

Elliptic solutions in the Neumann–Rosochatius system with mixed fluxRafael Hernández^{*} and Juan Miguel Nieto[†]*Departamento de Física Teórica I, Universidad Complutense de Madrid, 28040 Madrid, Spain*

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Closed strings spinning in $\text{AdS}_3 \times S^3 \times T^4$ with mixed Ramond-Ramond and Neveu-Schwarz-Neveu-Schwarz three-form fluxes are described by a deformation of the one-dimensional Neumann–Rosochatius integrable system. In this article we find general solutions to this system that can be expressed in terms of elliptic functions. We consider closed strings rotating either in S^3 with two different angular momenta or in AdS_3 with one spin. To find the solutions, we will need to extend the Uhlenbeck integrals of motion of the Neumann–Rosochatius system to include the contribution from the flux. In the limit of pure Neveu-Schwarz-Neveu-Schwarz flux, where the problem can be described by a supersymmetric Wess-Zumino-Witten model, we find exact expressions for the classical energy in terms of the spin and the angular momenta of the spinning string.

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I. INTRODUCTION

The $\text{AdS}_3/\text{CFT}_2$ correspondence relates string theory backgrounds containing an AdS_3 factor and two-dimensional conformal field theories with maximal supersymmetry. As in other examples of the AdS/CFT correspondence, an integrable structure is also present in the case of $\text{AdS}_3/\text{CFT}_2$. The first explicit proof that integrability is a symmetry of the $\text{AdS}_3/\text{CFT}_2$ correspondence came from the observation that the Green–Schwarz action of type IIB string theory with Ramond-Ramond (R-R) three-form flux compactified on $\text{AdS}_3 \times S^3 \times M_4$, with M_4 taken as either T^4 or $S^3 \times S^1$, is a classically integrable theory [1]. This discovery led immediately to an exhaustive analysis of various aspects of the $\text{AdS}_3/\text{CFT}_2$ correspondence using techniques inherited from other integrable systems [2–10] (see Ref. [11] for a comprehensive review). It was later on proved that integrability also remains a symmetry of more general string backgrounds including a mixture of R-R and Neveu-Schwarz-Neveu-Schwarz (NS-NS) three-form fluxes [12]. This result has been responsible for all the recent insight in the understanding of type IIB string theory on $\text{AdS}_3 \times S^3 \times T^4$ with mixed fluxes [13–23].

Whenever integrability is present in a system, it shows up in many different facets. In the case of the $\text{AdS}_5/\text{CFT}_4$ correspondence, an appealing approach to the search for the spectra of the theory came from the identification of the Lagrangian describing closed strings rotating in $\text{AdS}_5 \times S^5$ with the Neumann–Rosochatius integrable system [24]. The Neumann–Rosochatius system is an integrable model describing an oscillator on a sphere or a hyperboloid with a centrifugal potential term. In Ref. [25] it was shown that the presence of a nonvanishing NS-NS three-form flux introduces a deformation in the Lagrangian of the Neumann–Rosochatius system. Integrability of the deformed system

provides a systematic method to construct general solutions corresponding to closed string configurations rotating in $\text{AdS}_3 \times S^3 \times T^4$ with mixed R-R and NS-NS fluxes. The most immediate class of solutions that can be obtained in this way is the closed strings with constant radii found in Ref. [25] (string solutions with mixed fluxes were studied before using diverse approaches in Refs. [14–18]). The purpose of this article is to exploit the flux-deformed Neumann–Rosochatius system to construct more general solutions.

The plan of the article is the following. In Sec. II we will present the problem by considering an ansatz for a closed string rotating with two different angular momenta in S^3 with a NS-NS three-form flux. The resulting Lagrangian is the Neumann–Rosochatius system with an additional contribution coming from the nonvanishing flux term. We will find the deformation introduced by the flux term in the Uhlenbeck integrals of motion of the system. In Sec. III we will construct a general class of solutions with nonconstant radii that can be expressed in terms of Jacobian elliptic functions. In the limit of pure NS-NS flux, the problem can be described by a supersymmetric Wess-Zumino-Witten (WZW) model. In this limit the elliptic solutions reduce to trigonometric functions, and we can find compact expressions for the classical energy of the rotating strings in terms of the angular momenta. In Sec. IV we extend the analysis to the case where the string is rotating in AdS_3 . We conclude in Sec. V with several remarks and some discussion on our results. We include an Appendix where, in the limit of pure NS-NS flux, we solve the general case where the string is allowed to rotate both in AdS_3 and S^3 .

II. NEUMANN–ROSOCHATIUS SYSTEM WITH MIXED FLUX

In this article we will analyze the motion of closed strings rotating in $\text{AdS}_3 \times S^3 \times T^4$ with nonvanishing NS-NS

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three-form flux. We will consider no dynamics along the torus, and thus the background metric will be

$$ds^2 = -\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\phi^2 + d\theta^2 + \sin^2 \theta d\phi_1^2 + \cos^2 \theta d\phi_2^2, \quad (2.1)$$

and the NS-NS B-field will be

$$b_{t\phi} = q \sinh^2 \rho, \quad b_{\phi_1 \phi_2} = -q \cos^2 \theta, \quad (2.2)$$

where $0 \leq q \leq 1$. The limit $q = 0$ corresponds to the case of pure R-R flux, while setting $q = 1$ we are left with pure NS-NS flux. In the case of pure R-R flux, the sigma model for closed strings rotating in $\text{AdS}_3 \times S^3$ can be reduced to the Neumann–Rosochatius integrable system [24]. The presence of NS-NS flux introduces an additional term in the Lagrangian of the Neumann–Rosochatius system [25]. To exhibit this it is convenient to use the embedding coordinates of AdS_3 and S^3 , which are related to the global angles by

$$Y_1 + iY_2 = \sinh \rho e^{i\phi}, \quad Y_3 + iY_4 = \cosh \rho e^{i\theta}, \quad (2.3)$$

$$X_1 + iX_2 = \sin \theta e^{i\phi_1}, \quad X_3 + iX_4 = \cos \theta e^{i\phi_2}. \quad (2.4)$$

For simplicity in this section and Sec. III, we will restrict the motion of the strings to rotation on S^3 . The extension to strings spinning in AdS_3 will be considered in Sec. IV. We will thus take $Y_3 + iY_4 = e^{i\nu_0 \tau}$ and $Y_1 = Y_2 = 0$, together with an ansatz for a closed string rotating with two different angular momenta along S^3 ,

$$\begin{aligned} X_1 + iX_2 &= r_1(\sigma) e^{i\varphi_1(\tau, \sigma)}, \\ X_3 + iX_4 &= r_2(\sigma) e^{i\varphi_2(\tau, \sigma)}. \end{aligned} \quad (2.5)$$

