Semiclassical expansions in the Toda hierarchy and the hermitian matrix model *

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Abstract

An iterative algorithm for determining a class of solutions of the dispersionful 2-Toda hierarchy characterized by string equations is developed. This class includes the solution which underlies the large N-limit of the Hermitian matrix model in the one-cut case. It is also shown how the double scaling limit can be naturally formulated in this scheme

Key words: Toda hierarchy. String equations. Hermitian matrix model.
PACS number: 02.30.Ik.

*Partially supported by MEC project FIS2005-00319 and ESF programme MISGAM

1 Introduction

Since the pioneering works [1]-[2] the Toda hierarchy has become one of the paradigmatic examples of the relevance of integrable systems in the theory of random matrix models. As a consequence of the activity in this field a rich theory of the different facets of the Toda hierarchy has been developed.

The present work is motivated by the applications of the Toda hierarchy theory to the Hermitian matrix model. In this model the first integrable structure which emerges is the 1-Toda hierarchy [3]

$$\frac{\partial L}{\partial t_j} = [(L^j)_+, L]$$

on semi-infinite tridiagonal matrices

$$L = \Lambda + u_n + v_n \Lambda^T, \quad n \geq 0.$$

Here Λ is the standard shift matrix and $\mathcal{A}_+$ denotes the upper part (above the main diagonal) of semi-infinite matrices $\mathcal{A}$. This relationship may be conveniently described by considering infinite-dimensional deformations of monic orthogonal polynomials on the real line

$$P_n(z, \boldsymbol{t}) = z^n + \cdots, \quad \boldsymbol{t} := (t_1, t_2, \ldots), \quad n \geq 0,$$

with respect to an exponential weight:

$$\int_{-\infty}^{\infty} P_n(z, \boldsymbol{t})\, P_m(z, \boldsymbol{t})\, e^{V(z, \boldsymbol{t})}\, dz = h_n(\mathbf{t})\delta_{nm}, \quad V(z, \boldsymbol{t}) := \sum_{k \geq 1} (t_k + c_k)\, z^k,$$

where $\boldsymbol{c} := (c_1, c_2, \ldots)$ is a given set of complex constants. It turns out (see for instance [3]) that the functions

$$\psi_n(z, \boldsymbol{t}) := P_n(z, \boldsymbol{t})\, \exp(\sum_{k \geq 1} z^k\, t_k), \quad n \geq 0,$$

satisfy the linear system of the semi-infinite 1-Toda hierarchy

$$\frac{\partial \psi_n}{\partial t_k} = (L^k)_+\, \psi_n, \quad L\, \psi_n = z\, \psi_n, \quad k \geq 1, \quad n \geq 0,$$

and have a τ-function representation

$$\psi_n(z, \boldsymbol{t}) = \frac{\tau_n(\boldsymbol{t} - [z^{-1}])}{\tau_n(\boldsymbol{t})}\, z^n\, \exp(\sum_{k \geq 1} z^k\, t_k),$$

provided by the partition function of the Hermitian matrix model

$$\tau_N(\boldsymbol{t}) = Z_N(\boldsymbol{t}) := \int_{\mathbb{R}^N} \prod_{k=1}^{N} \Big(d\, x_k\, e^{V(x_k, \boldsymbol{t})} \Big) (\Delta(x_1, \cdots, x_N))^2, \tag{1}$$

where $\Delta(x_1, \cdots, x_n) := \prod_{i > j} (x_i - x_j)$.

Many exciting properties of the Hermitian model emerge in the analysis of its large N-limit [4]-[8]. One of the main tools supplied by the theory of the Toda hierarchy for such analysis [1] is the use of a pair of constraints called *string equations*

$$L = \Lambda + u_n + v_n\,\Lambda^T, \quad M = \sum_{k\geq 1} k\,(t_k + c_k)(L^{k-1})_+ \tag{2}$$

which are satisfied by the canonically conjugated operators

$$L\,\psi_n = z\,\psi_n, \quad \frac{\partial \psi_n}{\partial z} = M\,\psi_n.$$

The present paper deals with the analysis of the large N-limit of the partition function

$$Z_N(N\,\mathbf{t}) = \int_{\mathbb{R}^N} \prod_{k=1}^{N} \Big(d\,x_k\, e^{N\,V(x_k,\mathbf{t})}\Big)\big(\Delta(x_1,\cdots,x_N)\big)^2. \tag{3}$$

Here a small parameter $\epsilon := 1/N$ and rescaled variables $\mathbf{t} := \epsilon\,\boldsymbol{t}$ and constants $\mathbf{c} := \epsilon\,\boldsymbol{c}$ have been introduced. Since ϵ plays the role of the Planck constant $\hbar$, expansions in powers of ϵ are referred to as *semiclassical* expansions. The same slow variables $\mathbf{t} = \epsilon\,\boldsymbol{t}$ together with a continuous variable $x := \epsilon\,n$ are introduced to pass from the Toda hierarchy to its *dispersionful* formulation [10], which provides an interpolated continuous version of the Toda hierarchy. In this way, and due to the fact that $\tau_n(\boldsymbol{t}) = Z_n(\boldsymbol{t})$ is a τ-function of the semi-infinite 1-Toda hierachy, it is natural to expect that a τ-function $\tau(\epsilon, x, \mathbf{t})$ of the dispersionful 1-Toda hierarchy verifying

$$\tau(\epsilon, \epsilon\,n, \mathbf{t}) = Z_n(N\,\mathbf{t}), \tag{4}$$

should describe the large N-limit of the Hermitian model. In this paper we are concerned with the characterization of this solution of the dispersionful 1-Toda hierarchy.

The main result of our work is a scheme for obtaining solutions of the dispersionful 2-Toda hierarchy satisfying the system of string equations

$$\mathcal{L} = \bar{\mathcal{L}}, \quad \mathcal{M} + F(\mathcal{L}) = \overline{\mathcal{M}} + \overline{F}(\bar{\mathcal{L}}), \tag{5}$$

where $(\mathcal{L}, \mathcal{M})$ and $(\bar{\mathcal{L}}, \overline{\mathcal{M}})$ denote two pairs of Lax-Orlov operators and $(F, \bar{F})$ are two arbitrary functions. The first string equation represents the 1-Toda reduction condition and is satisfied by Lax operators of the form

$$\mathcal{L} = \bar{\mathcal{L}} = \Lambda + u + v\,\Lambda^{-1},$$

where now $\Lambda := \exp(\epsilon\,\partial_x)$, and (u, v) are characterized by semiclassical expansions

$$u = \sum_{k\geq 0} \epsilon^k\, u^{(k)}, \quad v = \sum_{k\geq 0} \epsilon^{2k}\, v^{(2k)}. \tag{6}$$

The point is that for $x = 1$ and $\bar{F} \equiv 0$ the constraints (5) interpolate (2), so that the corresponding solution of the dispersionful 1-Toda hierarchy is a candidate to the solution underlying the large N-limit of the Hermitian model. That it is the only possible candidate can be argued as follows:

1. Recent research [12]-[14] proved that the solutions of an extended version of the dispersionful 1-Toda hierarchy are determined by the leading order terms $(u^{(0)}, v^{(0)})$. In fact, the coefficients $(u^{(k)}, v^{(2k)})$ are rational functions of $(u^{(0)}, v^{(0)})$ and their x-derivatives (quasi-triviality property).

2. As it is shown in this paper, the terms $(u^{(0)}, v^{(0)})$ of the solution of (5) coincide with those characterizing the leading order in the large N-expansion (planar limit) of the Hermitian model.

Our strategy is inspired by previous results [15]-[18] on solution methods for dispersionless string equations. We also develop some useful standard technology of the theory of Lax equations [19]-[21] in the context of the dispersionful 1-Toda hierarchy. Thus we introduce two generating functions $\mathbb{R}$ and $\mathbb{T}$ related to the resolvent of the Lax operator which play a crucial role in our analysis.

The paper is organized as follows:

In the next section the basic theory of the dispersionful 2-Toda hierarchy and the method of string equations are discussed. In Section 3 we deal with the dispersionful 1-Toda hierarchy and its relationship with the Hermitian matrix model from the point of view of the continuous string equations (5). The generating functions $\mathbb{R}$ and $\mathbb{T}$ are introduced and are characterized by two important identities. Our main results are derived in Section 4 where a scheme for solving the string equations (5) in terms of semiclassical expansions is provided. In particular we prove that the leading terms of these expansions characterize the planar limit of the Hermitian matrix model. In Section 5 it is showed how the double scaling limit method can be naturally implemented in our scheme.

Applications of our method to normal matrix models which are also related to string equations of the Toda hierarchy [22]-[29] will be considered elsewhere.

