The inverse problem concerning symmetries of ordinary differential equations

ARTICLE *in* JOURNAL OF MATHEMATICAL PHYSICS · FEBRUARY 1988

Impact Factor: 1.18 · DOI: 10.1063/1.528000

Artemio Gonzalez-Lopez
Complutense University of Madrid

The inverse problem concerning symmetries of ordinary differential equations

F. Gonzalez-Gascon
Departamento de Metotos Matemáticos, Facultad de C. Fisicas, Universidad Complutense, Madrid-3, Spain

A. González-López
Department of Physics, Princeton University, Princeton, New Jersey 08544

(Received 13 August 1987; accepted for publication 21 October 1987)

It is shown that for any local Lie group G of transformations in $R \times R^n$ there exist differential systems of the form $\mathbf{x}^{(m} = \mathbf{f}(t,\mathbf{x},...,\mathbf{x}^{(m-1})$, which are symmetrical under G. The order $\underset{\sim}{m}$ of these systems is related to $\underset{\sim}{r}$, the number of essential parameters of G.

I. INTRODUCTION

In a recent paper[1] it was shown that for normal systems of differential equations of type

$$\mathbf{x}^{(m} = \mathbf{f}(t,\mathbf{x},\dot{\bar{x}},...,\bar{x}^{(m-1})\mathbf{x}\in\mathbb{R}^n, \quad (1)$$

the maximal number of its pointlike symmetry vectors is (i) infinite, when $m=1$; (ii) not greater than n^2+4n+3, when $m=2$; (iii) not greater than $2n^2+nm+2$, when $m>2$. Since the number $2n^2+nm+2$ increases without limit with $\underset{\sim}{m}$, the question arises of whether or not it is possible to find a system of type (1) symmetrical under a given group G, for a sufficiently high value of $\underset{\sim}{m}$. We prove that the reply to this question is affirmative. Our result is local, in the sense that the function f of (1), whose existence we prove, will be, in general, only locally defined.

Note that since a first-order ($m=1$) system always possesses an infinite number of pointlike symmetry vectors, one could naively expect to find for any G a first-order system possessing G among its symmetries. That this is not generally possible is seen if G is, for instance, a group acting transitively on the $(t,\mathbf{x},\dot{x})$ space. For an example see Part (1) of Sec. III.

Note also that the construction given here does not guarantee that G is the maximal group of pointlike symmetries G_M of (1), but only that $G\subset G_M$.

II. MAIN RESULT

Assume that $\mathbf{S}_i(t,\mathbf{x})$, $i=l,...,r$, is a basis of generators of G. Calling $\mathbf{S}_i^e$ the e-order extension of $\mathbf{S}_i$ we have[2]

$$[\mathbf{S}_i^{(e},\mathbf{S}_j^{(e}] = \sum_{k=1}^{r} c_{ijk}\mathbf{S}_k^{(e},$$
$$e=0,1,2,..., \quad i,j=1,...,r, \quad (2)$$

where c_{ijk} are the structure constants of G associated with the basis $\{\mathbf{S}_i(t,\mathbf{x})\}$. On the other hand, the necessary and sufficient condition in order that G be a symmetry group of equations (1) is[2]

$$\mathbf{S}_i^{(m}(\mathbf{x}^{(m}-\mathbf{f})\Big|_{\mathbf{x}^m=\mathbf{f}} = \mathbf{0}, \quad i=1,...,r. \quad (3)$$

Conditions (3) indicate that the manifold M^m of $(t,\mathbf{x},...,\mathbf{x}^{(m})$ space defined by Eq. (1) is invariant under the action of the vector fields $\mathbf{S}_1^{(m},...,\mathbf{S}_r^{(m}$. We are going to prove that given G, one can find a sufficiently high $\underset{\sim}{m}$ such that, for a certain $\mathbf{f}$, Eqs. (1) do possess G as a group of symmetries.

