Probability distributions for the phase difference

A. Luis and L. L. Sánchez-Soto
Departamento de Optica, Facultad de Ciencias Físicas, Universidad Complutense, 28040 Madrid, Spain
(Received 30 November 1994; revised manuscript received 21 June 1995)

In this work we analyze the quantum phase properties of pairs of electromagnetic field modes. Since phases differing by 2π are physically indistinguishable, we propose a general procedure to obtain the correct mod(2π) probability distributions for the phase difference. This allows us to investigate the properties of a number of phase approaches. This procedure provides deeper insight into the quantum nature of the phase difference. We relate this problem to the representation of nonbijective canonical transformations in quantum mechanics.

PACS number(s): 42.50.Dv, 03.65.−w

I. INTRODUCTION

The problem of quantum phase fluctuations of optical fields has a long history and has provoked many discussions [1]. There have been many attempts to properly introduce a satisfactory description of phase and significant progress has been achieved in the last fews years in clarifying the status of the quantum phase operator (for latest reviews see Refs. [2,3]).

Most of this previous work has been devoted to the properties of the phase for a single-mode field or, equivalently, for a single-harmonic oscillator. The more relevant conclusion is that there is no such phase operator, at least verifying simultaneously a polar decomposition, self-adjointness, and adequate commutation relations. This has allowed the introduction of several approaches depending on which of those criteria should be fulfilled [4].

Although the definition of the absolute phase is, alone, an interesting problem, from a practical point of view an absolute phase has no meaning. Since in real measurements we are always forced to deal with the difference with respect to a reference phase, the phase difference should be the fundamental quantity in describing the optical phase. It is worth emphasizing that the absence of a proper phase operator for a single mode is usually ascribed to the semiboundedness of the eigenvalue spectrum of the number operator. However, the conjugate variable to the phase difference is the number difference, that is not bounded from below. So, it is reasonable to expect that the phase difference will be free of the problems arising in the one-mode case.

Taking this into account, two different ways to describe the phase difference emerge. One focuses on the phase-difference variable itself, without any previous assumption about the absolute phases. We have pursued this issue in previous works [5–7], showing that the polar decomposition corresponding to the phase difference has a unitary solution, in contrast with the polar decomposition for the absolute phase. In fact, the solution has interesting commutation relations with the number difference [8].

The other way to proceed is to describe the phase difference in terms of previously introduced phase operators for the two systems. This approach is faced with an interesting difficulty. Due to its periodic character, adding and subtracting phases must be done carefully. The eigenvalue spectra of the sum and difference operators have widths of 4π, and this is not compatible with the idea that the phase must be 2π periodic. Thus there should be a way to cast the phase sum and difference into the 2π range. Such a casting procedure was proposed by Barnett and Pegg [9].

We should emphasize that although the 4π and 2π probability distributions are both valid, they give different values for the variances. The former explicitly reveals the existence of correlations between single-mode phases, while the latter is easier to interpret because in it the phase sum or difference is a single-valued variable.

Our aim here is to obtain a casting procedure more suited for a clear analysis of its implications and a comparison with the phase-difference operator arising directly from the polar decomposition.

The plan of this paper is as follows. In Sec. II we first consider the enlightening subject of the angle difference. In this case we have the same kind of problems linked to the periodicity, but with the advantage of having an angle operator. This fact makes the special behavior of the transformation to the angle sum and difference more transparent and allows an easier translation to the very similar phase problem. It also shows the nonbijective character of this transformation and the particularities that this fact introduces.

In Sec. III we apply these conclusions to the Pegg-Barnett formalism and in Sec. IV to the Q function. The consequences of the casting procedure are then discussed and compared.

II. ANGLE SUM AND DIFFERENCE

We begin our discussion with a brief description of the problem of the angle difference. For a system described by an angular momentum component L_z (like a particle constrained to move on a circle or a one-dimensional system obeying periodic conditions) the exponential of the angle $E=e^{i\varphi}$ is given by the lowering operator [10]

$$E|m\rangle=|m-1\rangle, \tag{2.1}$$

where $|m\rangle$ are the eigenvectors of L_z, the integer m running from $-\infty$ to $+\infty$.

This unitary operator E verifies the following commutation relation with L_z [11]:

1050-2947/96/53(1)/495(7)/$06.00

$$[E,L_z]=E, \tag{2.2}$$

and its eigenvectors are

$$|\varphi\rangle=\frac{1}{\sqrt{2\pi}}\sum_{m=-\infty}^{+\infty} e^{im\varphi}|m\rangle, \tag{2.3}$$

with $E|\varphi\rangle=e^{i\varphi}|\varphi\rangle$. This operator E, like L_z, is itself a complete set of commuting operators for the system.