2 String equations in the dispersionful 2-Toda hierarchy

2.1 The dispersionful 2-Toda hierarchy

The formulation of the dispersionful 2-Toda hierarchy [10] uses operators of the form

$$\mathcal{A} = \sum_{j\in\mathbb{Z}} a_j(\epsilon, x, \mathbf{t}, \bar{\mathbf{t}})\,\Lambda^j, \quad \Lambda := \exp(\epsilon\,\partial_x), \tag{7}$$

where x is a complex variable and the coefficients are in turn series in the small parameter ϵ

$$a_j(\epsilon, x, \mathbf{t}, \bar{\mathbf{t}}) = \sum_{k\in\mathbb{Z}} \epsilon^k\, a_j^{(k)}(x, \mathbf{t}, \bar{\mathbf{t}}).$$

Here $\mathbf{t} = (t_1, t_2, \ldots)$ and $\bar{\mathbf{t}} = (\bar{t}_1, \bar{t}_2, \ldots)$ denote two infinite sets of complex variables. The order in ϵ of $\mathcal{A}$ is defined by

$$\mathrm{ord}_\epsilon(\mathcal{A}) := \max\{-k \,|\, a_j^{(k)}(x, \mathbf{t}, \bar{\mathbf{t}}) \neq 0\}.$$

For example $\mathrm{ord}_\epsilon(\epsilon) = -1$ and $\mathrm{ord}_\epsilon(\Lambda) = 0$. In particular, zero-order operators are those with regular coefficents a_j as $\epsilon \to 0$. As usual the $\mathcal{A}_\pm$ parts of $\mathcal{A}$ will denote the truncations of Λ-series in the positive and strictly negative power terms, respectively. Given a function w depending on x, the following notation convention will be henceforth used

$$w_{[r]} := \Lambda^r w = w(x + r\,\epsilon), \quad r \in \mathbb{Z}.$$

The dispersionful 2-Toda hierarchy can be formulated in terms of a pair of formal wave functions of the form

$$
\begin{aligned}
&\Psi = \exp\Big(\frac{1}{\epsilon}\,\mathbb{S}\Big), \quad \mathbb{S} = \sum_{j=1}^{\infty} t_j z^j + x\,\log z - \sum_{j\geq 1} \frac{1}{j\,z^j} S_{j+1},\\
&\bar{\Psi} = z^{-1}\,\exp\Big(\frac{1}{\epsilon}\,\bar{\mathbb{S}}\Big), \quad \bar{\mathbb{S}} = \sum_{j=1}^{\infty} \bar{t}_j z^j - x\,\log z - \bar{S}_0 - \sum_{j\geq 1} \frac{1}{j\,z^j} \bar{S}_{j+1},
\end{aligned}
\tag{8}
$$

where

$$
S_j = \sum_{k\geq 0} \epsilon^k\, S_j^{(k)}(x, \mathbf{t}, \bar{\mathbf{t}}), \quad \bar{S}_j = \sum_{k\geq 0} \epsilon^k\, \bar{S}_j^{(k)}(x, \mathbf{t}, \bar{\mathbf{t}}),
$$

These functions $\mathbf{\Psi} = \Psi$, $\overline{\Psi}$ are assumed to satisfy the linear system

$$
\epsilon\,\frac{\partial \mathbf{\Psi}}{\partial t_j} = (\mathcal{L}^j)_+\,\mathbf{\Psi}, \quad \epsilon\,\frac{\partial \mathbf{\Psi}}{\partial \bar{t}_j} = (\bar{\mathcal{L}}^j)_-\,\mathbf{\Psi}, \tag{9}
$$

where the Lax operators $\mathcal{L}$ and $\bar{\mathcal{L}}$ are determined by the equations

$$
\mathcal{L}\,\Psi = z\,\Psi, \quad \bar{\mathcal{L}}\,\overline{\Psi} = z\,\overline{\Psi}, \tag{10}
$$

and are assumed [10] to be of zero order in ϵ. We will also use the Orlov operators $\mathcal{M}$ and $\overline{\mathcal{M}}$ characterized by

$$
\mathcal{M}\,\Psi = \epsilon\,\frac{\partial \Psi}{\partial z}; \quad \overline{\mathcal{M}}\,\overline{\Psi} = \epsilon\,\frac{\partial \overline{\Psi}}{\partial z}, \tag{11}
$$

which satisfy

$$
[\mathcal{L}, \mathcal{M}] = [\overline{\mathcal{L}}, \overline{\mathcal{M}}] = \epsilon.
$$

Using (8) and (10)-(11) one sees that the following expansions follow

$$
\begin{aligned}
&\mathcal{L} = \Lambda + u_0 + u_1\,\Lambda^{-1} + \cdots, \quad \mathcal{M} = \sum_{j=1}^{\infty} j\,t_j \mathcal{L}^{j-1} + x\,\mathcal{L}^{-1} + \sum_{j\geq 1} S_{j+1}\,\mathcal{L}^{-j-1}\\
&\bar{\mathcal{L}} = \bar{u}_{-1}\,\Lambda^{-1} + \bar{u}_0 + \bar{u}_1\,\Lambda + \cdots, \quad \overline{\mathcal{M}} = \sum_{j=1}^{\infty} j\,\bar{t}_j \bar{\mathcal{L}}^{j-1} - (x+\epsilon)\bar{\mathcal{L}}^{-1} + \sum_{j\geq 1} \bar{S}_{j+1}\,\bar{\mathcal{L}}^{-j-1}.
\end{aligned}
\tag{12}
$$

Furthermore, (9) can be rewritten in Lax form as

$$
\epsilon\,\frac{\partial K}{\partial t_j} = [(\mathcal{L}^j)_+, K], \quad \epsilon\,\frac{\partial K}{\partial \bar{t}_j} = [(\bar{\mathcal{L}}^j)_-, K], \tag{13}
$$

where $K = \mathcal{L}$, $\mathcal{M}$, $\bar{\mathcal{L}}$, $\overline{\mathcal{M}}$.

There is also a τ-function representation of the wave functions [10]

$$
\begin{aligned}
\Psi &= \exp\Big(\frac{1}{\epsilon}\,(\sum_{j=1}^{\infty} t_j z^j + x\,\log z)\Big)\,\frac{\tau(\epsilon, x, \mathbf{t}-\epsilon\,[z^{-1}], \bar{\mathbf{t}})}{\tau(\epsilon, x, \bar{\mathbf{t}})},\\
\bar{\Psi} &= z^{-1}\,\exp\Big(\frac{1}{\epsilon}\,(\sum_{j=1}^{\infty} \bar{t}_j z^j - x\,\log z)\Big)\,\frac{\tau(\epsilon, x+\epsilon, \mathbf{t}, \bar{\mathbf{t}}-\epsilon\,[z^{-1}])}{\tau(\epsilon, x, \mathbf{t}, \bar{\mathbf{t}})},
\end{aligned}
\tag{14}
$$

where $[z^{-1}] := (1/z,\ 1/2z^2,\ 1/3z^3, \ldots)$ and τ is of the form.

$$
\tau = \exp\Big(\frac{1}{\epsilon^2}\,\mathbb{F}\Big), \quad \mathbb{F} = \sum_{k\geq 0} \epsilon^k\, F^{(k)}(x, \mathbf{t}, \bar{\mathbf{t}}). \tag{15}
$$

The dispersionful 2-Toda hierachy arises as a continuum limit of the standard 2-Toda hierachy [9] in which the standard discrete variable n is substituted by a continuous variable x and two sets of *fast* continuous variables $\boldsymbol{t} := \epsilon^{-1}\mathbf{t}$, $\bar{\boldsymbol{t}} := \epsilon^{-1}\bar{\mathbf{t}}$ are introduced. Thus, the main dynamical objects (τ-functions, wave functions, Lax and Orlov operators) of both hierarchies are related by

$$
\tau(\epsilon, \epsilon\, n, \epsilon\, \boldsymbol{t}, \epsilon\, \bar{\boldsymbol{t}}) = \tau_n(\boldsymbol{t}, \bar{\boldsymbol{t}}) \tag{16}
$$

and

$$
\begin{array}{ll}
\Psi(z, \epsilon, \epsilon\, n, \epsilon\, \boldsymbol{t}, \epsilon\, \bar{\boldsymbol{t}}) = \psi_n(z, \boldsymbol{t}, \bar{\boldsymbol{t}}), & \bar{\Psi}(z, \epsilon, \epsilon\, n, \epsilon\, \boldsymbol{t}, \epsilon\, \bar{\boldsymbol{t}}) = \bar{\psi}_n(z, \boldsymbol{t}, \bar{\boldsymbol{t}}),\\
\mathcal{L}(z, \epsilon, \epsilon\, n, \epsilon\, \boldsymbol{t}, \epsilon\, \bar{\boldsymbol{t}}) = L(z, n, \boldsymbol{t}, \bar{\boldsymbol{t}}), & \bar{\mathcal{L}}(z, \epsilon\, n, \epsilon\, \boldsymbol{t}, \epsilon\, \bar{\boldsymbol{t}}) = \bar{L}(z, n, \boldsymbol{t}, \bar{\boldsymbol{t}}),\\
\mathcal{M}(z, \epsilon, \epsilon\, n, \epsilon\, \boldsymbol{t}, \epsilon\, \bar{\boldsymbol{t}}) = \epsilon\, M(z, n, \boldsymbol{t}, \bar{\boldsymbol{t}}), & \overline{\mathcal{M}}(z, \epsilon, \epsilon\, n, \epsilon\, \boldsymbol{t}, \epsilon\, \bar{\boldsymbol{t}}) = \epsilon\, \overline{M}(z, n, \boldsymbol{t}, \bar{\boldsymbol{t}}).
\end{array}
\tag{17}
$$

Our subsequent analysis uses an important result proved by Takasaki and Takebe (Proposition 2.7.11. in [10])