The idea of the proof is to eliminate the possible transitivity of the action of G^e on $D^e = \{(t,\mathbf{x},...,\mathbf{x}^{(e})\}$ for low values of $\underset{\sim}{e}$ by making $\underset{\sim}{e}$ bigger and bigger. This is made possible, essentially, due to property (2), implying that at any point of D^e the vector fields $\mathbf{S}_i^{(e}$ generate an involutive distribution $\mathscr{D}^e$ (Ref. 3) of dimension not greater than $\underset{\sim}{r}$. To avoid singularity points where the dimension of the distribution $\mathscr{D}^e$ changes value, we restrict conveniently the domain D^e in order that in this restricted domain $\widetilde{D}^e$, $\dim(\mathscr{D}^e)$ keeps a constant and maximum value $\underset{\sim}{d_e}$. Of course $\dim \mathscr{D}^{e-1} = d_e - 1$ in the projection of $\widetilde{D}^e$ along the $\mathbf{x}^{(e}$ axis. See Ref. 4 for details.

Therefore let $\mathbf{S}_1^{(e},...,\mathbf{S}_{d_e}^{(e}$ be a local basis of $\mathscr{D}^e$. Note that it might be necessary to renumber the generators of G for the basis of $\mathscr{D}^e$ to appear in this way. Conditions (3) for the symmetry of (1) under G now take the form

$$\mathbf{S}_i^{(m}(\mathbf{x}^{(m}-\mathbf{f})\Big|_{\mathbf{x}^{(m}=\mathbf{f}} = \mathbf{0}, \quad i=1,...,d. \quad (4)$$

Writing Eq. (1) in the implicit form

$$\mathbf{E}(t,\mathbf{x},...,\mathbf{x}^{(m)}) = 0, \quad (5)$$

where $\mathbf{E}$ is a vector of m components, Eqs. (4) take the form

$$\mathbf{S}_i^{(m}(\mathbf{E})_{\mathbf{E}=\mathbf{0}} = \mathbf{0}, \quad i=1,...,d_m. \quad (6)$$

A sufficient condition necessary for Eqs. (6) to be satisfied is that the m components of the function $\mathbf{E}$ of (6) be local first integrals of $\mathbf{S}_i^{(m}$, that is, if $\mathbf{E}$ satisfies

$$\mathbf{S}_i^{(m}(\mathbf{E}) = \mathbf{0}, \quad i=1,...,d_m. \quad (7)$$

But according to the Frobenius theorem[3] the number of locally independent first integrals I of an involutive distribution like $\mathscr{D}^m$ is $d_I = \dim(D^m) - d_m = 1 + n(1+m) - d_m$. Now, since $d_m \leqslant r$ it follows that $d_I \geqslant n$ for sufficiently large m. Assuming $d_I \geqslant n$, in order to satisfy (7) it is sufficient to choose $\underset{\sim}{n}$ first integrals I of $\mathbf{S}_i^{(m}$ such that they satisfy the additional requirement

$$\operatorname{rank}\left(\frac{\partial I_i}{\partial x_k^{(m}}\right) = n, \quad k=1,...,n. \quad (8)$$

Condition (8) guarantees, via the implicit function theorem, that the system of differential equations

$$I_1 = C_1, \quad \vdots \quad I_n = C_n \quad (9)$$

 0022-2488/88/030618-04$02.50

can be locally written in the normal form (1). The symbols $C_1,...,C_n$ in Eqs. (9) are real numbers, and they appear since Eqs. (7) are clearly equivalent to

$$\mathbf{S}_i^{(m}(\mathbf{E}-\mathbf{C})=\mathbf{0}, \tag{10}$$

for any $\mathbf{C}\in R^n$. Let us see that condition (8) can be satisfied if $\underline{m}$ is chosen such that

$$\dim(\mathscr{D}^m)=\dim(\mathscr{D}^{m-1}). \tag{11}$$

In fact, if (8) were not satisfied by an appropriate choice of $I_1,...,I_n$ between the d_I first integrals of $\mathscr{D}^m$, we would have

$$\sum_{k=1}^{n} a_k \frac{\partial I_i}{\partial x_k^{(m}}=0, \quad i=1,2,...,d_I, \tag{12}$$

where a_k are functions on D^m.