The angles are chosen as

$$\varphi_i(\tau, \sigma) = \omega_i \tau + \alpha_i(\sigma), \quad (2.6)$$

and the condition that the solutions lie on a three-sphere implies that

$$r_1^2 + r_2^2 = 1. \quad (2.7)$$

When we enter this ansatz in the world sheet action in the conformal gauge, we find

$$\begin{aligned} L_{S^3} &= \frac{\sqrt{\lambda}}{2\pi} \left[\sum_{i=1}^2 \frac{1}{2} [(r_i')^2 + r_i^2 (\alpha_i')^2 - r_i^2 \omega_i^2] \right. \\ &\quad \left. - \frac{\Lambda}{2} (r_1^2 + r_2^2 - 1) + q r_2^2 (\omega_1 \alpha_2' - \omega_2 \alpha_1') \right], \end{aligned} \quad (2.8)$$

where the prime stands for derivatives with respect to σ and Λ is a Lagrange multiplier. The equations of motion for the radial coordinates following from (2.8) are

$$r_1'' = -r_1 \omega_1^2 + r_1 \alpha_1'^2 - \Lambda r_1, \quad (2.9)$$

$$r_2'' = -r_2 \omega_2^2 + r_2 \alpha_2'^2 - \Lambda r_2 + 2q r_2 (\omega_1 \alpha_2' - \omega_2 \alpha_1'). \quad (2.10)$$

The equations for the angles can be easily integrated once,

$$\alpha_i' = \frac{v_i + q r_2^2 \epsilon_{ij} \omega_j}{r_i^2}, \quad i = 1, 2, \quad (2.11)$$

where v_i are some integration constants and we have chosen $\epsilon_{12} = +1$. These equations must be accompanied by the Virasoro constraints, which read

$$\sum_{i=1}^2 [r_i'^2 + r_i^2 (\alpha_i'^2 + \omega_i^2)] = w_0^2, \quad (2.12)$$

$$\sum_{i=1}^2 r_i^2 \omega_i \alpha_i' = \sum_{i=1}^2 v_i \omega_i = 0. \quad (2.13)$$

Furthermore, dealing with closed string solutions requires

$$\begin{aligned} r_i(\sigma + 2\pi) &= r_i(\sigma), \\ \alpha_i(\sigma + 2\pi) &= \alpha_i(\sigma) + 2\pi \bar{m}_i, \end{aligned} \quad (2.14)$$

where \bar{m}_i are integer numbers that behave as winding numbers. The energy and the two angular momenta of the string are given by

$$E = \sqrt{\lambda} w_0, \quad (2.15)$$

$$J_1 = \sqrt{\lambda} \int_0^{2\pi} \frac{d\sigma}{2\pi} (r_1^2 \omega_1 - q r_2^2 \alpha_2'), \quad (2.16)$$

$$J_2 = \sqrt{\lambda} \int_0^{2\pi} \frac{d\sigma}{2\pi} (r_2^2 \omega_2 + q r_2^2 \alpha_1'). \quad (2.17)$$

Before concluding our presentation of the flux-deformed Neumann–Rosochatius Lagrangian, we must note that there is a gauge freedom in the choice of the NS-NS two-form (2.2). As a consequence of this ambiguity, the flux term in (2.8) could have been written in the form $q[(2-c)r_2^2 - cr_1^2]/2$, where c is an arbitrary parameter. This parameter does not affect the equations of motion, but it contributes as a total derivative to the angular momenta, and thus it could show up if nontrivial boundary conditions were imposed.¹ However, for the spinning string solutions that we will consider in this article, the c -term plays no role because it can be absorbed in a redefinition of the v_i integration constants and a shift of the angular momenta.

¹See Ref. [14] for a detailed discussion on this point concerning the dyonic giant magnon solutions.

A. Integrals of motion

The integrability of the Neumann–Rosochatius system follows from the existence of a set of integrals of motion in involution, the Uhlenbeck constants. The Uhlenbeck constants were first found in Ref. [26] for the case $\alpha_i = \text{constant}$ and were extended to general values of α_i in Ref. [24]. In the case of a closed string rotating in S^3 , there are two integrals I_1 and I_2 , but as they must satisfy the constraint $I_1 + I_2 = 1$, we are left with a single independent constant,

$$I_1 = r_1^2 + \frac{1}{\omega_1^2 - \omega_2^2} \left[(r_1 r_2' - r_1' r_2)^2 + \frac{v_1^2}{r_1^2} r_2^2 + \frac{v_2^2}{r_2^2} r_1^2 \right]. \quad (2.18)$$

Furthermore, the Hamiltonian of the Neumann–Rosochatius system,

$$H = \frac{1}{2} \sum_{i=1}^2 [r_i'^2 + r_i^2 \alpha_i^2 + r_i^2 \omega_i^2], \quad (2.19)$$

can be written in terms of the Uhlenbeck constants and the integrals of motion v_i ,

$$H = \frac{1}{2} \sum_{i=1}^2 [\omega_i^2 I_i + v_i^2]. \quad (2.20)$$

When the NS-NS three-form is turned on, the Uhlenbeck constants should be deformed in some way. To find this deformation, we will assume that the extended constant can be written as

$$\bar{I}_1 = r_1^2 + \frac{1}{\omega_1^2 - \omega_2^2} \left[(r_1 r_2' - r_1' r_2)^2 + \frac{v_1^2}{r_1^2} r_2^2 + \frac{v_2^2}{r_2^2} r_1^2 + 2f \right], \quad (2.21)$$

where $f = f(r_1, r_2, q)$, with no dependence on r_1' or r_2' . This function can be determined if we impose that $\bar{I}_1' = 0$. After some immediate algebra, we find that

$$f' + \frac{(q^2 \omega_2^2 + 2q\omega_2 v_1) r_1'}{r_1^3} + q^2 (\omega_1^2 - \omega_2^2) r_1 r_1' = 0, \quad (2.22)$$

where we have used the constraint (2.7) together with

$$\begin{aligned} r_1 r_1' + r_2 r_2' &= 0, \\ r_1 r_1'' + (r_1')^2 + r_2 r_2'' + (r_2')^2 &= 0 \end{aligned} \quad (2.23)$$

and the equations of motion (2.9), (2.10), and (2.11). As all three terms in relation (2.22) are total derivatives, integration is immediate, and we readily conclude that the deformation of the Uhlenbeck constant is given by

$$\begin{aligned} \bar{I}_1 &= r_1^2 (1 - q^2) + \frac{1}{\omega_1^2 - \omega_2^2} \left[(r_1 r_2' - r_1' r_2)^2 \right. \\ &\quad \left. + \frac{(v_1 + q\omega_2)^2}{r_1^2} r_2^2 + \frac{v_2^2}{r_2^2} r_1^2 \right]. \end{aligned} \quad (2.24)$$

As in the absence of flux, the deformed constants satisfy the condition $\bar{I}_1 + \bar{I}_2 = 1$.² The Hamiltonian including the contribution from the NS-NS flux can also be written now using the deformed Uhlenbeck constants and the integrals of motion v_i ,

$$H = \frac{1}{2} \sum_{i=1}^2 [\omega_i^2 \bar{I}_i + v_i^2] + \frac{1}{2} q^2 (\omega_1^2 - \omega_2^2) - q\omega_1 v_2. \quad (2.25)$$