Theorem 1. *Suppose that*

$$
\mathcal{P}(\epsilon, x\,\Lambda^{-1}, \Lambda) = \sum_{k\in\mathbb{Z}} p_k(\epsilon, x\,\Lambda^{-1})\,\Lambda^k, \quad \mathcal{Q}(\epsilon, x\,\Lambda^{-1}, \Lambda) = \sum_{k\in\mathbb{Z}} q_k(\epsilon, x\,\Lambda^{-1})\,\Lambda^k,
$$

$$
\overline{\mathcal{P}}(\epsilon, x\,\Lambda^{-1}, \Lambda) = \sum_{k\in\mathbb{Z}} \bar{p}_k(\epsilon, x\,\Lambda^{-1})\,\Lambda^k, \quad \overline{\mathcal{Q}}(\epsilon, x\,\Lambda^{-1}, \Lambda) = \sum_{k\in\mathbb{Z}} \bar{q}_k(\epsilon, x\,\Lambda^{-1})\,\Lambda^k,
$$

are operators of zero order in ϵ verifying

$$
[\mathcal{P}, \mathcal{Q}] = [\overline{\mathcal{P}}, \overline{\mathcal{Q}}] = \epsilon
$$

If $(\mathcal{L}, \mathcal{M})$ and $(\bar{\mathcal{L}}, \overline{\mathcal{M}})$ are operators of zero order in ϵ of the form (12) which satisfy the pair of constraints

$$
\mathcal{P}(\epsilon, \mathcal{M}, \mathcal{L}) = \overline{\mathcal{P}}(\epsilon, \overline{\mathcal{M}}, \bar{\mathcal{L}}), \quad \mathcal{Q}(\epsilon, \mathcal{M}, \mathcal{L}) = \overline{\mathcal{Q}}(\epsilon, \overline{\mathcal{M}}, \bar{\mathcal{L}}), \tag{18}
$$

and

$$
[\mathcal{L}, \mathcal{M}] = [\bar{\mathcal{L}}, \overline{\mathcal{M}}] = \epsilon, \tag{19}
$$

then $(\mathcal{L}, \mathcal{M})$ and $(\bar{\mathcal{L}}, \overline{\mathcal{M}})$ are solutions of the Lax equations (13) of the 2-Toda hierarchy.

Constraints of the form (18) are called *string equations*. In this work we are interested in the particular example given by

$$\begin{cases} \mathcal{L} = \bar{\mathcal{L}}, \\ \mathcal{M} + F(\mathcal{L}) = \overline{\mathcal{M}} + \overline{F}(\bar{\mathcal{L}}), \end{cases} \tag{20}$$

where $F(\mathcal{L})$ and $\overline{F}(\bar{\mathcal{L}})$ are arbitrary functions of the form

$$F(\mathcal{L}) := \sum_{j\geq 1} j\, c_j\, \mathcal{L}^{j-1}, \quad \bar{F}(\bar{\mathcal{L}}) := \sum_{j\geq 1} j\, \bar{c}_j\, \bar{\mathcal{L}}^{j-1}.$$

3 The dispersionful 1-Toda hierarchy and the Hermitian model

The first string equation in (20) represents the so called *tridiagonal* (1-Toda) reduction of the dispersionful 2-Toda hierarchy and implies the following form of the Lax operators

$$\mathcal{L} = \bar{\mathcal{L}} = \Lambda + u + v\,\Lambda^{-1}. \tag{21}$$

Thus, as a consequence of the Lax equations, u and v depend on $(\mathbf{t}, \bar{\mathbf{t}})$ through the combination $\mathbf{t} - \bar{\mathbf{t}}$. Moreover (21) implies

$$(\Lambda + u + v\,\Lambda^{-1})\,\Psi = z\,\Psi, \quad (\Lambda + u + v\,\Lambda^{-1})\overline{\Psi} = z\,\overline{\Psi}, \tag{22}$$

so that

$$u = \epsilon^{-1}\,(S_{2[1]} - S_2), \quad \log v = \epsilon^{-1}\,(\bar{S}_{0[-1]} - \bar{S}_0). \tag{23}$$

In order to solve the string equations (20) it is required to characterize the action of the operators $(\mathcal{L}^j)_+$ and $(\bar{\mathcal{L}}^j)_-$ on the wave functions Ψ and $\overline{\Psi}$. This calculation is also needed to determine the integrable systems of the dispersionful 1-Toda hierarchy. We start by introducing the two series in z

$$p(z) = z - u + \mathcal{O}\Big(\frac{1}{z}\Big), \quad \bar{p}(z) = \frac{v_{[1]}}{z} + \mathcal{O}\Big(\frac{1}{z^2}\Big), \quad z \to \infty, \tag{24}$$

satisfying

$$\Lambda\,\Psi = p(z)\,\Psi, \quad \Lambda\,\overline{\Psi} = \bar{p}(z)\,\overline{\Psi}, \tag{25}$$

which according to (22) are determined by

$$\boldsymbol{p}(z) + u + \frac{v}{\boldsymbol{p}_{[-1]}(z)} = z, \tag{26}$$

where $\boldsymbol{p} = p, \bar{p}$. By using (22) it is clear that there are functions $\alpha_j, \beta_j, \bar{\alpha}_j, \bar{\beta}_j$, which depend polynomially in z, such that

$$\begin{aligned} \epsilon\,\frac{\partial \boldsymbol{\Psi}}{\partial t_j} &= (\mathcal{L}^j)_+ \boldsymbol{\Psi} = \alpha_j\,\boldsymbol{\Psi} + \beta_j\,\Lambda\boldsymbol{\Psi} = (\alpha_j + \beta_j\,\boldsymbol{p})\boldsymbol{\Psi}, \\ \epsilon\,\frac{\partial \boldsymbol{\Psi}}{\partial \bar{t}_j} &= (\bar{\mathcal{L}}^j)_- \boldsymbol{\Psi} = \bar{\alpha}_j\,\boldsymbol{\Psi} + \bar{\beta}_j\,\Lambda\boldsymbol{\Psi} = (\bar{\alpha}_j + \bar{\beta}_j\,\boldsymbol{p})\boldsymbol{\Psi}, \end{aligned} \tag{27}$$

and

$$\bar{\alpha}_j = z^j - \alpha_j, \quad \bar{\beta}_j = -\beta_j. \tag{28}$$

Hence, we have

$$\alpha_j + \beta_j\, p = \partial_{t_j}\mathbb{S}(z) = z^j + \mathcal{O}\Big(\frac{1}{z}\Big), \quad \alpha_j + \beta_j\, \bar{p} = \partial_{t_j}\bar{\mathbb{S}}(z) = -\partial_{t_j}\bar{S}_0 + \mathcal{O}\Big(\frac{1}{z}\Big), \tag{29}$$

so that

$$\alpha_j = \frac{1}{2}\Big(z^j - \partial_{t_j}\bar{S}_0 - (\beta_j\,(p+\bar{p}))_{\oplus}\Big), \quad \beta_j = \Big(\frac{z^j}{p-\bar{p}}\Big)_{\oplus}, \tag{30}$$

where $(\ )_{\oplus}$ and $(\ )_{\ominus}$ stand for the projections of z-series on the subspaces generated by the positive and strictly negative powers , respectively.

At this point it is useful to introduce the generating functions

$$\mathbb{R} := \frac{z}{p-\bar{p}} = \sum_{k\geq 0}\frac{R_k(u,v)}{z^k}, \quad \mathbb{T} := \frac{p+\bar{p}}{p-\bar{p}} = \sum_{k\geq 0}\frac{T_k(u,v)}{z^k}, \quad R_0 = T_0 = 1. \tag{31}$$

By substituting p and $\bar{p}$ by their expressions in terms of $\mathbb{R}$ and $\mathbb{T}$ in the identities

$$u = z + \frac{p\,p_{[-1]} - \bar{p}\,\bar{p}_{[-1]}}{\bar{p}_{[-1]} - p_{[-1]}}, \quad v = \frac{\bar{p}-p}{\bar{p}_{[-1]} - p_{[-1]}}\,\bar{p}_{[-1]}p_{[-1]}, \tag{32}$$

we obtain the following relations

$$\begin{cases} \mathbb{T}_{[1]} + \mathbb{T} + \dfrac{2}{z}(u_{[1]} - z)\,\mathbb{R}_{[1]} = 0, \\[2ex] \mathbb{T}^2 - \dfrac{4}{z^2}\,v_{[1]}\,\mathbb{R}\,\mathbb{R}_{[1]} = 1, \end{cases} \tag{33}$$

which allow us to compute recursively the coefficients of the series (31) as polynomials in u, v and their x-translations $u_{[r]}$ and $v_{[r]}$. Indeed, the system (33) implies

$$\begin{cases} 2T_{k+1} = -\displaystyle\sum_{i+j=k+1;i,j\geq 1} T_i\,T_j + 4v_{[1]}\sum_{i+j=k-1} R_i\,R_{j[1]}, \\[2ex] R_{k+1} = u\,R_k + \dfrac{1}{2}[T_{k+1} + T_{k+1\,[-1]}], \quad . \end{cases} \tag{34}$$