But (12) implies that the vector field $\mathbf{Z}$ defined by

$$\mathbf{Z}=\sum_{x=1}^{n} a_k \frac{\partial}{\partial x_k^{(m}} \tag{13}$$

has $I_1,...,I_{d_I}$ as first integrals. Therefore $\mathbf{Z}\in\mathscr{D}^m$ and we can write

$$\mathbf{Z}=\sum_{i=1}^{d_m} c_i \mathbf{S}_i^{(m}, \tag{14}$$

for certain functions c_i defined on D^m.

Projecting (14) on the vectors

$$\frac{\partial}{\partial t};\frac{\partial}{\partial x_i};\frac{\partial}{\partial x_i^{(1}},...,\frac{\partial}{\partial x_i^{(m-1}}, \quad \mathbf{0}=\sum_{i=1}^{d_m} c_i\mathbf{S}_i^{(m-1}. \tag{15}$$

But (15) and (11) are contradictory since from the fact that $\mathbf{S}_1^{(m},...,\mathbf{S}_{d_m}^{(m}$ are a basis of $\mathscr{D}^m$ it immediately follows (note that $\mathbf{S}_i^{m-1}$ does not depend on $\mathbf{x}^{(m)}$ that $\mathbf{S}_1^{(m-1},...,\mathbf{S}_{d_m}^{(m-1}$ generate $\mathscr{D}^{m-1}$. Hence by (16), dim $\mathscr{D}^{m-1}<d_m$ and (12) is contradicted. Therefore (11) implies (8).

It remains only to prove that (11) can always be satisfied by choosing $\underline{m}$ conveniently. But this follows from the fact[4] that

$$\dim(\mathscr{D}^r)\leqslant\dim(\mathscr{D}^{r+1}). \tag{16}$$

Indeed, since the dimension is a positive integer, nondecreasing by (16), and bounded by $\underline{r}$ (the number of parameters of the group) it is obvious that for a certain $\underline{m}$ (12) holds. Furthermore, $\underline{m}$ satisfies

$$m\leqslant r-\dim(\mathscr{D}^0)+1=r_1, \tag{17}$$

since the worst situation that can occur concerning (11) is that

$$\dim(\mathscr{D}^s)=\dim(\mathscr{D}^{s-1})+1, \quad s<m, \tag{18}$$

in which case (17) would hold with the equal sign.

Note that if we require $\underline{m}$ to be greater than (or equal to) a given k ($k=1,2,...$) then the above considerations lead to the inequality

$$m\leqslant r-\dim(\mathscr{D}^{k-1})+k=r_k, \tag{19}$$

the equality sign being valid only when the sequence of dimensions $\dim(\mathscr{D}^{k-1})$, $\dim(\mathscr{D}^k)$,... is strictly increasing by 1 at each step and condition (11) is fulfilled when $\dim(\mathscr{D}^m)=\dim(\mathscr{D}^{m-1})=r$.

Note also that calling $r(G)$ the minimum integer such that for $k\geqslant r(G)$, $\dim(\mathscr{D}^k)$ maintains a constant value, that is, $\dim(\mathscr{D}^k)=\dim(\mathscr{D}^{k'})$ for every $k,k'\geqslant r(G)$, by (11) we can say that for *every* $m>r(G)$ there are systems of differential equations of order $\underline{m}$ invariant under G.

Note finally that [see Eq. (11)] the construction given here actually assures the existence of n-parameter families of systems of type (1) invariant, for any value of the parameters, under G. Let us now see, with two examples, that condition (11) is not necessary for the existence of *individual* systems of type (1) invariant under G.