When we have two such systems, labeled 1 and 2, the exponentials of the angle sum E_+ and angle difference E_- are

$$E_+=E_1E_2, \quad E_-=E_1E_2^\dagger. \tag{2.4}$$

These unitary operators verify the commutation relations

$$\left[E_-,\frac{L_{1z}+L_{2z}}{2}\right]=0, \quad \left[E_+,\frac{L_{1z}+L_{2z}}{2}\right]=E_+,$$

$$\left[E_-,\frac{L_{1z}-L_{2z}}{2}\right]=E_-, \quad \left[E_+,\frac{L_{1z}-L_{2z}}{2}\right]=0, \tag{2.5}$$

showing that their canonically conjugate variables are the angular momentum sum and difference. The eigenvectors of E_+ and E_- are of the form $|\varphi_1,\varphi_2\rangle$ with eigenvalues $e^{i\varphi_+}=e^{i(\varphi_1+\varphi_2)}$ and $e^{i\varphi_-}=e^{i(\varphi_1-\varphi_2)}$, respectively.

Note that while (E_1,E_2), (L_{1z},L_{2z}), or $(L_{1z}+L_{2z},L_{1z}-L_{2z})$ are complete sets of commuting operators, this is not true for (E_+,E_-), since $|\varphi_1,\varphi_2\rangle$ and $|\varphi_1+\pi,\varphi_2+\pi\rangle$ have the same angle sum and difference.

Therefore another commuting operator must be considered to describe the system. We propose to use even and odd combinations of $|\varphi_1,\varphi_2\rangle$ and $|\varphi_1+\pi,\varphi_2+\pi\rangle$ to solve the degeneracy. So, we can take

$$|\varphi_+,\varphi_-,p\rangle=\frac{e^{-ip\varphi_1}}{2}[|\varphi_1,\varphi_2\rangle+(-1)^p|\varphi_1+\pi,\varphi_2+\pi\rangle], \tag{2.6}$$

with $p=0,1$; and

$$\varphi_1=\frac{\varphi_++\varphi_-}{2}, \quad \varphi_2=\frac{\varphi_+-\varphi_-}{2}, \tag{2.7}$$

and define an operator Π,

$$\Pi|\varphi_+,\varphi_-,p\rangle=p|\varphi_+,\varphi_-,p\rangle. \tag{2.8}$$

This operator, together with E_+ and E_-, gives a complete set of commuting operators. The associated basis is (2.6), allowing the resolution of the identity

$$I=\sum_p \int\int d\varphi_+ d\varphi_- |\varphi_+,\varphi_-,p\rangle\langle\varphi_+,\varphi_-,p|, \tag{2.9}$$

where φ_+ and φ_- run over 2π intervals.

To obtain the probability distribution function $\mathscr{P}$ cast into a 2π range for the angle sum and difference associated with a system state ρ we must add the contributions from each p value,

$$\mathscr{P}(\varphi_+,\varphi_-)=\sum_{p=0,1}\langle\varphi_+,\varphi_-,p|\rho|\varphi_+,\varphi_-,p\rangle. \tag{2.10}$$

Taking into account (2.6) and (2.7) we can express $\mathscr{P}(\varphi_+,\varphi_-)$ in terms of the probability distribution for the individual angles $P(\varphi_1,\varphi_2)=\langle\varphi_1,\varphi_2|\rho|\varphi_1,\varphi_2\rangle$ in the form

$$\mathscr{P}(\varphi_+,\varphi_-)=\frac{1}{2}\left[P\left(\frac{\varphi_++\varphi_-}{2},\frac{\varphi_+-\varphi_-}{2}\right)+P\left(\frac{\varphi_++\varphi_-}{2}+\pi,\frac{\varphi_+-\varphi_-}{2}+\pi\right)\right]. \tag{2.11}$$

We see that the probability distribution for the angle sum and difference cannot be obtained from the one associated with the individual angles simply by the corresponding transformation of the variables (2.7). This is because the same sum and difference can be obtained from two different values for the angles of each system such that the transformation becomes nonbijective. The true transformation is obtained only after adding these two contributions. In the Appendix we study this problem in the context of canonical transformations.

III. PEGG-BARNETT FORMALISM FOR THE PHASE DIFFERENCE

We now turn to our problem of the phase difference between two systems like two harmonic oscillators or a two-mode quantum field.

As discussed in the Introduction, one possible way to describe this variable is using a previous definition of the absolute phase for each system. Contrary to L_z, now the spectrum of the number operator N is bounded from below. This precludes the unitarity of the operator exponential of the phase E arising from the polar decomposition of the annihilation operator

$$a=E\sqrt{N}, \tag{3.1}$$

adding difficulties to its interpretation in describing the phase in quantum optics.

Despite this fact, the eigenvectors of E with unit-modulus eigenvalues $e^{i\phi}$,

$$|\phi\rangle=\frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty} e^{in\phi}|n\rangle, \tag{3.2}$$

known as Susskind-Glogower phase states, are considered to have a well-defined phase, and are the starting point for some approaches to the problem [12,13].

One of them, the Pegg-Barnett approach [14], is perhaps the most widely used in the last few years. It is equivalent to the prescription of a phase probability distribution for a system state ρ in terms of the Susskind-Glogower phase states (3.2)

$$P(\phi)=\langle\phi|\rho|\phi\rangle. \tag{3.3}$$

When we have a two-mode field the joint probability distribution is given by

$$P(\phi_1,\phi_2)=\langle\phi_1,\phi_2|\rho|\phi_1,\phi_2\rangle, \tag{3.4}$$

where $|\phi_1,\phi_2\rangle$ are two-mode Susskind-Glogower phase states. Since we are mainly interested in the phase difference, we may ask first for the corresponding probability distribution $\mathscr{P}(\phi_+,\phi_-)$ associated with the phase-sum and phase-difference variables. Finally we integrate over the phase sum.