III. SPINNING STRINGS IN S^3

In this section we will construct general solutions of the Neumann–Rosochatius system in the presence on NS-NS flux corresponding to closed strings spinning in S^3 with two different angular momenta. A convenient way to analyze this problem is to introduce an ellipsoidal coordinate [27]. The ellipsoidal coordinate ζ is defined as the root of the equation

$$\frac{r_1^2}{\zeta - \omega_1^2} + \frac{r_2^2}{\zeta - \omega_2^2} = 0. \quad (3.1)$$

If we choose the angular frequencies such that $\omega_1 < \omega_2$, the range of the ellipsoidal coordinate is $\omega_1^2 \leq \zeta \leq \omega_2^2$. Now we could enter ζ directly into the equations of motion to find the second-order differential equation for this coordinate. Instead we will follow Ref. [24] and use the Uhlenbeck constants to reduce the problem to a first-order differential equation. To do this we just need to write the Uhlenbeck integral in terms of the ellipsoidal coordinate. This is immediate once we note that

$$(r_1 r_2' - r_2 r_1')^2 = \frac{\zeta'^2}{4(\omega_1^2 - \zeta)(\zeta - \omega_2^2)}. \quad (3.2)$$

When we solve for ζ'^2 in the deformed Uhlenbeck constant (2.24), we conclude that

$$\zeta'^2 = -4P_3(\zeta), \quad (3.3)$$

where $P_3(\zeta)$ is the third-order polynomial

²Note that if we had included the parameter c in the Lagrangian the Uhlenbeck integrals of motion would have satisfied also this constraint because c can be absorbed in the v_i integrals of motion.

$$\begin{aligned}
 P_3(\zeta) &= (1 - q^2)(\zeta - \omega_1^2)^2(\zeta - \omega_2^2) \\
 &\quad + (\zeta - \omega_1^2)(\zeta - \omega_2^2)(\omega_1^2 - \omega_2^2)\bar{I}_1 \\
 &\quad + (\zeta - \omega_1^2)^2 v_2^2 + (\zeta - \omega_2^2)^2 (v_1 + q\omega_2)^2 \\
 &= (1 - q^2) \prod_{i=1}^3 (\zeta - \zeta_i). \tag{3.4}
 \end{aligned}$$

This polynomial defines an elliptic curve $s^2 + P_3(\zeta) = 0$. In fact if we change variables to

$$\zeta = \zeta_2 + (\zeta_3 - \zeta_2)\eta^2, \tag{3.5}$$

Eq. (3.3) becomes the differential equation for the Jacobian elliptic cosine,

$$\eta'^2 = (1 - q^2)(\zeta_3 - \zeta_1)(1 - \eta^2)(1 - \kappa + \kappa\eta^2), \tag{3.6}$$

where the elliptic modulus is given by $\kappa = (\zeta_3 - \zeta_2)/(\zeta_3 - \zeta_1)$. The solution is thus

$$\eta(\sigma) = \text{cn}\left(\sigma\sqrt{(1 - q^2)(\zeta_3 - \zeta_1)} + \sigma_0, \kappa\right), \tag{3.7}$$

with σ_0 an integration constant that can be set to zero by performing a rotation. Therefore, we conclude that

$$r_1^2(\sigma) = \frac{\zeta_3 - \omega_1^2}{\omega_2^2 - \omega_1^2} + \frac{\zeta_2 - \zeta_3}{\omega_2^2 - \omega_1^2} \text{sn}^2\left(\sigma\sqrt{(1 - q^2)(\zeta_3 - \zeta_1)}, \kappa\right). \tag{3.8}$$

We must stress that we need to order the roots in such a way that $\zeta_1 < \zeta_3$ to make sure that the argument of the elliptic sine is real. We also need $\zeta_2 < \zeta_3$ to have $\kappa > 0$, together with $\zeta_1 < \zeta_2$ to keep $\kappa < 1$. Furthermore, imposing that (3.8) must have codomain between 0 and 1 demands $\omega_1^2 \leq \zeta_{2,3} \leq \omega_2^2$. Note that this restriction does not apply to ζ_1 . This is a solution of circular type.³ The periodicity condition on the radial coordinates implies that⁴

$$\pi\sqrt{(1 - q^2)(\zeta_3 - \zeta_1)} = n\mathbf{K}(\kappa), \tag{3.9}$$

³In the absence of R-R flux and setting the v_i integrals to zero, this choice of parameters corresponds to solutions of circular type when I_1 is taken as negative, or solutions of folded type when I_1 is positive [24].

⁴There are four cases in which we have to alter this periodicity condition. When either $v_1 + q\omega_2 = 0$ or $v_2 = 0$, the condition becomes $\frac{\pi}{2}\sqrt{(1 - q^2)(\zeta_3 - \zeta_1)} = n\mathbf{K}(\kappa)$ because of changes of branch when we take the square root of (3.8). The two remaining cases correspond to the limit $\zeta_3 \rightarrow \zeta_2$, which is the constant radii case, and to the limit $\kappa \rightarrow 1$, where the periodicity of the elliptic sine becomes infinite. In both cases there is no periodicity condition. We will discuss these two limits below in this section.

where we have used that $2\mathbf{K}(\kappa)$ is the period of the square of the Jacobi sine, with $\mathbf{K}(\kappa)$ the complete elliptic integral of the first kind and n an integer number. We can use now Eq. (3.8) to write the winding numbers \bar{m}_i in terms of the integration constants v_i and the angular frequencies ω_i . From the periodicity condition on α_1 ,

$$2\pi\bar{m}_1 = \int_0^{2\pi} \alpha'_1 d\sigma = \int_0^{2\pi} \left(\frac{v_1}{r_1^2} + q\omega_2 \frac{r_2^2}{r_1^2}\right) d\sigma, \tag{3.10}$$

we can write

$$\frac{\bar{m}_1 + q\omega_2}{v_1 + q\omega_2} = \int_0^{2\pi} \frac{d\sigma}{2\pi r_1^2}. \tag{3.11}$$

Inserting (3.8) in this expression and performing the integration, we find

$$\begin{aligned}
 \bar{m}_1 + q\omega_2 &= \frac{(v_1 + q\omega_2)(\omega_2^2 - \omega_1^2)}{(\zeta_3 - \omega_1^2)\mathbf{K}(\kappa)} \Pi\left(\frac{\zeta_3 - \zeta_2}{\zeta_3 - \omega_1^2}, \kappa\right), \tag{3.12}
 \end{aligned}$$

where $\Pi(a, b)$ is the complete elliptic integral of the third kind. In a similar way, from the periodicity condition for α_2 ,

$$2\pi\bar{m}_2 = \int_0^{2\pi} \alpha'_2 d\sigma = \int_0^{2\pi} \left(\frac{v_2}{r_2^2} - q\omega_1\right) d\sigma, \tag{3.13}$$

we find that

$$\frac{\bar{m}_2 + q\omega_1}{v_2} = \int_0^{2\pi} \frac{d\sigma}{2\pi r_2^2}, \tag{3.14}$$

which we can integrate to get

$$\bar{m}_2 + q\omega_1 = \frac{v_2(\omega_2^2 - \omega_1^2)}{(\omega_2^2 - \zeta_3)\mathbf{K}(\kappa)} \Pi\left(-\frac{\zeta_3 - \zeta_2}{\omega_2^2 - \zeta_3}, \kappa\right). \tag{3.15}$$