III. EXAMPLES SHOWING THAT CONDITION (11) IS NOT NECESSARY

(1) Let us take as G the Poincaré group in $R\times R^2$ (two spatial dimensions). We shall see that (11) is not satisfied either for $m=1$ or for $m=2$. Nevertheless, as has been shown elsewhere[5] $\dddot{\mathbf{x}}=\overline{0}$ is a second-order differential system (in fact the only one) invariant under the Poincaré group in $R\times R^2$. In fact the six generators of the group are

$$\mathbf{S}_1=\frac{\partial}{\partial t};\quad \mathbf{S}_{1+c}=\frac{\partial}{\partial x_i};\quad \mathbf{S}_4=x_1\frac{\partial}{\partial x_2}-x_2\frac{\partial}{\partial x_1};$$

$$\mathbf{S}_{4+i}=-\left(x_i\frac{\partial}{\partial t}+t\frac{\partial}{\partial x_i}\right);\quad i=1,2. \tag{20}$$

Since $\dim(\mathscr{D}^0)=3$ the group acts transitively on $D^0=(t,x_1,x_2)$.

The corresponding generators of the first extension of G are given by

$$\begin{aligned}
&\mathbf{S}_1^{(1}=\frac{\partial}{\partial t};\quad \mathbf{S}_{1+i}^{(1}=\frac{\partial}{\partial x_i},\\
&\mathbf{S}_4^{(1}=\mathbf{S}_4+\dot{x}_1\frac{\partial}{\partial \dot{x}_2}-\dot{x}_2\frac{\partial}{\partial \dot{x}_1},\\
&\mathbf{S}_{4+i}^{(1}=\mathbf{S}_{4+i}+(\dot{x}_i^2-1)\frac{\partial}{\partial \dot{x}_i}+\dot{x}_i\dot{x}_j\frac{\partial}{\partial \dot{x}_j}.
\end{aligned} \tag{21}$$

One can immediately check that $\dim(\mathscr{D}^1)=5$. Indeed, the singular points of $\mathscr{D}^1$ are only those satisfying $1-\dot{x}_1^2-\dot{x}_2^2=0$. Therefore $\widetilde{D}_1=\{(t,x_1,x_2,\dot{x}_1,\dot{x}_2|1-\dot{x}_1^2-\dot{x}_2^2\neq 0\}$. Here G^1 acts transitively in each of the two unconnected components of $\widetilde{D}_1$ and also in the set $1-\dot{x}_1^2-\dot{x}_2^2=0$. Accordingly, it is impossible to find a single first-order system of type (1) symmetric under G.

The second-order extension of G is defined by

$$\begin{aligned}
&\mathbf{S}_1^{(2}=\frac{\partial}{\partial t};\quad \mathbf{S}_{1+i}^{(2}=\frac{\partial}{\partial x_i},\\
&\mathbf{S}_4^{(2}=\mathbf{S}_4^{(1}+\left(-\ddot{x}_2\frac{\partial}{\partial \ddot{x}_1}+\ddot{x}_1\frac{\partial}{\partial \ddot{x}_2}\right),\\
&\mathbf{S}_5^{(2}=\mathbf{S}_5^{(1}+\left(3\dot{x}_1\ddot{x}_1\frac{\partial}{\partial \ddot{x}_1}+(2\dot{x}_1\ddot{x}_2+\dot{x}_2\ddot{x}_1)\frac{\partial}{\partial \ddot{x}_2}\right),\\
&\mathbf{S}_6^{(2}=\mathbf{S}_6^{(1}+\left((2\dot{x}_2\ddot{x}_1+\dot{x}_1\ddot{x}_2)\frac{\partial}{\partial \ddot{x}_1}+3\dot{x}_2\ddot{x}_2\frac{\partial}{\partial \ddot{x}_2}\right).
\end{aligned} \tag{22}$$

We can see that $\dim\mathscr{D}^2=6$. Since $\dim\mathscr{D}^2=6>\dim(\mathscr{D}^1)=5$ condition (11) is not satisfied. But from this fact one cannot conclude, in general, that there are not systems of type (1), for $m = 2$, symmetrical under G. In fact the system $\ddot{\bar{x}} = \mathbf{0}$ is a system invariant under G.