At this point the reader is referred to the previous and careful analysis of Barnett and Pegg [9]. However, we think that the Barnett-Pegg procedure is perhaps obscure in the sense that the nonbijective character of the transformation is not explicit and must be supplied with a careful handling of the range variation of all the variables. Moreover, the general relation between $\mathscr{P}(\phi_+,\phi_-)$ and $P(\phi_1,\phi_2)$ is not easily obtained.

As is clear from the preceding section, the transformation to the phase sum and difference

$$\phi_+=\phi_1+\phi_2,\quad \phi_-=\phi_1-\phi_2, \tag{3.5}$$

or equivalently

$$\phi_1=\frac{\phi_++\phi_-}{2},\quad \phi_2=\frac{\phi_+-\phi_-}{2}, \tag{3.6}$$

is nonbijective because we are demanding ϕ_+ and ϕ_- to be 2π periodic and not 4π periodic, as seems necessary from the previous equations. Therefore $\mathscr{P}(\phi_+,\phi_-)$ cannot be obtained from $P(\phi_1,\phi_2)$ simply by means of the change of variables (3.6). The proper form for the transformation is obtained by adding the probabilities corresponding to (ϕ_1,ϕ_2) and $(\phi_1+\pi,\phi_2+\pi)$ which give the same phase sum and difference, getting [taking into account the Jacobean of (3.6)]

$$\mathscr{P}(\phi_+,\phi_-)=\frac{1}{2}\left[P\left(\frac{\phi_++\phi_-}{2},\frac{\phi_+-\phi_-}{2}\right)+P\left(\frac{\phi_++\phi_-}{2}+\pi,\frac{\phi_+-\phi_-}{2}+\pi\right)\right], \tag{3.7}$$

which is 2π periodic.

Before going on, some remarks seem in order. First, $\mathscr{P}(\phi_+,\phi_-)$ contains less information than $P(\phi_1,\phi_2)$. For example, in general $P(\phi_1)$ cannot be obtained from $\mathscr{P}(\phi_+,\phi_-)$. Contrary to the angle variable, here we do not have phase operators from which (3.7) could be directly derived as in (2.4). However, the analysis of Sec. II supports this transformation law. In fact, one way to overcome the difficulties caused by the semiboundedness of the number operator is, precisely, enlarging the Hilbert space so as to include fictitious negative number states [15,16]. Physical results are recovered only when we are restricted to states having null projection over them. With this extension we are then formally in the same situation considered in Sec. II and therefore we arrive at (3.7) in the same way.

If we were only interested in the calculation of mean values (such as $e^{ik\phi_+}e^{il\phi_-}$, with k and l integers) all of this analysis would not be necessary, since they can be obtained simply as

$$\langle e^{ik\phi_+}e^{il\phi_-}\rangle=\int\int d\phi_1 d\phi_2 e^{ik(\phi_1+\phi_2)}e^{il(\phi_1-\phi_2)}P(\phi_1,\phi_2). \tag{3.8}$$

Equation (3.8) can also be used to obtain $\mathscr{P}(\phi_+,\phi_-)$ by noting that we must get the same mean values for any periodic function of the phase sum and difference whether we use the variables (ϕ_+,ϕ_-) or (ϕ_1,ϕ_2); i.e.,

$$\int\int d\phi_+ d\phi_- e^{ik\phi_+}e^{il\phi_-}\mathscr{P}(\phi_+,\phi_-)=\int\int d\phi_1 d\phi_2 e^{ik(\phi_1+\phi_2)}e^{il(\phi_1-\phi_2)}P(\phi_1,\phi_2). \tag{3.9}$$

Since $\mathscr{P}(\phi_+,\phi_-)$ and $P(\phi_1,\phi_2)$ are periodic functions, these equalities determine $\mathscr{P}(\phi_+,\phi_-)$ completely, as can be shown using Fourier analysis. The final result is the same as Eq. (3.7).

From now on we shall deal exclusively with the phase difference whose probability distribution function is given by

$$\mathscr{P}(\phi_-)=\int d\phi_+\mathscr{P}(\phi_+,\phi_-), \tag{3.10}$$

and, for simplicity, in what follows we shall omit the subscript $-$ on it. A useful expression for $\mathscr{P}(\phi)$ is

$$\mathscr{P}(\phi)=\int d\theta\langle\theta+\phi,\theta|\rho|\theta+\phi,\theta\rangle. \tag{3.11}$$

Note that, as may be expected, $\mathscr{P}(\phi)$ is invariant under any phase-sum shift. Then the system state ρ and

$$e^{i\phi_0(N_1+N_2)}\rho e^{-i\phi_0(N_1+N_2)} \tag{3.12}$$

have the same $\mathscr{P}(\phi)$, as is clearly seen from (3.11).

