed points on $\mathrm{Sym}^n(X_1^m \times X_2^{n-m})$ if and only if it is conjugate to an element $(\mathbf{g}', s') \in G_n$ such that $s' = (s_1, s_2) \in S_m \times S_{n-m} \subset S_n$ and the element $(\mathbf{g}', s') = ((\mathbf{g}_1, \mathbf{g}_2), (s_1, s_2))$ has fixed points on $X_1^m \times X_2^{n-m}$ (and only on it). The centralizer of the element $(\mathbf{g}', s')$ is $C_{G_m}((g_1, s_1)) \times C_{G_{n-m}}((g_2, s_2))$. The components of $(X_1^m \times X_2^{n-m})^{\langle (\mathbf{g}', s')\rangle}$ are the products $(X_1^m)_\alpha^{\langle (\mathbf{g}_1, s_1)\rangle} \times (X_2^{n-m})_\beta^{\langle (\mathbf{g}_2, s_2)\rangle}$ of the components of $(X_1^m)^{\langle (\mathbf{g}_1, s_1)\rangle}$

and $(X_2^{n-m})^{\langle(\mathbf{g}_2,s_2)\rangle}$. The shift $F_{\alpha\beta}^{(\mathbf{g}',s')}$ is equal to $F_\alpha^{(\mathbf{g}_1,s_1)}+F_\beta^{(\mathbf{g}_2,s_2)}$. Therefore

$$\begin{aligned}
& [\mathrm{Sym}^n(X_1^m\times X_2^{n-m}),G_n]^k_{\underline{\varphi}} \\
= & \sum_{[(\mathbf{g}',s')]}\sum_{\alpha\beta}[(X_1^m\times X_2^{n-m})_{\alpha\beta}^{(\mathbf{g}',s')},C_{G_m}((\mathbf{g}_1,s_1))\times C_{G_{n-m}}((\mathbf{g}_2,s_2))]^{k-1}_{\underline{\varphi}}\cdot\mathbb{L}^{(F_\alpha^{(\mathbf{g}_1,s_1)}+F_\beta^{(\mathbf{g}_2,s_2)})} \\
= & \sum_{[(\mathbf{g}_1,s_1)]}\sum_{\alpha}[(X_1^m)_\alpha^{(\mathbf{g}_1,s_1)},C_{G_m}((\mathbf{g}_1,s_1))]^{k-1}_{\underline{\varphi}}\cdot\mathbb{L}^{F_\alpha^{(\mathbf{g}_1,s_1)}}\times \\
& \sum_{[(\mathbf{g}_2,s_2)]}\sum_{\beta}[(X_1^{n-m})_\beta^{(\mathbf{g}_2,s_2)},C_{G_{n-m}}((\mathbf{g}_2,s_2))]^{k-1}_{\underline{\varphi}}\cdot\mathbb{L}^{F_\beta^{(\mathbf{g}_2,s_2)}} \\
= & [X_1^m,G_m]^k_{\underline{\varphi}}\times[X_2^{n-m},G_{n-m}]^k_{\underline{\varphi}}\,.
\end{aligned}$$

□

Let $X$ be a $G$-manifold and let $c$ be an element of $G$ acting trivially on $X$. Let $r$ be a fixed positive integer. Denote by $G\cdot\langle a\rangle$ the group generated by $G$ and the additional element $a$ commuting with all the elements of $G$ and such that $\langle a\rangle\cap G=\langle c\rangle$, $c=a^r$. Define the action of the group $G\cdot\langle a\rangle$ on $X$ (an extension of the $G$-action) so that $a$ acts trivially.

**Lemma 3** (cf. [13, Lemma 4-1]) *In the described situation one has*

$$[X,G\cdot\langle a\rangle]^k_{\underline{\varphi}}=r^k[X,G]^k_{\underline{\varphi}}\,.$$