Note that since $\dim \mathscr{D}^2 = 6$ and the group has six parameters it is clear that for $k \geqslant 2$ we will have $\dim(\mathscr{D}^k) = 6$. Therefore (12) is satisfied for every $m > 2$ and by the results of Sec. II we can say that there are two-parameter families of m-order ($m > 2$) differential systems symmetrical under G.

(2) Let us take now as G the conformal group in $R \times R$. (Note that the space is now only one dimensional.)

The six generators of this group can be taken as

$$\mathbf{S}_1 = \frac{\partial}{\partial t}; \quad \mathbf{S}_2 = \frac{\partial}{\partial x}; \quad \mathbf{S}_3 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x},$$
$$\mathbf{S}_4 = t\frac{\partial}{\partial x} + x\frac{\partial}{\partial t}; \quad \mathbf{S}_5 = (t^2 + x^2)\frac{\partial}{\partial t} + 2tx\frac{\partial}{\partial x}, \tag{23}$$
$$\mathbf{S}_6 = 2tx\frac{\partial}{\partial t} + (t^2 + x^2)\frac{\partial}{\partial x},$$

and therefore the third-order extension of them will be given by

$$\mathbf{S}_1^{(3} = \frac{\partial}{\partial t}, \quad \mathbf{S}_2^{(3} = \frac{\partial}{\partial x},$$
$$\mathbf{S}_3^{(3} = \mathbf{S}_3 - \ddot{x}\frac{\partial}{\partial \ddot{x}} - 2\dddot{x}\frac{\partial}{\partial \dddot{x}},$$
$$\mathbf{S}_4^{(3} = \mathbf{S}_4 + (1 - \dot{x}^2)\frac{\partial}{\partial \dot{x}} - 2\ddot{x}\frac{\partial}{\partial \ddot{x}}$$
$$+ (-4\dot{x}\dddot{x} - 3\ddot{x}^2)\frac{\partial}{\partial \dddot{x}},$$
$$\mathbf{S}_5^{(3} = \mathbf{S}_5 + (2x - 2x\dot{x}^2)\frac{\partial}{\partial \dot{x}}$$
$$+ (2\dot{x} - 2\dot{x}^3 - 2\ddot{x}t - 6x\dot{x}\ddot{x})\frac{\partial}{\partial \ddot{x}} \tag{24}$$
$$+ \{-6\ddot{x}(2\dot{x}^2 + x\ddot{x}) - 4\dddot{x}(t + 2x\dot{x})\}\frac{\partial}{\partial \dddot{x}},$$
$$\mathbf{S}_6^{(3} = \mathbf{S}_6 + 2t(1 - \dot{x}^2)\frac{\partial}{\partial \dot{x}}$$
$$+ \{2(1 - \dot{x}^2) - 2\ddot{x}(x + 3t\dot{x})\}\frac{\partial}{\partial \ddot{x}}$$
$$+ \{-6\ddot{x}(2\dot{x} + t\ddot{x}) - 4\dddot{x}(x + 2t\dot{x})\}\frac{\partial}{\partial \dddot{x}}.$$

From (24) it follows immediately that

$$\dim \mathscr{D}^0 = 2; \quad \dim \mathscr{D}^1 = 3; \quad \dim \mathscr{D}^2 = 4; \quad \dim \mathscr{D}^3 = 5. \tag{25}$$

We see in (25) that condition (11) is not fulfilled for $m = 1$, $m = 2$, and $m = 3$. This implies, as we shall prove in Sec. IV, that there are no one-parameter families of differential equations of first-, second-, or third-order invariant under this group. Nevertheless, as we show now, there is one (and only one) third-order differential equation invariant under G.