This property means (as it occurs in the angle case) that the phase difference and the total photon number are compatible [17]. This compatibility can be expressed more explicitly by noting that (3.10) and (3.11) are also equivalent to

$$\mathscr{P}(\phi)=\sum_{n=0}^{\infty}\langle\phi^{(n)}|\rho|\phi^{(n)}\rangle, \tag{3.13}$$

where the vector

$$|\phi^{(n)}\rangle=\frac{1}{\sqrt{2\pi}}\sum_{n_1=0}^{n}e^{in_1\phi}|n_1,n-n_1\rangle \tag{3.14}$$

lies in the subspace $\mathscr{H}_n$ with total photon number n. Equation (3.13) allows us to extract a joint probability distribution function for the total number and the phase difference

$$\mathscr{P}(n,\phi)=\langle\phi^{(n)}|\rho|\phi^{(n)}\rangle, \tag{3.15}$$

and in this way we have the compatibility expressed as

$$\mathscr{P}(\phi)=\sum_{n=0}^{\infty}\mathscr{P}(n,\phi),\quad \mathscr{P}(n)=\int d\phi\mathscr{P}(n,\phi), \tag{3.16}$$

where $\mathscr{P}(n)$ is the probability of having n photons in the system. Consequently, we can independently study the phase-difference properties of any field state in each finite-dimensional subspace $\mathscr{H}_n$ without losing information.

Each $\mathscr{P}(n,\phi)$ is to some extent redundant. This can be shown by noting that $\mathscr{P}(n,\phi)$ cannot be arbitrary. Its more general form is, according to (3.15) and (3.14),

$$\mathscr{P}(n,\phi)=\sum_{k=-n}^{n} c_k e^{ik\phi}, \tag{3.17}$$

with $c_k=c_{-k}^*$. This has two interesting consequences. First, the mean value of $e^{im\phi}$ with $|m|>n$ is 0 in the subspace $\mathscr{H}_n$. Second, $\mathscr{P}(n,\phi)$ depends on $2n+1$ parameters and can be completely fixed by its values on $2n+1$ properly chosen points, such as, for example, $\phi_r^{(n)}=2\pi r/(2n+1)$ with r an integer running from $-n$ to n. We can then invert (3.17) and obtain the coefficients c_k as

$$c_k=\frac{1}{2n+1}\sum_{r=-n}^{n}\mathscr{P}(n,\phi_r^{(n)})e^{-i2\pi kr/(2n+1)}, \tag{3.18}$$

which allows us to express $\mathscr{P}(n,\phi)$ as

$$\mathscr{P}(n,\phi)=\frac{1}{2n+1}\sum_{k,r=-n}^{n}\mathscr{P}(n,\phi_r^{(n)})e^{ik(\phi-\phi_r^{(n)})}. \tag{3.19}$$

Thus the knowledge of the distribution $\mathscr{P}(n,\phi)$ for $2n+1$ values of the phase difference is enough to characterize the behavior of the phase difference within the $\mathscr{H}_n$ subspace.

We can now express any mean value in terms of these $\mathscr{P}(n,\phi_r^{(n)})$ as

$$\langle f(\phi)\rangle=\sum_{n=0}^{\infty}\frac{2\pi}{2n+1}\sum_{r=-n}^{n}\mathscr{P}(n,\phi_r^{(n)})f^{(n)}(\phi_r^{(n)}), \tag{3.20}$$

where $f^{(n)}(\phi)$ is the function arising from $f(\phi)$ after removing the Fourier frequencies higher than n. It is worth noting the close resemblance of these expressions with those obtained with a phase-difference operator arising from a polar decomposition [5]. Formally there are only two differences: the removing of Fourier frequencies and the number of phase-difference values, $2n+1$ instead of $n+1$. Otherwise, it turns up that they share more properties than expected at first glance. It seems that the phase difference has properties not evidently related to the absolute phase, as happens in the polar decomposition, here hidden in the casting procedure. A particular example is discussed in detail in Ref. [18].

IV. PHASE DIFFERENCE AND Q FUNCTION

When one focuses attention on realistic schemes for phase measurements (such as amplification with a linear laser amplifier [19], heterodyning [20], or beam splitting [21]) it becomes clear that, in fact, all these techniques are suited to determine the Q function for the signal field [22], defined as usual as

$$Q(\alpha)=\frac{1}{\pi}\langle\alpha|\rho|\alpha\rangle, \tag{4.1}$$

where $|\alpha\rangle$ is a coherent state.

In particular, a phase distribution is obtained from Q as a marginal distribution after integrating over the radial variable. Moreover, the Q function was recently shown [23] to be a special case of the quantum propensity [24], corresponding to the choice of the reference state as the vacuum state and of the phase-space *motion* as the Glauber displacement operator. Thus it is interesting to examine the corresponding casting procedure for this formalism that, to our best knowledge, has not been previously considered.

The joint distribution for the phase of a two-mode field is

$$P(\phi_1,\phi_2)=\int_0^\infty\int_0^\infty dr_1 dr_2 r_1 r_2 Q(r_1e^{i\phi_1},r_2e^{i\phi_2}), \tag{4.2}$$

where $\alpha_1=r_1e^{i\phi_1}$, $\alpha_2=r_2e^{i\phi_2}$. Then we can express it in terms of the phase sum and difference by means of Eq. (3.7) and finally integrate over the phase sum to get the probability distribution for the phase difference.