We can perform an identical computation to obtain the angular momenta. From Eq. (2.16) we get

$$\frac{J_1}{\sqrt{\lambda}} + qv_2 - q^2\omega_1 = \omega_1(1 - q^2) \int_0^{2\pi} \frac{d\sigma}{2\pi} r_1^2, \tag{3.16}$$

and therefore

$$\begin{aligned}
 \frac{J_1}{\sqrt{\lambda}} &= \frac{\omega_1(1 - q^2)}{\omega_2^2 - \omega_1^2} \left[\zeta_3 - \omega_1^2 - (\zeta_3 - \zeta_1) \left(1 - \frac{\mathbf{E}(\kappa)}{\mathbf{K}(\kappa)}\right) \right] \\
 &\quad - qv_2 + q^2\omega_1, \tag{3.17}
 \end{aligned}$$

with $\mathbf{E}(\kappa)$ the complete elliptic integral of the second kind. As before, Eq. (2.17) implies

$$\frac{J_2}{\sqrt{\lambda}} + qv_1 - q\bar{m}_1 = \omega_2(1 - q^2) \int_0^{2\pi} \frac{d\sigma}{2\pi} r_2^2, \quad (3.18)$$

and thus after integration we conclude that

$$\frac{J_2}{\sqrt{\lambda}} = \frac{\omega_2(1 - q^2)}{\omega_2^2 - \omega_1^2} \left[\omega_2^2 - \zeta_3 + (\zeta_3 - \zeta_1) \left(1 - \frac{E(\kappa)}{K(\kappa)} \right) \right] - qv_1 + q\bar{m}_1. \quad (3.19)$$

These expressions for the angular momenta can be used to rewrite the first Virasoro constrain (2.13) as

$$\omega_2 J_1 + \omega_1 J_2 = \sqrt{\lambda}(\omega_1 \omega_2 + q\omega_1 \bar{m}_1). \quad (3.20)$$

In principle we could now employ these relations to write the energy in terms of the winding numbers \bar{m}_i , the angular momenta J_i , and the Uhlenbeck constants. The first step to achieve this is to express the energy in terms of the frequencies ω_i and the constants v_i . Then we need to write the v_i in terms of the angular momenta using relations (3.17) and (3.19). However, solving these equations for arbitrary values of q is a very involved problem, and we will not present a discussion on it here. Instead in the following subsection, we will focus on the only case where a simple analysis is possible, which is that of pure NS-NS flux.

But before exploring our solutions in the regime of pure NS-NS flux, we will discuss an important limit where they can be reduced to simpler ones. It corresponds to the choices of parameters that make the discriminant of $P_3(\zeta)$ equal to zero. Our hierarchy of roots implies that there are only three cases able to fulfill this condition. The first corresponds to solutions with constant radii, where $\zeta_2 = \zeta_3$. These solutions were first constructed in Ref. [14] and later on recovered by deriving the corresponding finite-gap equations in Ref. [18] or by solving the equations of motion for the flux-deformed Neumann-Rosochatius system in Ref. [25]. The second case corresponds to the limit $\kappa = 1$, which is obtained when $\zeta_1 = \zeta_2$. These are the giant magnons analyzed in Ref. [17] for the $v_2 = 0$ case and in Ref. [15] for general values of v_2 (giant magnon solutions were also constructed in Refs. [14,18]). We must stress that in this case the periodicity condition cannot be imposed because the elliptic sine has an infinite period and the string does not close. Also factors $\sqrt{(1 - q^2)(\zeta_3 - \zeta_1)}/nK(\kappa)$ which had been canceled in the expressions for the angular momenta and windings do not cancel anymore. The third case corresponds to setting $\zeta_1 = \zeta_2 = \zeta_3$ and cannot be obtained unless we have equal angular frequencies, $\omega_1 = \omega_2$.

A. Solutions with pure NS-NS flux

The cubic term in the polynomial $P_3(\zeta)$ is dressed with a factor $1 - q^2$. Therefore, in the case of pure NS-NS three-form flux, the degree of the polynomial reduces to 2, and

the solution can be written using trigonometric functions. In this limit⁵

$$\zeta'^2 = -4P_2(\zeta), \quad (3.21)$$

with $P_2(\zeta)$ the second-order polynomial

$$\begin{aligned} P_2(\zeta) &= (\zeta - \omega_1^2)(\zeta - \omega_2^2)(\omega_1^2 - \omega_2^2)\bar{I}_1 + (\zeta - \omega_1^2)^2 v_2^2 \\ &\quad + (\zeta - \omega_2^2)^2 (v_1 + \omega_2)^2 \\ &= \omega^2(\zeta - \tilde{\zeta}_1)(\zeta - \tilde{\zeta}_2), \end{aligned} \quad (3.22)$$

where ω^2 is

$$\omega^2 = (\omega_1^2 - \omega_2^2)\bar{I}_1 + (v_1 + \omega_2)^2 + v_2^2. \quad (3.23)$$

The solution to Eq. (3.21) is given by

$$\zeta(\sigma) = \tilde{\zeta}_2 + (\tilde{\zeta}_1 - \tilde{\zeta}_2)\sin^2(\omega\sigma), \quad (3.24)$$

where we have set to zero an integration constant by performing a rotation. Therefore,

$$r_1^2(\sigma) = \frac{\tilde{\zeta}_2 - \omega_1^2}{\omega_2^2 - \omega_1^2} + \frac{\tilde{\zeta}_1 - \tilde{\zeta}_2}{\omega_2^2 - \omega_1^2} \sin^2(\omega\sigma). \quad (3.25)$$

Periodicity of the radial coordinates implies that ω must be a half-integer number.⁶ The relation between the winding numbers \bar{m}_i and the constants v_i and the frequencies ω_i is now rather simple. The periodicity condition for the angles implies

$$\begin{aligned} \bar{m}_1 + \omega_2 &= \frac{(v_1 + \omega_2)(\omega_1^2 - \omega_2^2)}{\sqrt{(\omega_2^2 - \tilde{\zeta}_1)(\omega_1^2 - \tilde{\zeta}_2)}} \\ &= \frac{\omega(v_1 + \omega_2)(\omega_1^2 - \omega_2^2)}{\sqrt{P_2(\omega_1^2)}} \\ &= \omega \times \text{sgn}(v_1 + \omega_2), \end{aligned} \quad (3.26)$$

⁵We can also take the limit directly in Eq. (3.7) if we take into account that ζ_1 goes to minus infinity when we set $q = 1$. In this limit the elliptic modulus vanishes, but the factor $(1 - q^2)\zeta_1$ in the argument of the elliptic sine remains finite, and we just need to recall that $\text{sn}(x, 0) = \sin x$.