**Proof**. We shall use the induction on $k$. For $k=0$ this is obvious (since $[X,G]^0_{\underline{\varphi}}=[X/G]$). Each conjugacy class of elements from $G\cdot\langle a\rangle$ is of the form $[g]a^s$, where $[g]\in G_*$, $0\le s<r$. The fixed point set of $ga^s$ coincides with $X^g$, the Zaslow shift $F_\alpha^{ga^s}$ at each component of $X^g$ coincides with $F_\alpha^g$ (since $a$ acts trivially). The centralizer $C_{G\cdot\langle a\rangle}(ga^s)$ is $C_G(g)\cdot\langle a\rangle$. Therefore

$$[X,G\cdot\langle a\rangle]^k_{\underline{\varphi}}=\sum_{[g]\in G_*}r\sum_{\alpha=1}^{N_g}[X_\alpha^g,C_{G(g)}\cdot\langle a\rangle]^{k-1}_{\underline{\varphi}}\cdot\mathbb{L}^{F_\alpha^g}=r^k[X,G]^k_{\underline{\varphi}}\,.$$

□

**Theorem 1** *Let $X$ be a smooth quasi-projective variety of dimension $d$ with a $G$-action. Then*

$$\sum_{n\ge0}[X^n,G_n]^k_{\underline{\varphi}}\cdot t^n=\left(\prod_{r_1,\ldots,r_k\ge1}\left(1-\mathbb{L}^{\Phi_k(\underline{r})d/2}\cdot t^{r_1r_2\cdots r_k}\right)^{r_2r_3^2\cdots r_k^{k-1}}\right)^{-[X,G]^k_{\underline{\varphi}}}, \tag{11}$$

*where*

$$\Phi_k(r_1,\ldots,r_k)=\varphi_1(r_1-1)+\varphi_2 r_1(r_2-1)+\ldots+\varphi_k r_1 r_2\cdots r_{k-1}(r_k-1).$$

**Proof**. To a big extend we shall follow the lines of the proof of Theorem A in [13]. We shall use the induction on the order $k$. For $k=1$ the equation coincides with the one from Proposition 1. Assume that the statement is proved for the generalized Euler characteristic of order $k-1$. One has

$$\sum_{n\geq 0}[X^n,G_n]^k_{\underline{\varphi}}\cdot t^n=\sum_{n\geq 0}t^n\left(\sum_{[(\mathbf{g},s)]\in G_{n*}}\sum_{comp}[(X^n)^{\langle(\mathbf{g},s)\rangle}_{comp},C_{G_n}((\mathbf{g},s))]^{k-1}_{\underline{\varphi}}\cdot\mathbb{L}^{F^{(\mathbf{g},s)}_{comp}}\right),$$

where the sums are over all the conjugacy classes $[(\mathbf{g},s)]$ of elements of $G_n$ and over all the components of $(X^n)^{\langle(\mathbf{g},s)\rangle}$ (or rather unions of components from an orbit of the $C_{G_n}((\mathbf{g},s))$-action on the components of it).

The conjugacy classes $[(\mathbf{g},s)]$ of elements of $G_n$ are characterized by their types. Let $a=(\mathbf{g},s)\in G_n$, $\mathbf{g}=(g_1,\ldots,g_n)$. Let $z=(i_1,\ldots,i_r)$ be one of the cycles in the permutation $s$. The *cycle-product* of the element $a$ corresponding to the cycle $z$ is the product $g_{i_r}g_{i_{r-1}}\ldots g_{i_1}\in G$. The conjugacy class of the cycle-product is well-defined by the element $\mathbf{g}$ and the cycle $z$ of the permutation $s$. For $[c]\in G_*$ and $r\geq 0$, let $m_r(c)$ be the number of $r$-cycles in the permutation $s$ whose cycle-products lie in $[c]$. One has

$$\sum_{[c]\in G_*,r\geq 1} r m_r(c)=n\,.$$

The collection $\{m_r(c)\}_{r,c}$ is called the *type* of the element $a=(\mathbf{g},s)\in G_n$. Two elements of the group $G_n$ are conjugate to each other if and only if they are of the same type.

In [13] (see also [14]) it is shown that, for an element $(\mathbf{g},s)\in G_n$ of type $\{m_r(c)\}$, the subspace $(X^n)^{\langle(\mathbf{g},s)\rangle}$ can be identified with

$$\prod_{[c]\in G_*}\prod_{r\geq 1}(X^{\langle c\rangle})^{m_r(c)}\,. \tag{12}$$

By [13, Theorem 3.5] the centralizer of the element $(\mathbf{g},s)\in G_n$ is isomorphic to

$$\prod_{[c]\in G_*}\prod_{r\geq 1}\left\{(C_G(c)\cdot\langle a_{r,c}\rangle)\wr S_{m_r(c)}\right\}$$

(acting on the product (12) component-wise) where $C_G(c)\cdot\langle a_{r,c}\rangle$ is the group generated by $C_G(c)$ and an element $a_{r,c}$ commuting with all the elements of $C_G(c)$ and such that $a_{r,c}^r = c$, $\langle a_{r,c}\rangle \cap C_G(c) = \langle c\rangle$, and $a_{r,c}$ acts on $(X^{\langle c\rangle})^{m_r(c)}$ trivially.