Instead of taking this direct way, we rather prefer to employ a useful relation between the two-mode coherent states $|\alpha_1,\alpha_2\rangle$ and the SU(2) coherent states $|n,\theta,\phi\rangle$ [25],

$$|\alpha_1,\alpha_2\rangle=\sum_{n=0}^{\infty}\frac{r^n e^{in\phi_2}}{\sqrt{n!}}e^{-r^2/2}|n,\theta,\phi\rangle, \tag{4.3}$$

where

$$|n,\theta,\phi\rangle=\sum_{n_1=0}^{n}\binom{n}{n_1}^{1/2}\left(\cos\frac{\theta}{2}\right)^{n_1}\left(\sin\frac{\theta}{2}\right)^{n-n_1}\times e^{in_1\phi}|n_1,n-n_1\rangle, \tag{4.4}$$

and

$$r=\sqrt{r_1^2+r_2^2},\quad \phi=\phi_1-\phi_2,\quad \tan\frac{\theta}{2}=\frac{r_2}{r_1}, \tag{4.5}$$

with $0\leqslant r<\infty$, $0\leqslant\theta<\pi$.

These vectors $|n,\theta,\phi\rangle$ are the SU(2) coherent states in the subspaces $\mathscr{H}_n$ corresponding to the realization of the algebra of this group in terms of the bosonic operators [26]

$$j_x=\frac{1}{2}(a_1^\dagger a_2+a_1a_2^\dagger),\qquad j_y=\frac{i}{2}(a_2^\dagger a_1-a_2a_1^\dagger),\qquad j_z=\frac{1}{2}(a_1^\dagger a_1-a_2^\dagger a_2), \tag{4.6}$$

with

$$\mathbf{j}^2=\frac{N}{2}\left(\frac{N}{2}+1\right), \tag{4.7}$$

and we have

$$|n,\theta,\phi\rangle=\exp\left(\frac{\theta}{2}e^{-i\phi}a_2^\dagger a_1-\frac{\theta}{2}e^{i\phi}a_1^\dagger a_2\right)|n,0\rangle. \tag{4.8}$$

We can use Eq. (4.3) to express the Q function in terms of the phase sum and difference as well as the variables r and θ. If we integrate over the phase sum and r we get a function of θ and the phase difference ϕ,

$$\mathscr{P}(\theta,\phi)=\sum_{n=0}^{\infty}\frac{n+1}{4\pi}\langle n,\theta,\phi|\rho|n,\theta,\phi\rangle\sin\theta=\sum_{n=0}^{\infty}\mathscr{P}(n,\theta,\phi). \tag{4.9}$$

It can be recognized that $\mathscr{P}(n,\theta,\phi)$ is (up to the volume element) the SU(2) Q function for the projection of the state over the subspace $\mathscr{H}_n$.

To obtain the distribution for the phase difference we simply integrate over θ,

$$\mathscr{P}(\phi)=\sum_{n=0}^{\infty}\int_0^{\pi}d\theta\mathscr{P}(n,\theta,\phi)=\sum_{n=0}^{\infty}\mathscr{P}(n,\phi)$$
$$=\sum_{n=0}^{\infty}\frac{n+1}{4\pi}\int_0^{\pi}d\theta\sin\theta\langle n,\theta,\phi|\rho|n,\theta,\phi\rangle, \tag{4.10}$$

and we get finally

$$\mathscr{P}(n,\phi)=\frac{1}{2\pi}\sum_{n_1,n_1'=0}^{n}G_{n_1,n_1'}^{(n)}e^{i(n_1-n_1')\phi}$$
$$\times\langle n_1',n-n_1'|\rho|n_1,n-n_1\rangle, \tag{4.11}$$

where

$$G_{n_1,n_1'}^{(n)}=\frac{\Gamma\left(\frac{n_1+n_1'}{2}+1\right)\Gamma\left(n+1-\frac{n_1+n_1'}{2}\right)}{\sqrt{n_1!n_1'!(n-n_1)!(n-n_1')!}}. \tag{4.12}$$

This structure is the same as the one found in Eq. (3.15). They only differ from the coefficients $G_{n_1,n_1'}^{(n)}$ that are replaced by 1 in the Pegg-Barnett approach. Otherwise we have the same compatibility between the phase difference and the total photon number (3.16). Therefore the discussion made from (3.16) to (3.20) could be translated here in the same terms. Note that while Eq. (3.13) represents a sharp phase-difference measurement, Eq. (4.10) represents an unsharp (or noisy) measurement, as discussed in Refs. [27,28].

We can also introduce a similar joint distribution for the total number and the number difference in the form

$$\mathscr{P}(n,\theta)=\int d\phi\mathscr{P}(n,\theta,\phi)$$
$$=\frac{n+1}{4\pi}\int d\phi\langle n,\theta,\phi|\rho|n,\theta,\phi\rangle\sin\theta. \tag{4.13}$$

This could be justified as in Eq. (4.10), since the connection between the polar angle θ (or its cosine) and the number difference is equivalent to the relation between the azimuthal angle ϕ and the phase difference. This distribution could be very interesting theoretically and perhaps tightly connected with the experiment. However, surely it would not be named a probability distribution for the number difference nor called for to support a continuous range of variation for it. A similar situation arises for the field quadratures because the integration of Q over the real or imaginary part of α does not give their probability distribution functions.