⁶An important exception to this condition happens when $\omega = J + \bar{m}_2$, which is a solution even if it is not a half-integer. This value corresponds to the case of constant radii, where [25]

$$\begin{aligned} \bar{I}_1 &= - \left| \frac{2(v_1 + \omega_2)v_2}{\omega_2^2 - \omega_1^2} \right|, & J_1 &= \frac{\bar{m}_2 J}{\bar{m}_2 - \bar{m}_1}, \\ J_2 &= \frac{\bar{m}_1 J}{\bar{m}_1 - \bar{m}_2}, & E &= J - \bar{m}_1. \end{aligned}$$

$$\begin{aligned}\bar{m}_2 + \omega_1 &= \frac{v_2(\omega_1^2 - \omega_2^2)}{\sqrt{(\omega_2^2 - \zeta_1)(\omega_2^2 - \zeta_2)}} \\ &= \frac{\omega v_2(\omega_1^2 - \omega_2^2)}{\sqrt{P_2(\omega_2^2)}} \\ &= \omega \times \text{sgn}(v_2).\end{aligned}\quad (3.27)$$

From the definition of the angular momenta, we find

$$\frac{J_1}{\sqrt{\lambda}} = \omega_1 - v_2, \quad \frac{J_2}{\sqrt{\lambda}} = \bar{m}_1 - v_1. \quad (3.28)$$

We can now write the energy as a function of the winding numbers and the angular momenta. A convenient way to do this is recalling the relation between the energy and the Uhlenbeck constant. If we assume that both $v_1 + \omega_2$ and v_2 are positive (the extension to the other possible signs of $v_1 + \omega_2$ and v_2 is immediate) and we combine Eqs. (2.25) and (3.23), we can write

$$E^2 = \lambda(\omega^2 + \omega_1^2 - \omega_2^2 - 2v_1\omega_2 - 2v_2\omega_1), \quad (3.29)$$

and thus using relations (3.26)–(3.28), we conclude that

$$\begin{aligned}E^2 &= \lambda\bar{m}_1^2 + (2\sqrt{\lambda}J_1 - \lambda(\omega - \bar{m}_2))(\omega - \bar{m}_2) \\ &\quad + 2\sqrt{\lambda}J_2(\omega - \bar{m}_1).\end{aligned}\quad (3.30)$$

Now we can use the Virasoro constraint (2.13) to write

$$\begin{aligned}J_1 &= \frac{(J - \sqrt{\lambda}\omega)(\omega - \bar{m}_2)}{\bar{m}_1 - \bar{m}_2}, \\ J_2 &= \frac{J(\bar{m}_1 - \omega) + \sqrt{\lambda}\omega(\omega - \bar{m}_2)}{\bar{m}_1 - \bar{m}_2},\end{aligned}\quad (3.31)$$

where $J = J_1 + J_2$ is the total angular momentum. Replacing these expressions in (3.30), we obtain the energy as a function of the winding numbers and the total momentum,

$$E^2 = \lambda(\bar{m}_1^2 - \bar{m}_2^2 + 4\omega\bar{m}_2 - 3\omega^2) - 2\sqrt{\lambda}J(\bar{m}_1 + \bar{m}_2 - 2\omega). \quad (3.32)$$

IV. SPINNING STRINGS IN AdS₃

In this section we will analyze the case where the strings are spinning in AdS₃. We will include no dynamics along S³ so that the conserved charges of the solutions in this section will be the energy and spin. We will therefore choose the ansatz

$$\begin{aligned}Y_3 + iY_0 &= z_0(\sigma)e^{i\phi_0(\tau,\sigma)}, \\ Y_1 + iY_2 &= z_1(\sigma)e^{i\phi_1(\tau,\sigma)},\end{aligned}\quad (4.1)$$

where the angles are chosen as

$$\phi_a(\tau, \sigma) = w_a\tau + \beta_a(\sigma), \quad (4.2)$$

with $a = 0, 1$. In this case periodicity requires

$$\begin{aligned}z_a(\sigma + 2\pi) &= z_a(\sigma), \\ \beta_a(\sigma + 2\pi) &= \beta_a(\sigma) + 2\pi\bar{k}_a.\end{aligned}\quad (4.3)$$

When we enter this ansatz in the world sheet action in the conformal gauge, we find

$$\begin{aligned}L_{\text{AdS}_3} &= \frac{\sqrt{\lambda}}{4\pi} \left[g^{ab}(z'_a z'_b + z_a z_a \beta'_b{}^2 - z_a z_a w_b^2) \right. \\ &\quad \left. - \frac{\tilde{\Lambda}}{2}(g^{ab}z_a z_b + 1) - 2qz_1^2(w_0\beta'_1 - w_1\beta'_0) \right],\end{aligned}\quad (4.4)$$

where we have taken $g = \text{diag}(-1, 1)$ and $\tilde{\Lambda}$ is the Lagrange multiplier needed to impose that the solutions lie on AdS₃,

$$z_1^2 - z_0^2 = -1. \quad (4.5)$$

The equations of motion for the radial coordinates are

$$z_0'' = z_0\beta_0'^2 - z_0w_0^2 - \tilde{\Lambda}z_0, \quad (4.6)$$

$$z_1'' = z_1\beta_1'^2 - z_1w_1^2 - \tilde{\Lambda}z_1 - 2qz_1(w_0\beta'_1 - w_1\beta'_0), \quad (4.7)$$

and the equations for the angles are

$$\beta'_a = \frac{u_a + qz_1^2\varepsilon_{ab}w_b}{g^{aa}z_a^2}, \quad (4.8)$$

with u_a some integrals of motion and $\varepsilon_{10} = +1$. The Virasoro constraints now read

$$z_0'^2 + z_0^2(\beta_0'^2 + w_0^2) = z_1'^2 + z_1^2(\beta_1'^2 + w_1^2), \quad (4.9)$$

$$z_1^2w_1\beta'_1 - z_0^2w_0\beta'_0 = u_0w_0 + u_1w_1 = 0. \quad (4.10)$$

The energy and the spin are given by

$$E = \sqrt{\lambda} \int_0^{2\pi} \frac{d\sigma}{2\pi} (z_0^2w_0 - qz_1^2\beta'_1), \quad (4.11)$$

$$S = \sqrt{\lambda} \int_0^{2\pi} \frac{d\sigma}{2\pi} (z_1^2w_1 - qz_1^2\beta'_0). \quad (4.12)$$