For example, the first few coefficients are:

$$\begin{aligned} &T_1 = 0, \quad R_1 = u, \quad T_2 = 2v_{[1]}, \quad R_2 = u^2 + v_{[1]} + v, \\ &T_3 = 2v_{[1]}(u + u_{[1]}), \quad R_3 = u^3 + 2uv_{[1]} + 2uv + u_{[1]}v_{[1]} + u_{[-1]}v, \\ &T_4 = 2v_{[1]}\,(u_{[1]}^2 + uu_{[1]} + u^2 + v_{[2]} + v_{[1]} + v). \end{aligned}$$

In this way, by taking into account the second equation of (34), one finds

$$\begin{aligned} \partial_{t_j}\mathbb{S}(z) = \alpha_j + \beta_j\,p &= z^j - \frac{1}{2}\partial_{t_j}\bar{S}_0 - \frac{z}{2\,\mathbb{R}}\,(z^{j-1}\,\mathbb{R})_{\ominus} + \Big(\frac{z}{2\,\mathbb{R}}\,\mathbb{T}\,(z^{j-1}\,\mathbb{R})_{\oplus}\Big)_{\ominus} \\ &= z^j - \frac{1}{2}(\partial_{t_j}\bar{S}_0 + R_j) - \frac{1}{2\,z}T_{j+1[-1]} + \mathcal{O}\Big(\frac{1}{z^2}\Big), \end{aligned} \tag{35}$$

so that

$$\partial_{t_j}\bar{S}_0 = -R_j, \quad \partial_{t_j}S_2 = \frac{1}{2}\,T_{j+1[-1]}$$

and then from (23) we get that the flows of the dispersionful 1-Toda hierarchy can be expressed as

$$\epsilon\,\partial_{t_j}u = \frac{1}{2}\,(T_{j+1} - T_{j+1[-1]}), \quad \epsilon\,\partial_{t_j}v = v\,(R_j - R_{j[-1]}). \tag{36}$$

Furthermore, our calculation implies the following useful relations

$$\begin{aligned}
(\mathcal{L}^j)_-\Psi &= \Big(-\frac{1}{2}R_j + \frac{z}{2\,\mathbb{R}}\,(z^{j-1}\,\mathbb{R})_\ominus - \Big(\frac{z}{2\,\mathbb{R}}\,\mathbb{T}\,(z^{j-1}\,\mathbb{R})_\oplus\Big)_\ominus\Big)\,\Psi,\\
(\mathcal{L}^j)_+\overline{\Psi} &= \Big(\frac{1}{2}R_j + \frac{z}{2\,\mathbb{R}}\,(z^{j-1}\,\mathbb{R})_\ominus + \Big(\frac{z}{2\,\mathbb{R}}\,\mathbb{T}\,(z^{j-1}\,\mathbb{R})_\oplus\Big)_\ominus\Big)\,\overline{\Psi},
\end{aligned} \tag{37}$$

for $j \geq 1$. In particular, by taking the second equation of (34) into account one finds that as $z \to \infty$

$$\begin{aligned}
(\mathcal{L}^j)_-\Psi &= \Big(\frac{1}{2z}\,T_{j+1[-1]} + \mathcal{O}\Big(\frac{1}{z^2}\Big)\Big)\,\Psi,\\
(\mathcal{L}^j)_+\overline{\Psi} &= \Big(R_j + \frac{1}{2z}\,T_{j+1} + \mathcal{O}\Big(\frac{1}{z^2}\Big)\Big)\overline{\Psi}.
\end{aligned} \tag{38}$$

We observe that since $R_0 = 1$, $T_1 = 0$ these last equations hold for $j \geq 0$.

By following the analysis of [21] it can be seen that $\mathbb{R}$ and $\mathbb{T}$ are closely related to the resolvent of the Lax operator $\mathcal{L}$

$$\mathcal{R} := (z - \mathcal{L})^{-1}.$$

Thus, from Lemmas 3.5 and 3.18 of [21] one proves that

$$\Big(z - \frac{2\,\mathbb{R}}{(1+\mathbb{T})}\,\Lambda\Big)\,\mathcal{R}_+ = \mathbb{R}, \quad \operatorname{Res}\mathcal{R}_+ = \frac{\mathbb{R}}{z},$$

where $\operatorname{Res}(\sum c_k\,\Lambda^k) := c_0$

τ-function representation

It follows from (14) and(22) that the functions u and v can be written in terms of the τ-function as

$$u = \epsilon\,\frac{\partial}{\partial t_1}\,\log\,\frac{\tau(\epsilon, x+\epsilon, \mathbf{t})}{\tau(\epsilon, x, \mathbf{t})}, \quad v = \frac{\tau(\epsilon, x+\epsilon, \mathbf{t})\,\tau(\epsilon, x-\epsilon, \mathbf{t})}{\tau^2(\epsilon, x, \mathbf{t})}, \tag{39}$$

where we have set $\mathbf{t} - \bar{\mathbf{t}} \to \mathbf{t}$. On the other hand, it can be proved [13]-[14] that the ϵ-expansion of the τ-functions of the dispersionful 1-Toda hierarchy is of the form

$$\tau = \exp\Big(\frac{1}{\epsilon^2}\,\mathbb{F}\Big), \quad \mathbb{F} = \sum_{k\geq 0}\epsilon^{2k}\,F^{(2k)}. \tag{40}$$

As a consequence u and v can be expanded as

$$u = \sum_{k\geq 0} \epsilon^k \, u^{(k)}, \quad v = \sum_{k\geq 0} \epsilon^{2k} \, v^{(2k)}. \tag{41}$$

Let us introduce the *reduced* $\mathbb{S}$ and $\mathbb{M}$ functions

$$\mathbb{S}_r := -\sum_{j\geq 1} \frac{1}{j\, z^j} S_{j+1}, \quad \mathbb{M}_r := \frac{\partial \mathbb{S}_r}{\partial z}.$$

From (14) we see that

$$\mathbb{F}(\epsilon, x, \mathbf{t} - \epsilon\,[z^{-1}]) - \mathbb{F}(\epsilon, x, \mathbf{t}) = \epsilon\, \mathbb{S}_r(\epsilon, z, x, \mathbf{t}), \tag{42}$$

and by differentiating this equation with respect to z we obtain

$$\sum_{j\geq 1} \frac{1}{z^{j+1}} \frac{\partial}{\partial t_j} \mathbb{F}(\epsilon, x, \mathbf{t}) = \mathbb{M}_r(\epsilon, z, x, \mathbf{t} + \epsilon\,[z^{-1}]). \tag{43}$$

This identity can be rewritten in in a more convenient form. Indeed (42) implies

$$\mathbb{S}_r(\epsilon, z, x, \mathbf{t}) - \mathbb{S}_r(\epsilon, z, x, \mathbf{t} - \epsilon\,[z'^{-1}]) = \mathbb{S}_r(\epsilon, z', x, \mathbf{t}) - \mathbb{S}_r(\epsilon, z', x, \mathbf{t} - \epsilon\,[z^{-1}]),$$

and by differentiating with respect to z and then taking the limit $z' \to z$ one finds

$$\mathbb{M}_r(\epsilon, z, x, \mathbf{t} - \epsilon\,[z^{-1}]) = \mathbb{M}_r(\epsilon, z, x, \mathbf{t}) + \epsilon \sum_{j\geq 1} \frac{1}{z^{j+1}} \frac{\partial \mathbb{S}_r}{\partial t_j}(\epsilon, z, x, \mathbf{t} - \epsilon\,[z^{-1}]).$$

Thus (43) becomes

$$\sum_{j\geq 1} \frac{1}{z^{j+1}} \frac{\partial \mathbb{F}}{\partial t_j} = \mathbb{M}_r - \epsilon \sum_{j\geq 1} \frac{1}{z^{j+1}} \frac{\partial \mathbb{S}_r}{\partial t_j}. \tag{44}$$

3.1 The Hermitian matrix model

Let us write the partition function of the Hermitian matrix model in terms of slow variables $\mathbf{t} := \epsilon\, \boldsymbol{t}$, where $\epsilon = 1/N$

$$Z_n(N\,\mathbf{t}) = \int_{\mathbb{R}^n} \prod_{k=1}^{n} \Big(d\,x_k\, e^{N\,V(x_k,\mathbf{t})}\Big)(\Delta(x_1, \cdots, x_n))^2, \quad V(z, \mathbf{t}) := \sum_{k\geq 1} (t_k + \mathbf{c}_k)\, z^k. \tag{45}$$

The large N-limit of the model is determined by the asymptotic expansion of $Z_n(N\,\mathbf{t})$ for $n = N$ as $N \to \infty$

$$Z_N(N\,\mathbf{t}) = \int_{\mathbb{R}^N} \prod_{k=1}^{N} \Big(d\,x_k\, e^{N\,V(x_k,\mathbf{t})}\Big)(\Delta(x_1, \cdots, x_N))^2, \tag{46}$$

It is well-known [3] that $Z_n(\boldsymbol{t})$ is a τ-function of the semi-infinite 1-Toda hierachy , then in view of (16) we may look for a τ-function $\tau(\epsilon, x, \mathbf{t})$ of the dispersionful 1-Toda hierarchy verifying

$$\tau(\epsilon, \epsilon\, n, \mathbf{t}) = Z_n(N\,\mathbf{t}), \tag{47}$$

and consequently

$$\tau(\epsilon, 1, \mathbf{t}) = Z_N(N\,\mathbf{t}). \tag{48}$$

The point is that for

$$x = 1, \quad \bar{t}_j = \bar{c}_j = 0, \quad j \geq 1, \tag{49}$$

the system (20) of continuous string equations interpolates the discrete system (2). Hence the solution of the dispersionful 1-Toda hierarchy provided by (20) can be expected to correspond to the τ-function verifying (47) and, as a consequence, to describe the the large N-limit of the Hermitian matrix model.