Although the Q function cannot be the substitute of a phase operator [22], it has the advantage of corresponding to an experimental realization and provides a complete knowledge of the system state.

Using the concept of area of overlap and interference in phase space, Schleich, Horowicz, and Varro [29] suggested a theoretical procedure for determining phase distributions. Their approach consists in calculating the phase distribution by averaging the Wigner distribution function over the field amplitude. The casting procedure for this Wigner function is also necessary and can be obtained along the same lines we have followed for the Q function. This point should be important in the recent studies of optical homodyne tomography [30] to characterize, from the experimental point of view, the state of the field.

V. CONCLUSIONS

In this paper we have developed a procedure giving the phase-difference probability distribution function for a two-mode field state in terms of previous approaches for the one-mode field phase, such as the Pegg-Barnett formalism and the Q function. In this procedure we have encountered the difficulty of the nonbijective character of the transformation from the individual phases to the phase sum and phase difference.

After solving this problem, obtaining the corresponding transformation law for the probability distributions, we have studied some general properties of the phase-difference probability distribution. We have shown that it is compatible with the total photon number. This could be expected from the classical Poisson bracket verified by the corresponding variables, but it has further consequences. We have shown that its value on a numerable set of points completely fixes the whole probability distribution. When the state involves a finite number of photons this set of points is also finite.

We have also compared these results with the corresponding ones obtained from a phase-difference operator arising from a polar decomposition for a two-mode field. Although very different in their origin and in some of their properties, we have shown that these approaches are closer than expected at first sight.

ACKNOWLEDGMENTS

The authors wish to thank Professor Ryszard Tanaś, Professor Alberto Galindo, and Dr. Adam Miranowicz for a critical reading of the manuscript and for stimulating discussions.

APPENDIX: CANONICAL TRANSFORMATION TO ANGLE SUM AND DIFFERENCE

In this Appendix we wish to consider the transformation relating one set of coordinates of the phase space of the system $(\varphi_1,\varphi_2,L_{1z},L_{2z})$ to another one $(\varphi_+,\varphi_-,L_+,L_-)$ defined by

$$\varphi_+ = \varphi_1 + \varphi_2, \quad \varphi_- = \varphi_1 - \varphi_2,$$
$$L_+ = \frac{L_{1z}+L_{2z}}{2}, \quad L_- = \frac{L_{1z}-L_{2z}}{2}. \tag{A1}$$

This transformation is a canonical one; i.e., it preserves the Poisson brackets and thereby (φ_+, L_+) and (φ_-, L_-) are conjugate variables.

It is clear that a similar transformation in position and momentum for instance will not need any special caution. Even if the range of variation of the position would be a finite interval, we could always properly accommodate the range of variation of the sum and difference variables.

However, in the angle case we are forced to think about φ_+ and φ_- as 2π-periodic variables. This necessary restriction makes the transformation nonbijective since, as discussed, the points (φ_1, φ_2) and $(\varphi_1+\pi, \varphi_2+\pi)$ map on the same point (φ_+, φ_-).

The equivalence between phase-space coordinates related by a canonical transformation is expressed in quantum mechanics by a unitary transformation [31]. This means that now we have two Hilbert spaces $\mathscr{H}_+$ and $\mathscr{H}_-$ associated with angle and momentum operators $(\bar{E}_+, \bar{L}_+)$ and $(\bar{E}_-, \bar{L}_-)$ related to the original ones $\mathscr{H}_1$ and $\mathscr{H}_2$, associated with (E_1, L_{1z}) and (E_2, L_{2z}) via a unitary operator $U: \mathscr{H}_1 \otimes \mathscr{H}_2 \rightarrow \mathscr{H}_+ \otimes \mathscr{H}_-$ such that

$$\bar{E}_+ = U E_1 E_2 U^\dagger, \quad \bar{E}_- = U E_1 E_2^\dagger U^\dagger,$$
$$\bar{L}_+ = U \frac{L_{1z}+L_{2z}}{2} U^\dagger, \quad \bar{L}_- = U \frac{L_{1z}-L_{2z}}{2} U^\dagger. \tag{A2}$$

The knowledge of U provides us complete information about the transformation we are studying.

It is clear that some difficulties must appear in this definition of U. Intimately linked with the nonbijectivity, we find that the transformation must relate operators with different spectra [32]. Relations (A2) seem to impose half-integer values to L_+ and L_- (that is, 4π periodicity for the corresponding angles) contrary to what we have supposed. Therefore the transformation U posed in (A2) cannot be unitary.