As in the previous section, in order to construct general solutions for strings rotating in AdS₃, it will be convenient to introduce an analytical continuation of the ellipsoidal coordinates. The definition of this coordinate μ can be

directly borrowed from the definition for the sphere with a change of sign,

$$\frac{z_1^2}{\mu - w_1^2} - \frac{z_0^2}{\mu - w_0^2} = 0. \quad (4.13)$$

If we order the frequencies such that $w_1 > w_0$, the range of the ellipsoidal coordinate will be $w_1^2 \leq \mu$. Now we can again make use of the Uhlenbeck constants to obtain a first-order differential equation for this coordinate. In the case of the Neumann–Rosochatius system in AdS₃, the Uhlenbeck integrals satisfy the constraint $F_1 - F_0 = -1$, and thus we are again left with a single independent constant. To obtain the deformation of, say, F_1 by the NS-NS flux, we can proceed in the same way as in Sec. II. After some immediate algebra, we conclude that

$$\begin{aligned} \bar{F}_1 = z_1^2(1 - q^2) + \frac{1}{w_1^2 - w_0^2} & \left[(z_1 z_0' - z_1' z_0)^2 \right. \\ & \left. + \frac{(u_0 + qw_1)^2}{z_0^2} z_1^2 + \frac{u_1^2}{z_1^2} z_0^2 \right]. \end{aligned} \quad (4.14)$$

The Hamiltonian can also be written now using the deformed Uhlenbeck constants and the integrals of motion u_a ,

$$H = \frac{1}{2} \sum_{a=0}^1 [g_{aa} w_a^2 \bar{F}_a - u_a^2] + qu_1 w_0. \quad (4.15)$$

Now we need to note that

$$(z_1 z_0' - z_0 z_1')^2 = \frac{\mu'^2}{4(\mu - w_1^2)(\mu - w_0^2)}. \quad (4.16)$$

When we solve for μ'^2 in the deformed integral, we find that

$$\mu'^2 = -4Q_3(\mu), \quad (4.17)$$

where $Q_3(\mu)$ is the third-order polynomial,

$$\begin{aligned} Q_3(\mu) &= (1 - q^2)(\mu - w_1^2)^2(\mu - w_0^2) \\ &+ (\mu - w_1^2)(\mu - w_0^2)(w_0^2 - w_1^2)\bar{F}_1 \\ &+ (\mu - w_1^2)^2(u_0 + qw_1)^2 + (\mu - w_0^2)^2 u_1^2 \\ &= (1 - q^2) \prod_{i=1}^3 (\mu - \mu_i). \end{aligned} \quad (4.18)$$

This equation is an analytic continuation of the spherical one, so we can write

$$z_0^2(\sigma) = \frac{\mu_3 - w_0^2}{w_1^2 - w_0^2} + \frac{\mu_2 - \mu_3}{w_1^2 - w_0^2} \operatorname{sn}^2 \left(\sigma \sqrt{(1 - q^2)(\mu_3 - \mu_1)}, \nu \right), \quad (4.19)$$

where the elliptic modulus is $\nu = (\mu_3 - \mu_2)/(\mu_3 - \mu_1)$. As in the case of strings rotating in S^3 , we must perform now an analysis of the roots of the polynomial. We need to choose $\mu_3 > \mu_1$ to make sure that the argument of the elliptic sine is real, and $\mu_3 > \mu_2$ to have $\nu > 0$, together with $\mu_2 > \mu_1$ to keep $\nu < 1$. Furthermore, the AdS₃ condition (4.5) implies that $z_0^2 \geq 1$, which constrains μ_2 and μ_3 to be greater than or equal to w_1^2 . This restriction does not apply to μ_1 . Note that this hierarchy of roots implies that not all possible combinations of the parameters u_i, w_i , and \bar{F}_1 are allowed. The periodicity condition on the radial coordinates now implies that⁷

$$\pi \sqrt{(1 - q^2)(\mu_3 - \mu_1)} = n' \mathbf{K}(\nu), \quad (4.20)$$

with n' an integer number. From the periodicity condition on β_1 ,

$$2\pi \bar{k}_1 = \int_0^{2\pi} \beta_1' d\sigma = \int_0^{2\pi} \left(\frac{u_1}{z_1^2} + qw_0 \right) d\sigma, \quad (4.21)$$

we can write

$$\frac{\bar{k}_1 - qw_0}{u_1} = \int_0^{2\pi} \frac{d\sigma}{2\pi z_1^2}. \quad (4.22)$$

Performing the integration we find

$$\bar{k}_1 - qw_0 = \frac{u_1(w_1^2 - w_0^2)}{(\mu_3 - w_1^2)\mathbf{K}(\nu)} \Pi \left(\frac{\mu_3 - \mu_2}{\mu_3 - w_1^2}, \nu \right). \quad (4.23)$$

The periodicity condition for β_0 implies that

$$2\pi \bar{k}_0 = \int_0^{2\pi} \beta_0' d\sigma = \int_0^{2\pi} \left(-\frac{u_0}{z_0^2} + qw_1 \frac{z_1^2}{z_0^2} \right) d\sigma. \quad (4.24)$$

Now we must recall that we are working in AdS₃ instead of its universal covering. The time coordinate should therefore be single valued, and thus we have to exclude windings along the time direction. When we set $\bar{k}_0 = 0$, Eq. (4.24) becomes

⁷Again there are four different cases where this condition must be modified. When either $u_0 + qw_1 = 0$ or $u_1 = 0$, the periodicity condition becomes $\frac{\pi}{2} \sqrt{(1 - q^2)(\mu_3 - \mu_1)} = n' \mathbf{K}(\nu)$ because of changes of branch when we take the square root of (4.19). The other two cases are the degenerate limits where there is no periodicity.

$$\frac{qw_1}{u_0 + qw_1} = \int_0^{2\pi} \frac{d\sigma}{2\pi} \frac{1}{z_0^2}, \quad (4.25)$$

which we can integrate to get

$$qw_1 = \frac{(u_0 + qw_1)(w_1^2 - w_0^2)}{(\mu_3 - w_0^2)\mathbf{K}(\nu)} \Pi\left(\frac{\mu_3 - \mu_2}{\mu_3 - w_0^2}, \nu\right). \quad (4.26)$$

In the same way, we can perform an identical computation to obtain the energy and the spin. From Eq. (4.11) we get

$$\frac{E}{\sqrt{\lambda}} + qu_1 - q^2w_0 = (1 - q^2)w_0 \int_0^{2\pi} \frac{d\sigma}{2\pi} z_0^2, \quad (4.27)$$

and thus

$$\begin{aligned} \frac{E}{\sqrt{\lambda}} &= q^2w_0 - qu_1 \\ &+ \frac{(1 - q^2)w_0}{w_1^2 - w_0^2} \left[\mu_3 - w_0^2 - (\mu_3 - \mu_1) \left(1 - \frac{\mathbf{E}(\nu)}{\mathbf{K}(\nu)} \right) \right]. \end{aligned} \quad (4.28)$$