The components of $(X^{\langle c\rangle})^{m_r(c)}$ (with respect to the $C_G(c)\cdot\langle a_{r,c}\rangle$-action) are $\mathrm{Sym}^{m_{r,c}}\left(\prod_{\alpha=1}^{N_\alpha}(X_\alpha^{\langle c\rangle})^{m_{r,c}(\alpha)}\right)$, where $\sum_{\alpha=1}^{N_\alpha} m_{r,c}(\alpha) = m_r(c)$. Here and bellow the sum over *comp* means the summation over all the components indicated in the summands. Therefore

$$\sum_{n\geq 0}[X^n, G_n]^k_{\underline{\varphi}}\cdot t^n = \sum_{n\geq 0} t^n\left(\sum_{[(\mathbf{g},s)]\in G_{n*}}\sum_{comp}[(X^n)^{\langle(\mathbf{g},s)\rangle}_{comp}, C_{G_n}((\mathbf{g},s))]^{k-1}_{\underline{\varphi}}\cdot\mathbb{L}^{F^{(\mathbf{g},s)}_{comp}}\right)$$

$$= \sum_{n\geq 0} t^n\cdot\left(\sum_{\{m_r(c)\}}\sum_{comp}[\prod_{[c],r}\{(X^{\langle c\rangle})^{m_r(c)}\}_{comp}, \prod_{[c],r}\{(C_G(c)\cdot\langle a_{r,c}\rangle)\wr S_{m_r(c)}\}]^{k-1}_{\underline{\varphi}}\cdot\mathbb{L}^{F^{(\mathbf{g},s)}_{comp}}\right)$$

$$= \sum_{n\geq 0} t^n\cdot\left(\sum_{\{m_{r,c}(\alpha)\}}\{\prod_{[c],r}[\mathrm{Sym}^{m_{r,c}}(\prod_{\alpha=1}^{N_\alpha}(X_\alpha^{\langle c\rangle})^{m_{r,c}(\alpha)}), (C_G(c)\cdot\langle a_{r,c}\rangle)\wr S_{m_r(c)}]^{k-1}_{\underline{\varphi}}\times \mathbb{L}^{\phi_k(\sum_{[c],r}\sum_{\alpha=1}^{N_\alpha} m_{r,c}(\alpha)(F^c_\alpha+\frac{(r-1)d}{2}))}\}\right)$$

Iterating Lemma 2 one gets

$$= \sum_{\{m_{r,c}(\alpha)\}} t^{\sum r m_{r,c}(\alpha)}\prod_{[c],r}\left\{\prod_{\alpha=1}^{N_\alpha}[(X_\alpha^{\langle c\rangle})^{m_{r,c}(\alpha)}, C_G(c)\cdot\langle a_{r,c}\rangle)\wr S_{m_r(c)}]^{k-1}_{\underline{\varphi}}\times \mathbb{L}^{\phi_k\left(\sum_{[c],r}\sum_{\alpha=1}^{N_\alpha} m_{r,c}(\alpha)(F^c_\alpha+\frac{(r-1)d}{2})\right)}\right\}$$

$$= \prod_{[c],r}\left(\prod_{\alpha=1}^{N_\alpha}\left(\sum_{\{m_{r,c}(\alpha)\}} t^{r m_{r,c}(\alpha)}[(X_\alpha^{\langle c\rangle})^{m_{r,c}(\alpha)}, C_G(c)\cdot\langle a_{r,c}\rangle)\wr S_{m_r(c)}]^{k-1}_{\underline{\varphi}}\times \mathbb{L}^{\phi_k\left(\sum_{[c],r}\sum_{\alpha=1}^{N_\alpha} m_{r,c}(\alpha)(F^c_\alpha+\frac{(r-1)d}{2})\right)}\right)\right)$$