We may express the $1/N$-expansions of the main objects of the hermitian matrix model in terms of objects in the dispersionful 1-Toda hierarchy. For instance, from (44) the *one-loop correlator*

$$W(z) := \frac{1}{N} \sum_{j\geq 0} \frac{1}{z^{j+1}} \langle trM^j \rangle = \frac{1}{z} + \frac{1}{N^2} \sum_{j\geq 1} \frac{1}{z^{j+1}} \frac{\partial \log\, Z_N(N\,\mathbf{t})}{\partial t_j},$$

becomes

$$W(z) = \frac{1}{z} + \mathbb{M}_r(\epsilon, z, 1, \mathbf{t}) - \epsilon \sum_{j\geq 1} \frac{1}{z^{j+1}} \frac{\partial \mathbb{S}_r}{\partial t_j}(\epsilon, z, 1, \mathbf{t}). \tag{50}$$

Loop correlators of higher order can be obtained from $W(z)$ by application of the *loop-insertion* operator $d/dV(z)$ [6]

$$W(z_1, \ldots, z_s) = \frac{d}{dV(z_s)} \cdots \frac{d}{dV(z_2)}\, W(z_1)$$
$$\frac{d}{dV(z)} := \sum_{j\geq 1} \frac{1}{z^{j+1}} \frac{\partial}{\partial t_j}.$$

4 Semiclassical expansions

We now turn to the solutions of the system of string equations (20). The first equation is solved by setting

$$\mathcal{L} = \bar{\mathcal{L}} = \Lambda + u + v\,\Lambda^{-1},$$

which is in agreement with the asymptotic form (12) required for $\mathcal{L}$ and $\bar{\mathcal{L}}$.

Let us consider the second string equation of (20). We look for solutions $\mathcal{M}$ and $\overline{\mathcal{M}}$ verifying asymptotic expansions of the form (12). To this end we first set

$$\mathcal{M} + F(\mathcal{L}) = \overline{\mathcal{M}} + \bar{F}(\overline{\mathcal{L}}) = \sum_{j=1}^{\infty} j\,(t_j + c_j)\,(\mathcal{L}^{j-1})_+ + \sum_{j=1}^{\infty} j\,(\bar{t}_j + \bar{c}_j)\,(\overline{\mathcal{L}}^{j-1})_-,$$

which, taking into account the first string equation, leads to

$$\begin{aligned} \mathcal{M} &= \sum_{j=1}^{\infty} j\,t_j\,\mathcal{L}^{j-1} + \sum_{j=1}^{\infty} j\,\Big((\bar{t}_j + \bar{c}_j) - (t_j + c_j)\Big)\,(\mathcal{L}^{j-1})_-, \\ \overline{\mathcal{M}} &= \sum_{j=1}^{\infty} j\,\bar{t}_j\,\mathcal{L}^{j-1} - \sum_{j=1}^{\infty} j\,\Big((\bar{t}_j + \bar{c}_j) - (t_j + c_j)\Big)\,(\mathcal{L}^{j-1})_+. \end{aligned} \tag{51}$$

In order to satisfy (12) and (19) we introduce auxiliary functions of the form

$$\Psi=\exp\frac{1}{\epsilon}\Big(\sum_{j=1}^{\infty}t_j z^j+x\,\log z-\sum_{j\geq 1}\frac{1}{j\,z^j}S_{j+1}\Big),$$

$$\overline{\Psi}=\exp\frac{1}{\epsilon}\Big(\sum_{j=1}^{\infty}\bar{t}_j z^j-(x+\epsilon)\,\log z-\bar{S}_0-\sum_{j\geq 1}\frac{1}{j\,z^j}\bar{S}_{j+1}\Big), \tag{52}$$

and impose

$$\mathcal{L}\,\Psi=z\,\Psi,\quad \mathcal{M}\,\Psi=\epsilon\,\frac{\partial\Psi}{\partial z},$$

$$\bar{\mathcal{L}}\,\overline{\Psi}=z\,\overline{\Psi},\quad \overline{\mathcal{M}}\,\overline{\Psi}=\epsilon\,\frac{\partial\overline{\Psi}}{\partial z}. \tag{53}$$

Our aim is to determine $u,v,\mathcal{M}$ and $\overline{\mathcal{M}}$ from (53). Now, with the help of (38), we have that the equations (53) for the Orlov operators read

$$\frac{x}{z}+\sum_{j\geq 2}\frac{1}{z^j}S_j=\sum_{j=1}^{\infty}j\,\Big((\bar{t}_j+\bar{c}_j)-(t_j+c_j)\Big)\Big(\frac{1}{2z}\,T_{j[-1]}+\mathcal{O}\Big(\frac{1}{z^2}\Big)\Big),$$

$$-\,\frac{x+\epsilon}{z}+\sum_{j\geq 2}\frac{1}{z^j}\bar{S}_j=-\sum_{j=1}^{\infty}j\,\Big((\bar{t}_j+\bar{c}_j)-(t_j+c_j)\Big)\Big(R_{j-1}+\frac{1}{2z}\,T_j+\mathcal{O}\Big(\frac{1}{z^2}\Big)\Big). \tag{54}$$

Matching the coefficients of z^{-1} in both sides of these two equations provides the same relation. Another relation is supplied by identifying the coefficients of the constant terms in the second equation of (54). Hence we get a system of two equations to determine (u,v)

$$\begin{cases}\sum_{j=1}^{\infty}j\,\Big((\bar{t}_j+\bar{c}_j)-(t_j+c_j)\Big)R_{j-1}=0,\\[2ex] \dfrac{1}{2}\,\sum_{j=1}^{\infty}j\,\Big((\bar{t}_j+\bar{c}_j)-(t_j+c_j)\Big)\,T_{j[-1]}=x.\end{cases} \tag{55}$$

By equating the coefficients of the remaining powers of z in (54) we characterize the functions S_j and $\overline{S}_j$ for $j\geq 1$ in terms of (u,v). Moreover, as it is proved below, the solution (u,v) provided by (55) is of the form

$$u=\sum_{k\geq 0}\epsilon^k\,u^{(k)}(x,\mathbf{t},\bar{\mathbf{t}}),\quad v=\sum_{k\geq 0}\epsilon^k\,v^{(k)}(x,\mathbf{t},\bar{\mathbf{t}}),$$

with $v^{(2k+1)}=0,\forall k\geq 0$. Thus, by solving (55) we characterize operators $(\mathcal{L},\mathcal{M})$ and $(\bar{\mathcal{L}},\overline{\mathcal{M}})$ which satisfy (20) and all the requirements of Theorem 1. Therefore, they are solutions of the Lax equations for the dispersionful 2-Toda hierarchy.

We observe that, as it is noticed by Takasaki and Takebe in [10], solving the system of string equations (20) does not determine the coefficient $\bar{S}_0$ in (52) and therefore it does not determine a wave function $\overline{\Psi}$ of the dispersionful 1-Toda hierarchy.

4.1 An iterative scheme for determining (u,v)

It is convenient to write (55) in the form

$$\oint_\gamma \frac{dz}{2\pi i\, z}\, U_z\, \mathbb{R}(z) \,=0, \quad \oint_\gamma \frac{dz}{2\pi i}\, U_z\, \mathbb{T}(z) \,=-2\,(x+\epsilon), \tag{56}$$

where U denotes the function

$$U(z,\mathbf{t},\bar{\mathbf{t}}) := \sum_{j=1}^{\infty} \Big((t_j+c_j)-(\bar{t}_j+\bar{c}_j)\Big)\, z^j, \tag{57}$$

and γ is a large positively oriented closed path. Now by using the first identity of (33) and the two equations of (56) we find

$$\oint_\gamma \frac{dz}{2\pi i}\, U_z\, (\mathbb{T}+\mathbb{T}_{[-1]}) = \oint_\gamma \frac{dz}{\pi i\, z}\,(z-u)\, U_z\, \mathbb{R} = \oint_\gamma \frac{dz}{\pi i}\, U_z\, \mathbb{R} = -4\,x-2\,\epsilon,$$

so that (55) reduces to a pair of equations involving $\mathbb{R}$ only

$$\begin{cases} \displaystyle\oint_\gamma \frac{dz}{2\pi i\, z}\, U_z(z)\, \mathbb{R}(z) \,=0, \\[2ex] \displaystyle\oint_\gamma \frac{dz}{2\pi i}\, U_z(z)\, \mathbb{R}(z) \,=-2\,x-\epsilon. \end{cases} \tag{58}$$

These equations together with the system (33)

$$\begin{cases} \mathbb{T}_{[1]}+\mathbb{T}+\dfrac{2}{z}(u_{[1]}-z)\,\mathbb{R}_{[1]}=0, \\[2ex] \mathbb{T}^2-\dfrac{4}{z^2}\, v_{[1]}\,\mathbb{R}\,\mathbb{R}_{[1]}=1, \end{cases} \tag{59}$$

give rise an iterative scheme for characterizing (u,v) as Taylor series in ϵ

$$u=\sum_{k\geq 0}\epsilon^k\, u^{(k)}(x,\mathbf{t},\bar{\mathbf{t}}), \quad v=\sum_{k\geq 0}\epsilon^k\, v^{(k)}(x,\mathbf{t},\bar{\mathbf{t}}).$$