Repeating the same steps with (4.12), we obtain an expression for the spin,

$$\frac{S}{\sqrt{\lambda}} - qu_0 = (1 - q^2)w_1 \int_0^{2\pi} \frac{d\sigma}{2\pi} z_1^2, \quad (4.29)$$

and thus

$$\frac{S}{\sqrt{\lambda}} = qu_0 + \frac{(1 - q^2)w_1}{w_1^2 - w_0^2} \left[\mu_3 - w_1^2 - (\mu_3 - \mu_1) \left(1 - \frac{\mathbf{E}(\nu)}{\mathbf{K}(\nu)} \right) \right]. \quad (4.30)$$

These expressions for the energy and the spin can be used to rewrite the first Virasoro constrain (4.10) as

$$w_1E - w_0S = \sqrt{\lambda}w_0w_1, \quad (4.31)$$

which is already a very closed expression. But what we want is a relation involving only E , S , and \bar{k}_1 . As in the case of strings spinning in S^3 , obtaining this relation for arbitrary values of q is again a lengthy and complicated problem, and we will not discuss it here. We will consider instead in the following subsection the case where a simple treatment is possible which is the limit of pure NS-NS flux of the above solutions.

To conclude our discussion, we are going to briefly discuss the choices of parameters that make the discriminant of $Q_3(\mu)$ equal to zero. As in the previous section, there are only three possible cases where the discriminant vanishes. The first is the constant radii case, where $\mu_2 = \mu_3$. However, this limit is not always well defined. The second case corresponds to the limit $\kappa = 1$, and it is obtained when

$\mu_1 = \mu_2$. In this case there is no periodicity condition because the elliptic sine has infinite period, and thus the string does not close. Also the factors $\sqrt{(1 - q^2)(\mu_3 - \mu_1)}/n\mathbf{K}(\nu)$ do not cancel anymore in the expressions for the energy, the spin, and the winding number. The third case corresponds to $\mu_1 = \mu_2 = \mu_3$ and requires setting $w_0 = w_1$.

A. Solutions with pure NS-NS flux

As in the case of strings rotating in S^3 , in the limit of pure NS-NS three-form flux, the above solutions can be written in terms of trigonometric functions. Now (4.17) reduces to

$$\mu'^2 = -4Q_2(\mu), \quad (4.32)$$

with $Q_2(\mu)$ the second-order polynomial

$$\begin{aligned} Q_2(\mu) &= (\mu - w_1^2)(\mu - w_0^2)(w_0^2 - w_1^2)\bar{F}_1 + (\mu - w_0^2)^2u_1^2 \\ &+ (\mu - w_1^2)^2(u_0 + w_1)^2 \\ &= \omega'^2(\mu - \tilde{\mu}_1)(\mu - \tilde{\mu}_2), \end{aligned} \quad (4.33)$$

where ω'^2 is

$$\omega'^2 = (w_0^2 - w_1^2)\bar{F}_1 + (u_0 + w_1)^2 + u_1^2. \quad (4.34)$$

Thus, we conclude that

$$z_0^2(\sigma) = \frac{\tilde{\mu}_2 - w_0^2}{w_1^2 - w_0^2} + \frac{\tilde{\mu}_1 - \tilde{\mu}_2}{w_1^2 - w_0^2} \sin^2(\omega'\sigma). \quad (4.35)$$

The periodicity condition on the radial coordinates implies now that ω' should be a half-integer number. The frequencies w_a and the integration constants u_a are related to the energy, the spin, and the winding number \bar{k}_1 by

$$\begin{aligned} w_1 &= \omega' \text{sgn}(u_0 + w_1), \\ \omega' &= (\bar{k}_1 - w_0) \text{sgn}(u_1), \end{aligned} \quad (4.36)$$

$$S = \sqrt{\lambda}u_0,$$

$$E = \sqrt{\lambda}(w_0 - u_1) = \frac{w_0}{w_1}S + \sqrt{\lambda}w_0. \quad (4.37)$$

Recalling now the Virasoro condition (4.9), the spin can be written as

$$S = \sqrt{\lambda} \frac{(\bar{k}_1 - \omega')^2 \omega'}{2\bar{k}_1(2\omega' - \bar{k}_1)}, \quad (4.38)$$

while the energy is given by

$$E = \sqrt{\lambda} \frac{\bar{k}_1^3 - 3\bar{k}_1^2 \omega' + \bar{k}_1 \omega'^2 + \omega'^3}{2\bar{k}_1(\bar{k}_1 - 2\omega')}. \quad (4.39)$$

We must note that we still have to impose a restriction on the parameters. This restriction comes from imposing that the discriminant of $Q_2(\mu)$ must be positive and taking the region in the parameter space with the correct hierarchy of roots. This condition can be written as

$$\begin{aligned} |2(u_0 + w_1)u_1| &\leq |\bar{F}_1(w_1^2 - w_0^2)| \\ &= |\omega'^2 - (u_0 + w_1)^2 - u_1^2|. \end{aligned} \quad (4.40)$$

The inequality is saturated in the cases of constant radii.

V. CONCLUDING REMARKS

In this article we have found a general class of solutions of the flux-deformed Neumann–Rosochatius system. The solutions that we have constructed correspond to closed strings with nonconstant radii rotating in $\text{AdS}_3 \times S^3 \times T^4$ with mixed R-R and NS-NS three-form fluxes. We have considered the cases where the string is rotating either in S^3 with two different angular momenta or in AdS_3 with one spin. The corresponding solutions can be expressed in terms of Jacobian elliptic functions. In the limit of pure NS-NS flux, the elliptic functions reduce to trigonometric functions. This reduction of the problem allows us to write rather compact expressions for the classical energy of the spinning string as a function of the corresponding conserved quantities.

The simplification in the limit of pure NS-NS flux is an appealing result already present in the case of the constant radii solutions studied in Ref. [25]. For the solutions that we have constructed in this article, the reduction implied by the presence of pure NS-NS flux appears as a consequence of the degeneration of the elliptic curve governing the dynamics of the problem. Furthermore, in this limit the theory can be described by a WZW model. It would be quite interesting to explore this limit in more detail and the relation of our approach to the solution of the WZW model. An important issue in this direction concerns the fate of the scattering matrix of the problem in the case of pure NS-NS flux. A complementary question for a better understanding of the quantum corrections to the spinning strings should come from the analysis of the spectrum of small quadratic fluctuations around the circular solutions.

Another interesting question is the extension of our analysis to other possible deformations of the backgrounds of type IIB string theory. An engaging case is that of the η -deformation of the $\text{AdS}_5 \times S^5$ background [28]. This is a much more complicated problem than the deformation by flux that we have studied in here because the deformation

comes from the breaking of the isometries of the metric down to the Cartan algebra. However, closed strings rotating in η -deformed $\text{AdS}_5 \times S^5$ have been shown recently to lead to an integrable extension of the Neumann–Rosochatius system [29]. It would be interesting to continue our analysis in this article to find general solutions of this η -deformed Neumann–Rosochatius system.