Indeed, invariance under $\mathbf{S}_1$ and $\mathbf{S}_2$ implies that the third-order equation will have the form

$$\dddot{x} = f(\dot{x}, \ddot{x}). \tag{26}$$

Invariance under $\mathbf{S}_3$ implies

$$\ddot{x}\frac{\partial f}{\partial \ddot{x}} = 2f, \tag{27}$$

that is,

$$f = a(\dot{x})\ddot{x}^2, \tag{28}$$

where $a(\dot{x})$ is an arbitrary function.

Invariance under $\mathbf{S}_4$ implies

$$(1 - \dot{x}^2)\frac{da}{d\dot{x}} - 6\dot{x}a = -4\dot{x}a - 3, \tag{29}$$

and therefore,

$$a(\dot{x}) = (3\dot{x} - b)/(\dot{x}^2 - 1). \tag{30}$$

Finally invariance under $\mathbf{S}_5$ implies $b = 0$. The resulting third-order differential equation is automatically invariant under $\mathbf{S}_6$.

Therefore we have obtained the differential equation

$$\dddot{x} = \frac{3\dot{x}\ddot{x}^2}{\dot{x}^2 - 1}, \tag{31}$$

which is the only one invariant under the conformal group in $R \times R$.

Note that since $\mathscr{D}^1$ does not act transitively on the whole $(t, x, \dot{x})$ space it is possible to have also first-order differential equations invariant under G. This is precisely what happens with the two differential equations

$$\dot{x} = 1, \quad \dot{x} = -1, \tag{32}$$

which are the only ones (of first order) invariant under G.

Observe that the set $\{(t, x, \dot{x}) \mid 1 - \dot{x}^2 = 0\}$ defines the singular points of $\mathscr{D}^1$, that is, the points where $\mathscr{D}^1$ has (in this case) dimension 2.

Although $\mathscr{D}^2$ also has singular points, where its dimension does not attain the maximum value, it is easy to check by direct computation that there do not exist second-order equations invariant under the conformal group in $R \times R$. (The generators $\mathbf{S}_1^{2}, \mathbf{S}_2^{2}, \mathbf{S}_3^{2}, \mathbf{S}_4^{(2}$ imply $\ddot{x} = 0$, but this equation is incompatible with the two generators $\mathbf{S}_5^{(2}$ and $\mathbf{S}_6^{(2}$.)

IV. *n*-PARAMETER FAMILIES OF EQUATIONS INVARIANT UNDER *G*

The above examples show that condition (11) is, in general, not necessary for the existence of isolated systems of order $\underline{m}$ invariant under G. We prove here that (11) is necessary and sufficient for the existence of n-parameter families of equations, each of them being invariant under G.

In fact, the necessary and sufficient conditions in order that the n-parameter family defined by

$$\mathbf{E}(t, \bar{x}, \dot{\bar{x}}, \ldots, \bar{x}^{(m)}) = \bar{c}, \quad \det\left(\frac{\partial \bar{E}}{\partial \bar{x}^{(m)}}\right) \neq 0, \tag{33}$$

be invariant under G are

$$\mathbf{S}_i^{(m}(\mathbf{E})_{|\mathbf{E} = \mathbf{C}} = \mathbf{0}, \quad i = 1, \ldots, d_m. \tag{34}$$

Since Eqs. (34) hold as identities in $\mathbf{C} \in R^n$ we must have

$$\mathbf{S}_i^{(m}(\mathbf{E}) = \mathbf{0}. \tag{35}$$

Let us see that (35) and

$$\dim(\mathscr{D}^{m-1}) < \dim(\mathscr{D}^m) \tag{36}$$

are contradictory.