By the induction one gets

$$= \prod_{[c],r}\prod_{\alpha=1}^{N_\alpha}\left(\prod_{r_1,\ldots,r_{k-1}\geq 1}\left(1-\mathbb{L}^{\Phi_{k-1}(r)\frac{d}{2}}(\mathbb{L}^{\varphi_k(F_\alpha^c+\frac{(r-1)d}{2})}t^r)^{r_1\cdots r_{k-1}}\right)^{r_2\cdot r_3^2\cdots r_{k-1}^{k-2}}\right)^{-[X_\alpha^{\langle c\rangle},C_G(c)\cdot\langle a_{r,c}\rangle]_{\underline{\varphi}}^{k-1}}$$

$$=\left(\prod_{r,r_1,\ldots,r_{k-1}\geq 1}\left(1-\mathbb{L}^{\Phi_{k-1}(r)\frac{d}{2}}\mathbb{L}^{\varphi_k(r_1\cdots r_{k-1}\frac{(r-1)d}{2})}t^{r_1\cdots r_{k-1}\cdot r}\right)^{r_2\cdot r_3^2\cdots r_{k-1}^{k-2}}\right)^{-\sum\limits_{[c],\alpha}[X_\alpha^{\langle c\rangle},C_G(c)\cdot\langle a_{r,c}\rangle]_{\underline{\varphi}}^{k-1}\mathbb{L}^{\phi_k F_\alpha^c}}$$

(Here we use the properties of the power structure.)

$$= \left(\prod_{r_1,\ldots,r_k\geq 1}\left(1-\mathbb{L}^{(\Phi_{k-1}(r)+\varphi_k r_1\cdots r_{k-1}(r_k-1))\frac{d}{2}}t^{r_1\cdots r_{k-1}\cdot r_k}\right)^{r_2\cdot r_3^2\cdots r_{k-1}^{k-2}}\right)^{-r_k^{k-1}\sum\limits_{[c],\alpha}[X_\alpha^{\langle c\rangle},C_G(c)]_{\underline{\varphi}}^{k-1}\mathbb{L}^{\phi_k F_\alpha^c}}$$

$$= \left(\prod_{r_1,\ldots,r_k\geq 1}\left(1-\mathbb{L}^{\Phi_k(\underline{r})d/2}t^{r_1 r_2\cdots r_k}\right)^{r_2 r_3^2\cdots r_k^{k-1}}\right)^{-[X,G]_{\underline{\varphi}}^k}.$$

In the last two equations $r$ is substituted by $r_k$. □

**Remark**. For $\underline{\varphi}=\underline{0}$, i.e. if $\varphi_i=0$ for all $i$, the definition of the higher order generalized Euler characteristics does not demand $X$ to be smooth. This way one gets the definition of a sort of higher order inertia classes and the statement of Theorem 1 holds for an arbitrary $G$-variety $X$.

Since $\chi([X^n,G_n]_{\underline{\varphi}}^k)=\chi^{(k)}(X,G)$, $\chi(\mathbb{L})=1$, taking the Euler characteristic of the both sides of the equation (11) one gets Theorem A of [13]:

$$\sum_{n\geq 0}\chi^{(k)}(X^n,G_n)\cdot t^n=\left(\prod_{r_1,\ldots,r_k\geq 1}(1-t^{r_1 r_2\cdots r_k})^{r_2 r_3^2\cdots r_k^{k-1}}\right)^{-\chi^{(k)}(X,G)}.$$

Let $e_{\underline{\varphi}}^{(k)}(X,G;u,v):=e([X,G]_{\underline{\varphi}}^k;u,v)$ be the higher order Hodge–Deligne polynomial of $(X,G)$ (of order $k$). Applying the Hodge–Deligne polynomial homomorphism, one gets a generalization of the main result in [14]:

$$\sum_{n\geq 0}e_{\underline{\varphi}}^{(k)}(X^n,G_n;u,v)\cdot t^n=\left(\prod_{r_1,\ldots,r_k\geq 1}\left(1-(uv)^{\Phi_k(\underline{r})d/2}\cdot t^{r_1 r_2\cdots r_k}\right)^{r_2 r_3^2\cdots r_k^{k-1}}\right)^{-e_{\underline{\varphi}}^{(k)}(X,G;u,v)}.$$

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