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APPENDIX: SPINNING STRINGS IN $\text{AdS}_3 \times S^3$ WITH PURE NS-NS FLUX

In this Appendix we will consider the case where the string is allowed to rotate both in AdS_3 and S^3 . We will restrict the analysis to the limit of pure NS-NS flux. As the pieces in the Lagrangian describing motion in AdS_3 and S^3 are decoupled, the equations for the corresponding coordinates are directly given by (2.9)–(2.11) and (4.6)–(4.8). The coupling between the AdS_3 and S^3 factors comes from the Virasoro constraints,

$$\begin{aligned} z_0^2 + z_0^2(\beta_0^2 + w_0^2) &= z_1^2 + z_1^2(\beta_1^2 + w_1^2) \\ &+ \sum_{i=1}^2 (r_i^2 + r_i^2(\alpha_i^2 + \omega_i^2)), \end{aligned} \quad (\text{A1})$$

$$z_1^2 w_1 \beta_1' + \sum_{i=1}^2 r_i^2 \omega_i \alpha_i' = z_0^2 w_0 \beta_0'. \quad (\text{A2})$$

The second Virasoro constrain can be rewritten as

$$\begin{aligned} \omega_2 J_1 + \omega_1 J_2 + w_1 E - w_0 S \\ = \sqrt{\lambda}(\omega_1 \omega_2 + w_0 w_1 + q \omega_1 \bar{m}_1). \end{aligned} \quad (\text{A3})$$

With these relations at hand, together with Eqs. (3.26), (3.27), (4.36), and (4.37), it is immediate to write the angular momenta and the energy as functions of ω , ω' ; the winding numbers \bar{m}_1 , \bar{m}_2 , and \bar{k}_1 ; the spin S ; and the total angular momentum J . In the case where $w_0 + \bar{k}_1 = -w_1 = -\omega'$, we conclude that

$$\begin{aligned} J_1 &= [-\bar{k}_1^2(\sqrt{\lambda}\omega' + 2S) + 2\bar{k}_1(\sqrt{\lambda}\omega'^2 + 2\omega'S + (\bar{m}_2 - \omega)(\sqrt{\lambda}\omega - J)) \\ &+ \omega'(\sqrt{\lambda}(\bar{m}_1^2 - \bar{m}_2^2 - \omega'^2 + \omega^2) - 2(\bar{m}_1 - \bar{m}_2)J)] / (2(\bar{m}_1 - \bar{m}_2)(\bar{k} - 2\omega')), \end{aligned} \quad (\text{A4})$$

$$J_2 = [\bar{k}_1^2(\sqrt{\lambda}\omega' + 2S) - 2\bar{k}_1(\sqrt{\lambda}\omega'^2 + 2\omega'S - \bar{m}_1J + \omega(\sqrt{\lambda}\bar{m}_2 - \sqrt{\lambda}\omega + J)) - \omega'(\sqrt{\lambda}(\bar{m}_1^2 - \bar{m}_2^2 - \omega'^2 + \omega^2) + 2(\bar{m}_1 - \bar{m}_2)J)]/(2(\bar{m}_1 - \bar{m}_2)(\bar{k}_1 - 2\omega')), \quad (\text{A5})$$

$$E = [\sqrt{\lambda}(\bar{k}_1^2 + \bar{m}_1^2 - (\bar{m}_2 - 3\omega)(\bar{m}_2 - \omega)) - 2\bar{k}_1(2\sqrt{\lambda}\omega' + S) + \omega'(3\sqrt{\lambda}\omega' + 4S) - 2J(\bar{m}_1 + \bar{m}_2 - 2\omega)]/(2(\bar{k}_1 - 2\omega')). \quad (\text{A6})$$

If we choose $w_0 + \bar{k}_1 = w_1 = \omega'$, we find

$$J_1 = [-\bar{k}_1^2(\sqrt{\lambda}\omega' + 2S) + 2\bar{k}_1(-\sqrt{\lambda}\omega'^2 - 2\omega'S + (\bar{m}_2 - \omega)(\sqrt{\lambda}\omega - J)) + \omega'(\sqrt{\lambda}(\bar{m}_1^2 - \bar{m}_2^2 - \omega'^2 + 4\bar{m}_2\omega - 3\omega^2) - 2(\bar{m}_1 + \bar{m}_2 - 2\omega)J)]/(2\bar{k}_1(\bar{m}_1 - \bar{m}_2)), \quad (\text{A7})$$

$$J_2 = [\bar{k}_1^2(\sqrt{\lambda}\omega' + 2S) + 2\bar{k}_1(\sqrt{\lambda}\omega'^2 + 2\omega'S + \bar{m}_1J - \omega(\bar{m}_2 - \omega + J)) - \omega'(\sqrt{\lambda}(\bar{m}_1^2 - \bar{m}_2^2 - \omega'^2 + 4\bar{m}_2\omega - 3\omega^2) - 2(\bar{m}_1 + \bar{m}_2 - 2\omega)J)]/(2\bar{k}_1(\bar{m}_1 - \bar{m}_2)), \quad (\text{A8})$$

$$E = [\sqrt{\lambda}(\bar{k}_1^2 + \bar{m}_1^2 - (\bar{m}_2 - 3\omega)(\bar{m}_2 - \omega) - \omega'^2) - 2\bar{k}_1S - 2J(\bar{m}_1 + \bar{m}_2 - 2\omega)]/(2\bar{k}_1). \quad (\text{A9})$$

When we take the limit $\bar{k}_1 \rightarrow 0$, $S \rightarrow 0$ and $\sqrt{\lambda}\omega' \rightarrow E$, we recover the expressions from Sec. III in both cases. In a similar way, when we set to zero the angular momenta, the winding numbers, and ω , we recover the analysis in Sec. IV. We can also reproduce the solutions of constant radii analyzed in Ref. [25]. In this case, when $w_0 + \bar{k}_1 = -w_1$ the angular momenta are given by

$$J_1 = \frac{\bar{k}_1S + \bar{m}_2J}{\bar{m}_2 - \bar{m}_1}, \quad J_2 = \frac{\bar{k}_1S + \bar{m}_1J}{\bar{m}_1 - \bar{m}_2}, \quad (\text{A10})$$

and the energy reduces to

$$E = -S \pm (J - \sqrt{\lambda}\bar{m}_1). \quad (\text{A11})$$

In the case where $w_0 + \bar{k}_1 = w_1$, the angular momenta are

$$J_1 = \frac{\bar{k}_1S + \bar{m}_2J}{\bar{m}_2 - \bar{m}_1}, \quad J_2 = \frac{\bar{k}_1S + \bar{m}_1J}{\bar{m}_1 - \bar{m}_2}, \quad (\text{A12})$$

and the energy becomes

$$E = S \pm \sqrt{(J - \lambda\bar{m}_1)^2 - 4\sqrt{\lambda}\bar{k}_1S}. \quad (\text{A13})$$

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