The first step of the method is to determine the expansions

$$\mathbb{R}(z)=\sum_{k\geq 0}\epsilon^k\, R^{(k)}, \quad \mathbb{T}(z)=\sum_{k\geq 0}\epsilon^k\, T^{(k)}, \tag{60}$$

in terms of (u,v). It can be done by equating the coefficients of powers of ϵ in (59). Indeed, the coefficients of ϵ^0 leads to

$$R^{(0)} \,=\, \frac{z}{\big((z-u^{(0)})^2-4v^{(0)}\big)^{\frac{1}{2}}}, \quad T^{(0)} \,=\, \frac{z-u^{(0)}}{\big((z-u^{(0)})^2-4v^{(0)}\big)^{\frac{1}{2}}}, \tag{61}$$

and the coefficients of ϵ^l $(l \geq 1)$ yield the following system

$$T^{(l)} - (z - u^{(0)})\frac{R^{(l)}}{z} = \frac{1}{2} \sum_{\substack{i+j=l \\ j \geq 1}} \Big(\frac{(-1)^j}{j!}\partial_x^j T^{(i)} + 2u^{(j)}\frac{R^{(i)}}{z}\Big),$$

$$\begin{aligned} T^{(0)}T^{(l)} - 4v^{(0)}\frac{R^{(0)}}{z}\frac{R^{(l)}}{z} = {} & 2 \sum_{\substack{i+j+k=l \\ j<l}} \Big(\sum_{i_1+i_2=i} \frac{1}{i_2!}\partial_x^{i_2} v^{(i_1)}\Big)\frac{R^{(j)}}{z}\Big(\sum_{\substack{k_1+k_2=k \\ k_1<l}} \frac{1}{k_2!}\partial_x^{k_2}\frac{R^{(k_1)}}{z}\Big) \\ & -\frac{1}{2}\sum_{\substack{i+j=l \\ i,j\geq 1}} T^{(i)}T^{(j)}. \end{aligned} \tag{62}$$

Some comments concerning these formulas are in order

i) The equations (62) determine each pair $(T^{(l)},\ R^{(l)}/z)$ from $(T^{(j)},\ R^{(j)}/z)$ with $j = 0, 1, \ldots, l-1$.

ii) The equations (62) are linear with respect to $T^{(l)}$, $R^{(l)}/z$. Moreover, by taking into account (61), we see that the determinant of the coefficients of $T^{(l)}$ and $R^{(l)}/z$ in (62) is

$$[(z - u^{(0)})^2 - 4v^{(0)}]^{\frac{1}{2}}.$$

Hence it follows that the functions $R^{(l)}/z$ can be written as linear combinations of

$$\frac{z}{\big((z-u^{(0)})^2 - 4v^{(0)}\big)^{r+\frac{1}{2}}}, \qquad \frac{1}{\big((z-u^{(0)})^2 - 4v^{(0)}\big)^{r+\frac{1}{2}}}, \qquad r = 1, 2, \ldots, l+1$$

with coefficients depending on $u^{(j)}$ and $v^{(j)}$ with $j = 0, 1, \ldots, l$ and their x-derivatives only.

Now let us go back to the system (58) and find (u, v). By substituting the ϵ expansion of $\mathbb{R}$ in (58) we get a system of two equations for each $R^{(l)}$

$$\begin{cases} \displaystyle\oint_\gamma \frac{dz}{2\pi i\, z} U_z(z)\, R^{(l)}(z) \ = 0, \\[2ex] \displaystyle\oint_\gamma \frac{dz}{2\pi i} U_z(z)\, R^{(l)}(z) \ = -2\, x\, \delta_{l0} - \delta_{l1}, \end{cases} \tag{63}$$

which determine each pair $(u^{(l)}, v^{(l)})$ recursively. Furthermore, we can eliminate the explicit dependence on $(x, \mathbf{t}, \bar{\mathbf{t}})$ in the corresponding expressions since, by differentiating with respect to x the equations (63) for $l = 0$

$$\begin{cases} \displaystyle\frac{1}{2\pi i}\oint_\gamma dz \frac{U_z}{\big((z-u^{(0)})^2 - 4v^{[0]}\big)^{\frac{1}{2}}} = 0, \\[2ex] \displaystyle\frac{1}{2\pi i}\oint_\gamma dz \frac{z\, U_z}{\big((z-u^{(0)})^2 - 4v^{(0)}\big)^{\frac{1}{2}}} = -2x, \end{cases} \tag{64}$$

all the integrals of the form

$$\frac{1}{2\pi i}\oint_\gamma dz \frac{U_z}{\left((z-u^{(0)})^2-4v^{(0)}\right)^{r+\frac{1}{2}}}, \quad \frac{1}{2\pi i}\oint_\gamma dz \frac{z\,U_z}{\left((z-u^{(0)})^2-4v^{(0)}\right)^{r+\frac{1}{2}}}, \tag{65}$$

can be expressed in terms of $(u^{(0)}, v^{(0)})$ and their x-derivatives. We observe that (65) are the variables introduced in [5] to determine the large N-expansion of the hermitian matrix model.

Some important relations among the coefficients of the semiclassical expansions under consideration are found by realizing that given a solution $(u(\epsilon,x),\, v(\epsilon,x),\, \mathbb{R}(\epsilon,z,x),\, \mathbb{T}(\epsilon,z,x))$ of (58)-(59), then

$$\tilde{u}(\epsilon,x) := u(-\epsilon, x+\epsilon), \qquad \tilde{v}(\epsilon,x) := v(-\epsilon,x),$$

$$\tilde{\mathbb{R}}(\epsilon,z,x) := \mathbb{R}(-\epsilon,z,x+\epsilon) \qquad \tilde{\mathbb{T}}(\epsilon,z,x) := \mathbb{T}(-\epsilon,z,x+2\epsilon),$$

satisfies (58)-(59) as well. Thus, since the solution of (58)-(59) is uniquely determined by $(u^{(0)},\, v^{(0)})$ we deduce that

$$\tilde{u}(\epsilon,x) = u(\epsilon,x), \qquad \tilde{v}(\epsilon,x) = v(\epsilon,x),$$

$$\tilde{\mathbb{R}}(\epsilon,z,x) = \mathbb{R}(\epsilon,z,x), \qquad \tilde{\mathbb{T}}(\epsilon,z,x) = \mathbb{T}(\epsilon,z,x).$$

Hence we find

$$\begin{aligned} u^{(2j-1)} &= \frac{1}{2}\sum_{k=1}^{2j-1}\frac{(-1)^{k+1}}{k!}\partial_x^k u^{(2j-1-k)}, & v^{(2j-1)} &= 0, \\ R^{(2j-1)} &= \frac{1}{2}\sum_{k=1}^{2j-1}\frac{(-1)^{k+1}}{k!}\partial_x^k R^{(2j-1-k)}, & T^{(2j-1)} &= \frac{1}{2}\sum_{k=1}^{2j-1}\frac{(-1)^{k+1}2^k}{k!}\partial_x^k T^{(2j-1-k)}, \end{aligned} \tag{66}$$

for $j = 1, 2, \ldots$.

4.1.1 Examples of calculations

Using (66) for $j = 1$, it is immediately found that

$$R^{(1)} = \frac{z\left(\left(z-u^{(0)}\right)u_x^{(0)} + 2\,v_x^{(0)}\right)}{2\left((z-u^{(0)})^2-4\,v^{(0)}\right)^{\frac{3}{2}}}, \quad T^{(1)} = \frac{4\,v^{(0)}\,u_x^{(0)} + 2\,\left(z-u^{(0)}\right)\,v_x^{(0)}}{\left((z-u^{(0)})^2-4\,v^{(0)}\right)^{\frac{3}{2}}}, \tag{67}$$

and

$$u^{(1)} = \frac{1}{2}u_x^{(0)}, \qquad v^{(1)} = 0. \tag{68}$$

With the help of *Mathematica*, one obtains

$$
\begin{aligned}
\frac{R^{(2)}}{z} &= \frac{4(z-u^{(0)})u^{(2)} + 8v^{(2)} + {u_x^{(0)}}^2 + 2v_{xx}^{(0)}}{4\left((z-u^{(0)})^2 - 4v^{(0)}\right)^{\frac{3}{2}}} \\
&+ \frac{\frac{7}{2}v^{(0)}{u_x^{(0)}}^2 + \frac{5}{2}(z-u^{(0)})v_x^{(0)}u_x^{(0)} + (z-u^{(0)})v^{(0)}u_{xx}^{(0)} + 3{v_x^{(0)}}^2 + 2v^{(0)}v_{xx}^{(0)}}{\left((z-u^{(0)})^2 - 4v^{(0)}\right)^{\frac{5}{2}}} \\
&+ \frac{10(z-u^{(0)})u_x^{(0)}v^{(0)}v_x^{(0)} + 10{v^{(0)}}^2{u_x^{(0)}}^2 + 10v^{(0)}{v_x^0}^2}{\left((z-u^{(0)})^2 - 4v^{(0)}\right)^{\frac{7}{2}}},
\end{aligned}
\tag{69}
$$