This is a simple example of the kind of problems arising in the representation in quantum mechanics of nonbijective canonical transformations [33]. Although U cannot be unitary, we can nevertheless find isometric mappings if we restrict the definition to certain subspaces of $\mathscr{H}_1 \otimes \mathscr{H}_2$. This can be accomplished by using the concept of the ambiguity group; i.e., the group connecting the set of points in the original space mapped on the same one in the new space [32]. Here this group V has only two elements: the identity and a joint π rotation on both angles; i.e.,

$$V_k = e^{ik\pi(L_{1z}+L_{2z})}, \tag{A3}$$

where $k=0,1$. Note that this group leaves invariant all the operators E_1E_2, $E_1E_2^\dagger$, $L_{1z}+L_{2z}$, and $L_{1z}-L_{2z}$ in the definition (A2) of the transformation.

If we want to find subspaces that could be isometrically mapped in $\mathscr{H}_+ \otimes \mathscr{H}_-$ verifying (A2) up to constants, we must restrict ourselves to subspaces where the action of the ambiguity group V becomes a constant phase factor, that is, the subspaces carrying the unitary representations of the group. Here we have two of these subspaces; we shall call them $\mathscr{E}_0$ and $\mathscr{E}_1$. The subspace $\mathscr{E}_0$ is spanned by the eigenvectors of L_{1z} and L_{2z} $\{|2n,2m\rangle, |2n+1,2m+1\rangle\}$, while $\mathscr{E}_1$ is spanned by $\{|2n+1,2m\rangle, |2n,2m+1\rangle\}$, with n and m integers running from $-\infty$ to $+\infty$. Note that in avoiding the nonbijectivity with these restrictions, we also remove the problem caused by the difference of the spectra. The subspace $\mathscr{E}_0$ has only eigenvalues of L_{1z} and L_{2z} whose sum or difference is even, and then the spectra of the operators involved in (A2) are equal. On the other hand, $\mathscr{E}_1$ contains only eigenvalues whose sum or difference is odd, and the spectra can be made equal simply adding to $\bar{L}_+$ and $\bar{L}_-$ in (A2) a half-integer constant.

Now it is possible to find two isometric mappings U_p from $\mathscr{E}_p$ $(p=0,1)$ to $\mathscr{H}_+ \otimes \mathscr{H}_-$ verifying (A2) up to constants. They are given by

$$U_0 = \sum_{n,m=-\infty}^{+\infty} \left(|\overline{n+m}, \overline{n-m}\rangle\langle 2n,2m| + |\overline{n+m+1}, \overline{n-m}\rangle\langle 2n+1,2m+1| \right),$$
$$U_1 = \sum_{n,m=-\infty}^{+\infty} \left(|\overline{n+m}, \overline{n-m}\rangle\langle 2n+1,2m| + |\overline{n+m}, \overline{n-m-1}\rangle\langle 2n,2m+1| \right), \tag{A4}$$

with $U_pU_p^\dagger = I$ while $U_p^\dagger U_p$ is the projector on the subspace $\mathscr{E}_p$.

Despite this, we may be interested in a truly unitary transformation defined over the whole space, in order to have a complete description of the system in terms of the angle sum and difference. To do this we need to enlarge the final space by adding a new variable, usually called the ambiguity spin. With this, the final space will be of the form $\mathscr{H}_+ \otimes \mathscr{H}_- \otimes \mathscr{V}$, where $\mathscr{V}$ is the Hilbert space needed to accommodate this new variable.

Clearly, the role of the ambiguity spin is to provide a different image for each subspace $\mathscr{E}_p$ and simultaneously equalize the spectra. Considering now $U: \mathscr{H}_1 \otimes \mathscr{H}_2 \rightarrow \mathscr{H}_+ \otimes \mathscr{H}_- \otimes \mathscr{V}$, $\mathscr{V}$ being a two-dimensional space, we have

$$U = |e_0\rangle U_0 + |e_1\rangle U_1, \tag{A5}$$

where $|e_0\rangle$ and $|e_1\rangle$ are an orthonormal basis in $\mathscr{V}$. U is a unitary operator performing the transformation

$$\bar{E}_+ = U E_1 E_2 U^\dagger, \quad \bar{E}_- = U E_1 E_2^\dagger U^\dagger,$$
$$\bar{L}_+ + \frac{\bar{\Pi}}{2} = U \frac{L_{1z}+L_{2z}}{2} U^\dagger, \quad \bar{L}_- + \frac{\bar{\Pi}}{2} = U \frac{L_{1z}-L_{2z}}{2} U^\dagger, \tag{A6}$$

where $\bar{\Pi}|e_p\rangle = p|e_p\rangle$.

With this unitary transformation we immediately obtain the probability distribution function associated with the angle sum and difference. Calling $|\bar{\varphi}_+,\bar{\varphi}_-\rangle$ the eigenstates of $\underline{E}_+$ and $\underline{E}_-$, we have

$$\begin{aligned}\mathscr{P}(\varphi_+,\varphi_-)&=\sum_{p=0,1}\langle\bar{\varphi}_+,\bar{\varphi}_-,e_p|U\rho U^\dagger|\bar{\varphi}_+,\bar{\varphi}_-,e_p\rangle\\&=\sum_{p=0,1}\langle\bar{\varphi}_+,\bar{\varphi}_-|U_p\rho U_p^\dagger|\bar{\varphi}_+,\bar{\varphi}_-\rangle\\&=\sum_{p=0,1}\langle\varphi_+,\varphi_-,p|\rho|\varphi_+,\varphi_-,p\rangle,\end{aligned}\tag{A7}$$

where we have used [with the definition (2.6)]

$$U|\varphi_+,\varphi_-,p\rangle=|\bar{\varphi}_+,\bar{\varphi}_-,e_p\rangle.\tag{A8}$$

Here we have arrived at (2.10) and therefore at the same transformation law (2.11) in a different way.