In fact, as explained in Sec. II, if $\mathbf{S}_1^{(m},...,\mathbf{S}_{d_m}^{(m}$ is a basis of $\mathscr{D}^m$ then $\mathbf{S}_1^{(m-1},...,\mathbf{S}_{d_m}^{(m-1}$ is a basis of $\mathscr{D}^{m-1}$. Therefore if (36) holds we must have

$$\sum_{i=1}^{d_m} c_i(t,\mathbf{a},...,\mathbf{a}^{(m-1)})\mathbf{S}_i^{(m-1} = \mathbf{0}, \tag{37}$$

where *not* all of the c_i are equal to zero.

Therefore

$$\sum_{i=1}^{d_m} c_i \mathbf{S}_i^{(m}(\mathbf{E}) \overset{(37)}{=} \sum_{i=1}^{d_m} c_i \sum_{j=1}^{n} \psi_{ij}^{(m} \frac{\partial(\mathbf{E})}{\partial x_j^{(m}} \overset{(35)}{=} \mathbf{0}, \tag{38}$$

contradicting the hypothesis of $\det(\partial \mathbf{E}/\partial \mathbf{x}^{(m})\neq 0$ imposed in (33). Note that this hypothesis concerning the determinant is essential in order to be able to apply the implicit function theorem to the variables $x_1^{(m},...,x_n^{(m}$ and put Eqs. (33) in the normal form (1). To conclude, we give an example of a one-parameter family of second-order differential equations invariant under the Poincaré group in $R\times R$.

In this case, the generators of G can be taken as

$$\mathbf{S}_1 = \frac{\partial}{\partial t}, \quad \mathbf{S}_2 = \frac{\partial}{\partial x}, \quad \mathbf{S}_3 = x\frac{\partial}{\partial t} + t\frac{\partial}{\partial x}. \tag{39}$$

The first and second extensions are given by

$$\mathbf{S}_1^{(1} = \frac{\partial}{\partial t}; \quad \mathbf{S}_2^{(1} = \frac{\partial}{\partial x}; \quad \mathbf{S}_3^{(1} = \mathbf{S}_3 + (1-\dot{x}^2)\frac{\partial}{\partial \dot{x}}, \tag{40}$$

and

$$\mathbf{S}_1^{(2} = \frac{\partial}{\partial t}; \quad \mathbf{S}_2^{(2} = \frac{\partial}{\partial x}; \quad \mathbf{S}_3^{(2} = \mathbf{S}_3^{(1} + (-3\dot{x}\ddot{x})\frac{\partial}{\partial \ddot{x}}. \tag{41}$$

One can immediately see that

$$\dim \mathscr{D}^0 = 2, \quad \dim \mathscr{D}^1 = 3. \tag{42}$$

Since the group has three parameters it is clear that $\dim \mathscr{D}^k = 3$ for any $k\geqslant 1$. Therefore there are one-parameter families of differential equations of order m (for any $m\geqslant 2$) invariant under this group. Taking for simplicity $m=2$, the symmetry of the equation

$$\ddot{x} = f(t,x,x) \tag{43}$$

under $\mathbf{S}_1$ and $\mathbf{S}_2$ implies

$$\frac{\partial f}{\partial t} = 0; \quad \frac{\partial f}{\partial x} = 0, \tag{44}$$

that is, $f(t,x,x) = g(\dot{x})$. The symmetry under the boosts S_3 implies

$$-3\dot{x}g = \frac{dg}{d\dot{x}}(1-\dot{x}^2) \tag{45}$$

which leads, after integration, to the one-parameter family

$$\ddot{x} = c(1-\dot{x}^2)^{3/2}. \tag{46}$$

[1] F. González-Gascón and A. González-López, J. Math. Phys. **24**, 2006 (1983).

[2] P. Olver, *Applications of Lie Groups to Differential Equations* (Springer, New York, 1986).

[3] F. Warner, *Foundations of Differential Geometry and Lie Groups* (Freeman, San Francisco, 1971).

[4] A. González-López, Ph.D. thesis, Universidad Complutense, Madrid, 1984 (unpublished).

[5] F. González-Gascón and A. González-López, Hadronic J. **6**, 841 (1983).