which leads to

$$
\begin{aligned}
u^{(2)} &= \frac{u_{xx}^{(0)}}{4} + \frac{v^{(0)}(7{u_x^{(0)}}^2u_{xx}^{(0)} - 4u_{xx}^{(0)}v_{xx}^{(0)} - 2u_x^{(0)}\,v_{xxx}^{(0)}) + v_x^{(0)}({u_x^{(0)}}^3 - 2u_x^{(0)}\,v_{xx}^{(0)} + 2v^{(0)}u_{xxx}^{(0)})}{24({v_x^{(0)}}^2 - v^{(0)}{u_x^{(0)}}^2)} \\
&+ \frac{v^{(0)}u_x^{(0)}({u_x^{(0)}}^4v_x^{(0)} + 4v^{(0)}u_{xx}^{(0)}({u_x^{(0)}}^3 - 2u_x^{(0)}v_{xx}^{(0)}) + 4v_x^{(0)}(v^{(0)}{u_{xx}^{(0)}}^2 + {v_{xx}^{(0)}}^2 - {u_x^{(0)}}^2v_{xx}^{(0)}))}{24\left({v_x^{(0)}}^2 - v^{(0)}\,{u_x^{(0)}}^2\right)^2} \\
v^{(2)} &= -\frac{{v^{(0)}}^2u_x^{(0)}({u_x^{(0)}}^5 + 4{u_x^{(0)}}^2v_x^{(0)}u_{xx}^{(0)} - 4v_{xx}^{(0)}({u_x^{(0)}}^3 + 2v_x^{(0)}u_{xx}^{(0)}) + 4u_x^{(0)}(v^{(0)}{u_{xx}^{(0)}}^2 + {v_{xx}^{(0)}}^2))}{24\left({v_x^{(0)}}^2 - v^{(0)}{u_x^{(0)}}^2\right)^2} \\
&- \frac{v^{(0)}({u_x^{(0)}}^4 - 3{u_x^{(0)}}^2v_{xx}^{(0)} + 2u_x^{(0)}(2\,v_x^{(0)}\,u_{xx}^{(0)} + v^{(0)}\,u_{xxx}^{(0)}) + 2(v^{(0)}\,{u_{xx}^{(0)}}^2 + {v_{xx}^{(0)}}^2 - v_x^{(0)}\,v_{xxx}^{(0)}))}{24({v_x^{(0)}}^2 - v^{(0)}{u_x^{(0)}}^2)}
\end{aligned}
$$

A further coefficient can be easily computed by taking $j=2$ in (66). Thus we obtain

$$
u^{(3)} \;=\; \frac{1}{2}u_x^{(2)} - \frac{1}{24}u_{xxx}^{(0)}, \quad v^{(3)} \;=\; 0.
$$

4.2 The classical limit

In the classical limit $\epsilon \to 0$ the functions (u, v) reduce to the first terms $(u^{(0)}, v^{(0)})$ of their semiclassical expansions and verify the equations of the dispersionless 1-Toda hierarchy

$$
\partial_{t_j}u = \frac{1}{2}\,\partial_x\,(r_{j+1} - u\,r_j), \quad \partial_{t_j}v = v\,\partial_x\,r_j, \tag{70}
$$

where r_j are the coefficients of the Laurent expansion of $R := R^{(0)}$

$$
R := \frac{z}{p-\bar{p}} = \frac{z}{\sqrt{(z-u)^2 - 4v}} = \sum_{k\geq 0}\frac{r_k(u,v)}{z^k}, \quad r_0 = 1, \tag{71}
$$

and we have taken into account (see (61)) that $\mathbb{T} = T^{(0)} = (z-u)\, R/z$. Here $p := p^{(0)}$ and $\bar{p} := \bar{p}^{(0)}$ are given by

$$
\begin{aligned}
p(z) &= \frac{1}{2}\Big((z-u) + \sqrt{(z-u)^2 - 4v}\Big) = z - u - \frac{v}{z} + \cdots \\
\bar{p}(z) &= \frac{1}{2}\Big((z-u) - \sqrt{(z-u)^2 - 4v}\Big) = \frac{v}{z} + \cdots .
\end{aligned}
\tag{72}
$$

According to (64), (u, v) are determined by

$$
\begin{cases}
\displaystyle\oint_\gamma \frac{dz}{2\pi i} \frac{U_z}{\sqrt{(z-u)^2 - 4v}} = 0, \\[2ex]
\displaystyle\oint_\gamma \frac{dz}{2\pi i} \frac{z\, U_z}{\sqrt{(z-u)^2 - 4v}} = -2\, x,
\end{cases}
\tag{73}
$$

which can be also expressed as *hodograph* type equations

$$
\begin{cases}
\sum_{j=1}^\infty j\, \Big((\bar{t}_j + \bar{c}_j) - (t_j + c_j)\Big) r_{j-1} = 0, \\[2ex]
\dfrac{1}{2}\, \sum_{j=1}^\infty j\, \Big((\bar{t}_j + \bar{c}_j) - (t_j + c_j)\Big)\, r_j = x.
\end{cases}
\tag{74}
$$

4.3 The planar limit of the Hermitian matrix model

From (50) the one-point correlator $W(z)$ is given by

$$
W(z) = \frac{1}{z} + \mathbb{M}_r(\epsilon, z, 1, \mathbf{t}) - \epsilon \sum_{j\geq 1} \frac{1}{z^{j+1}} \frac{\partial \mathbb{S}_r}{\partial t_j}(\epsilon, z, 1, \mathbf{t}),
$$

so that by using the first equations of (9) and (51) one finds

$$
\begin{aligned}
W(z) &= \sum_{j=0}^\infty (j+1) \left(\tilde{t}_{j+1} - \frac{\epsilon}{(j+1)\, z^{j+1}}\right) (\alpha_j + \beta_j\, p(z) - z^j) \\
&= -\sum_{j=1}^\infty j \left(\tilde{t}_j - \frac{\epsilon}{j\, z^j}\right) \left(-\frac{1}{2} R_{j-1} + \frac{z}{2\,\mathbb{R}}\, (z^{j-2}\,\mathbb{R})_\ominus - \left(\frac{z}{2\,\mathbb{R}}\, \mathbb{T}\, (z^{j-2}\,\mathbb{R})_\oplus\right)_\ominus \right),
\end{aligned}
\tag{75}
$$

where

$$
\tilde{t}_j := t_j + c_j.
$$

We are going to show that the solution of the dispersionless 1-Toda hierarchy determined by (74) and (49) describes the planar limit of the Hermitian matrix model in the *one-cut* case where the density of eigenvalues

$$
\rho(z) = M(z)\, \sqrt{(z-a)(z-b)},
$$

is supported on a single interval $[a,b]$. As it is known (see for instance [33]-[34]) these objects are related to the first term $W^{(0)}$ of the large N-expansion of W in the form

$$W^{(0)} = -\frac{1}{2}V_z(z) + i\pi\,\rho(z), \quad V(z) := \sum_{k\geq 1} \tilde{t}_k\, z^k.$$

According to (61), in the classical limit $\mathbb{T} = T^{(0)} = (z-u)\,R/z$ and then from (75) it follows

$$W^{(0)} = \frac{1}{2}\sum_{j=1}^{\infty} j\,\tilde{t}_j\, r_{j-1} - \frac{1}{2}\sum_{j=1}^{\infty} j\,\tilde{t}_j\, z^{j-1} + \frac{1}{2}(p-\bar{p})\sum_{j=2}^{\infty} j\,\tilde{t}_j\left(z^{j-2}\,R\right)_{\oplus},$$

with $x=1$ in all x-dependent functions. Due to (49) the first hodograph equation (74) implies that the first term in the last equation vanishes. Therefore the expressions for the density of eigenvalues and the end-points of its support provided the above solution of the dispersionless 1-Toda hierarchy are

$$\begin{gathered} \rho(z) := \frac{1}{2\pi i}\Big(\frac{V_z}{\sqrt{(z-a)(z-b)}}\Big)_{\oplus}\sqrt{(z-a)(z-b)}, \\ a := u - 2\sqrt{v}, \quad b := u + 2\sqrt{v}, \end{gathered} \tag{76}$$

where $x=1$ in all x-dependent functions. Moreover, from (73), they are determined by the equations

$$\begin{cases} \displaystyle\oint_{\gamma} \frac{dz}{2\pi i}\frac{V_z}{\sqrt{(z-a)(z-b)}} = 0, \\ \displaystyle\oint_{\gamma} \frac{dz}{2\pi i}\frac{z\,V_z}{\sqrt{(z-a)(z-b)}} = -2. \end{cases} \tag{77}$$

They coincide with the equations for the planar limit contribution to the partition function of the hermitian model [30]-[34].

5 Critical points and the double scaling limit

As we have seen the characterization of (u,v) as semiclassical expansions relies on the determination of smooth leading terms $(u^{(0)}, v^{(0)})$, which are defined implicitly by the hodograph equations (64). However, near critical points the functions $(u^{(0)}, v^{(0)})$ are multivalued and have singular x-derivatives. Thus the semiclassical expansions are not longer valid and a different procedure must be used. In this subsection we indicate how the so called *double scaling limit* method (see for instance [35]) can be formulated in our scheme.