It could be thought that the representation of arbitrary canonical transformations is a rather obscure subject arising only in very special situations far from any practical significance. Then the ambiguity spin appears as a curiosity imposed by a theoretical procedure, but otherwise spurious. However, this example shows that this is not always the case. Moreover, now the results of Sec. II are clearer. The reason is that the nonbijectivity merely translates the fact that the angle sum and difference are not by themselves a complete set of commuting operators. Then, the ambiguity spin is essentially the other operator needed to complete this set.

[1] P. Carruthers and M. M. Nieto, Rev. Mod. Phys. **40**, 441 (1968).
[2] Phys. Scr. **T48**, 1 (1993), special issue on quantum phase.
[3] A. Lukš and V. Peřinová, Quantum Opt. **6**, 125 (1994).
[4] J. Bergou and B. G. Englert, Ann. Phys. (N.Y.) **209**, 479 (1991).
[5] A. Luis and L. L. Sánchez-Soto, Phys. Rev. A **48**, 4702 (1993).
[6] A. Luis and L. L. Sánchez-Soto, Opt. Commun. **105**, 84 (1994).
[7] A. Luis, L. L. Sánchez-Soto, and R. Tanaś, Phys. Rev. A **51**, 1634 (1995).
[8] D. Ellinas, J. Mod. Opt. **38**, 2393 (1991).
[9] S. M. Barnett and D. T. Pegg, Phys. Rev. A **42**, 6713 (1990).
[10] S. M. Barnett and D. T. Pegg, Phys. Rev. A **41**, 3427 (1990).
[11] D. Judge and J. T. Lewis, Phys. Lett. **5**, 190 (1963).
[12] L. Susskind and J. Glogower, Physics **1**, 49 (1964).
[13] J. M. Lévy-Leblond, Rev. Mex. Fis. **22**, 15 (1973); Ann. Phys. (N.Y.) **101**, 319 (1976).
[14] D. T. Pegg and S. M. Barnett, Europhys. Lett. **6**, 483 (1988); J. Mod. Opt. **36**, 7 (1989).
[15] R. G. Newton, Ann. Phys. (N.Y.) **124**, 327 (1980).
[16] S. M. Barnett and D. T. Pegg, J. Phys. A **19**, 3849 (1986).
[17] J. C. Garrison and J. Wong, J. Math. Phys. **11**, 2242 (1970); A. Galindo, Lett. Math. Phys. **8**, 495 (1984); **9**, 263 (1985).
[18] A. Luis and L. L. Sánchez-Soto, Phys. Rev. A **51**, 859 (1995).
[19] H. Paul, Fortschr. Phys. **22**, 657 (1974).
[20] J. H. Shapiro and S. S. Wagner, IEEE J. Quantum Electron. **QE-20**, 803 (1984).
[21] J. W. Noh, A. Fougères, and L. Mandel, Phys. Rev. Lett. **67**, 1426 (1991); Phys. Rev. A **45**, 424 (1992); **46**, 2840 (1992).
[22] U. Leonhardt and H. Paul, Phys. Rev. A **47**, R2460 (1993); **48**, 4598 (1993).
[23] D. Burak and K. Wódkiewicz, Phys. Rev. A **46**, 2744 (1992).
[24] K. Wódkiewicz, Phys. Lett. A **115**, 304 (1986).
[25] F. T. Arecchi, E. Courtens, R. Gilmore, and H. Thomas, Phys. Rev. A **6**, 2211 (1972).
[26] J. Schwinger, in *Quantum Theory of Angular Momentum*, edited by L. C. Bidenharn and H. van Dan (Academic Press, New York, 1965).
[27] U. Leonhardt, J. A. Vaccaro, B. Böhmer, and H. Paul, Phys. Rev. A **51**, 84 (1995).
[28] P. Busch, M. Grabowski, and P. J. Lahti, Ann. Phys. (N.Y.) **237**, 1 (1995).
[29] W. Schleich, R. J. Horowicz, and S. Varro, Phys. Rev. A **40**, 7405 (1989).
[30] D. T. Smithey, M. Beck, A. Faridani, and M. G. Raymer, Phys. Rev. Lett. **70**, 1244 (1993).
[31] P. A. M. Dirac, *Quantum Mechanics* (Clarendon, Oxford, 1958).
[32] P. A. Mello and M. Moshinsky, J. Math. Phys. **16**, 2017 (1975); M. Moshinsky and T. H. Seligman, Ann. Phys. (N.Y.) **114**, 243 (1978); **120**, 402 (1979); J. Math. Phys. **22**, 1338 (1981); P. Kramer, M. Moshinsky, and T. H. Seligman, *ibid.* **19**, 683 (1978).
[33] A. Luis and L. L. Sánchez-Soto, J. Phys. A **24**, 2083 (1991).