To simplify the discussion we set $u\equiv 0$ and

$$t_{2j-1} = c_j = 0, \quad j\geq 1; \quad \bar{t}_j = \bar{c}_j = 0, \quad j\geq 1, \tag{78}$$

so that the Lax operator is of the form

$$\mathcal{L} = \Lambda + v\,\Lambda^{-1}, \tag{79}$$

and we are only considering the Toda flows associated with the even times t_{2j}. After eliminating $\mathbb{R}$ in (33), one sees that the generating function $\mathbb{U} := \mathbb{T}_{[-1]}$ satisfies the identity

$$v\Big(\mathbb{U} + \mathbb{U}_{[-1]}\Big)\Big(\mathbb{U} + \mathbb{U}_{[1]}\Big) = z^2(\mathbb{U}^2 - 1). \tag{80}$$

This leads to expansions of the form

$$\mathbb{U} = \sum_{j\geq 0} \frac{U_{2j}}{z^{2j}}, \quad \mathbb{U} = \sum_{k\geq 0} \epsilon^{2k}\, U^{(k)}. \tag{81}$$

On the other hand, the system (55) reduces to

$$-\sum_{j=1}^{\infty} j\, t_{2j}\, U_{2j} = -\frac{1}{4\pi i}\oint_{\gamma} dz\, V_z\, \mathbb{U} = x, \tag{82}$$

where $\quad V = \sum_{k\geq 1} t_{2k}\, z^{2k}$. Thus , the solution v is found from (80) and (82). In particular, the leading term $v^{(0)}$ is implicitly determined by the hodograph equation

$$H(\boldsymbol{t}_{even}, v^{(0)}) = x, \tag{83}$$

where

$$H(\boldsymbol{t}_{even}, v) := -\frac{1}{4\pi i}\oint_{\gamma} dz\, V_z\, U^{(0)} = -\frac{1}{4\pi i}\oint_{\gamma} dz \frac{z\, V_z}{(z^2 - 4v)^{\frac{1}{2}}}.$$

Given a general m-th order critical point $v_c := v_c(\boldsymbol{t}_{even})$ satisfying

$$\frac{\partial H}{\partial v}\Big|_{v_c} = \ldots = \frac{\partial^{m-1} H}{\partial v^{m-1}}\Big|_{v_c} = 0, \quad \frac{\partial^m H}{\partial v^m}\Big|_{v_c} \neq 0,$$

the method of the double scaling limit introduces a new small parameter $\tilde{\epsilon}$ and a new variable $\tilde{x}$ given by

$$\tilde{\epsilon} := \epsilon^{\frac{2}{2m+1}}, \quad x = H(v_c) + \tilde{\epsilon}^m\, \tilde{x}, \tag{84}$$

and generates solutions to (80) and (82) of the form

$$v = v_c\Big(1 + \sum_{k\geq 1} \tilde{\epsilon}^k\, u^{(k)}\Big), \quad \mathbb{U} = \sum_{k\geq 0} \tilde{\epsilon}^k\, \tilde{U}^{(k)}. \tag{85}$$

To prove it, we first observe that $\epsilon\, \partial_x = \tilde{\epsilon}^{1/2}\, \partial_{\tilde{x}}$, so that (80) can be rewritten as

$$v \sum_{n\geq 1} \tilde{\epsilon}^n \Big(\frac{4}{(2n)!}\mathbb{U}\, \partial_x^{2n}\mathbb{U} + \sum_{k+l=2n;\, k,l\geq 1} \frac{(-1)^k}{k!\, l!}\, \partial_x^k \mathbb{U}\, \partial_x^l \mathbb{U}\Big) = (z^2 - 4\, v)\, \mathbb{U}^2 - z^2, \tag{86}$$

and by substituting the expansions (85) in this identity and equating $\tilde{\epsilon}$-powers one can express each coefficient $\tilde{U}^{(n)}$ in the form

$$\tilde{U}^{(n)} = \sum_{r=1}^{n} \frac{z\, v_c^r\, G_{n,r}}{(z^2 - 4\, v_c)^{\frac{2r+1}{2}}}, \tag{87}$$

where the coefficients $G_{n,r}$ are differential polynomials in $u^{(k)}$, $1 \leq k \leq n-r+1$ and their $\tilde{x}$-derivatives . In particular

$$G_{n,1} = 2\, u^{(n)},$$

and the first few $\tilde{U}^{(n)}$ are

$$\begin{aligned}
\tilde{U}^{(0)} &= \frac{z}{(z^2-4v_c)^{\frac{1}{2}}}, \qquad \tilde{U}^{(1)} = \frac{2\, v_c\, z\, u^{(1)}}{(z^2-4v_c)^{\frac{3}{2}}},\\
\tilde{U}^{(2)} &= \frac{2\, v_c\, z\, u^{(2)}}{(z^2-4\, v_c)^{\frac{3}{2}}} + \frac{2\, {v_c}^2\, z\, \left(3\, {u^{(1)}}^2 + \partial_{\tilde{x}}^2 u^{(1)}\right)}{(z^2-4\, v_c)^{\frac{5}{2}}},\\
\tilde{U}^{(3)} &= \frac{2\, v_c\, z\, u^{(3)}}{(z^2-4\, v_c)^{\frac{3}{2}}} + \frac{{v_c}^2\, z \left(12\, u^{(1)}\, \left(6\, u^{(2)} + \partial_{\tilde{x}}^2 u^{(1)}\right) + 12\, \partial_{\tilde{x}}^2 u^{(2)} + \partial_{\tilde{x}}^4 u^{(1)}\right)}{6\, (z^2-4v_c)^{\frac{5}{2}}}\\
&\quad + \frac{2\, {v_c}^3\, z\, \left(10\, (u^{(1)})^3 + 5\, (\partial_{\tilde{x}} u^{(1)})^2 + 10\, u^{(1)}\, \partial_{\tilde{x}}^2 u^{(1)} + \partial_{\tilde{x}}^4 u^{(1)}\right)}{(z^2-4\, v_c)^{\frac{7}{2}}}.
\end{aligned}$$

Notice that $\tilde{U}^{(0)}(v) = U^{(0)}(v)$.

By substituting (84)-(85) in (82) we get the system

$$\begin{cases} \displaystyle\oint_\gamma dz\, V_z\, \tilde{U}^{(j)} = 0, \quad j = 1, \ldots, m-1, \\ \displaystyle -\frac{1}{4\pi i} \oint_\gamma dz\, V_z\, \tilde{U}^{(n)} = \delta_{nm}\, \tilde{x}, \quad n \geq m. \end{cases} \tag{88}$$

Since v_c is a m-th order critical point of (83) we have that

$$\oint_\gamma dz\, \frac{z\, V_z}{(z^2-4\, v_c)^{\frac{2j+1}{2}}} = 0, \quad j = 1, \ldots, m-1.$$

Hence, in view of (87), the first $m-1$ equations in (88) are identically satisfied while the remaining ones become

$$-\sum_{r=m}^{n} \frac{v_c^r\, G_{n,r}}{4\pi i} \oint_\gamma dz\, \frac{z\, V_z}{(z^2-4\, v_c)^{\frac{2r+1}{2}}} = \delta_{nm}\, \tilde{x}, \quad n \geq m. \tag{89}$$

For $n = m$ we get the equation which determines the the leading contribution $u^{(1)}$ in the double scaling limit

$$G_{m,m}(u^{(1)}) = K_m\, \tilde{x}, \quad K_m^{-1} := {v_c}^m \oint \frac{dz}{4\pi i} \frac{V_z z}{(z^2-4v_c)^{\frac{2m+1}{2}}}. \tag{90}$$

For example

$$\begin{aligned}
m = 2, &\qquad 2\left(3\, {u^{(1)}}^2 + \partial_{\tilde{x}}^2 u^{(1)}\right) = K_2\, \tilde{x};\\
m = 3, &\qquad 2\left(10\, (u^{(1)})^3 + 5\, (\partial_{\tilde{x}} u^{(1)})^2 + 10\, u^{(1)}\, \partial_{\tilde{x}}^2 u^{(1)} + \partial_{\tilde{x}}^4 u^{(1)}\right) = K_3\, \tilde{x}
\end{aligned}$$

For $n \geq m+1$ the equations of the system (89) characterize the coefficients $u^{(k)}$ for $k \geq 2$.

The differential equations (90) for $u^{(1)}$ are essentially the stationary KdV equations [36]. Indeed, from (86) and taking into account (87) one gets ($G_i := G_{i,i}$, $G_i' := \partial_{\tilde{x}} G_i, ...$)

$$2v_c \sum_{i+j=m-1} G_i\, G_j'' - \sum_{i+j=m} G_i\, G_j + 4 v_c u^{(1)} \sum_{i+j=m-1} G_i\, G_j - v_c \sum_{i+j=m-1} G_i'\, G_j' = 0,$$

which, up to trivial rescalings, coincides with the equation verified by the coefficients of the expansion of the resolvent diagonal R of the Schõdinger operator

$$R\, R'' - 2(z^2 - u)\, R^2 - \frac{1}{2} R'^2 + 2\, z^2 = 0, \quad R = 1 + \sum_{j \geq 1} \frac{R_j}{z^{2j}}.$$

Acknowledgements

The authors wish to thank the Spanish Ministerio de Educación y Ciencia (research project FIS2005-00319) for its finantial support. This work is also part of the MISGAM programme of the European Science Foundation.

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