Soliton-radiation interaction in nonlinear integrable lattices

L. Martínez Alonso
Departamento de Métodos Matemáticos de la Física, Facultad de Ciencias Físicas, Universidad Complutense, 28040 Madrid, Spain
(Received 13 February 1987)

The effect of the radiation modes on soliton motion in nonlinear lattices is investigated. A method based on the inverse scattering transform is developed, which enables us to characterize the position shifts of solitons due to their interaction with the radiation component. Applications to the Toda and Langmuir lattices are presented.

I. INTRODUCTION

The subject of integrable nonlinear lattices is an important branch of the inverse scattering transform (IST) method.[1,2] Many infinite families of these integrable systems can be derived and classified by means of discrete versions of the Lax-pair technique,[2,3] and some of their members are interesting models for vastly different physical situations. As in the continuous context, two types of (nonlinear) normal modes are manifested in the long-time behavior of discrete integrable systems: namely, solitons and radiation. Analysis of their dynamical properties is of relevance in several areas as, for instance, the classical statistical mechanics of soliton-bearing models.[4] In this paper we are concerned with the effect of radiation modes on soliton motion in nonlinear lattices. Our approach rests on the technique proposed in Ref. 5 for describing soliton-radiation interactions in continuous systems. Thus, the main component involved in our strategy is the asymptotic analysis of Gel'fand-Levitan-Marchenko (GLM) equations of discrete type. As a result the procedure can be applied to any nonlinear lattice integrable through the IST method.

The paper is organized as follows. Section II is devoted to a description of some basic aspects of the IST method associated with a linear difference equation. More concretely, we consider the difference-equation eigenvalue problem which gives rise to the Toda and Langmuir hierarchies of nonlinear lattices. Section III deals with the characterization of the asymptotic trajectories of solitons in the presence of radiation and the derivation of the position shifts caused by the soliton-radiation interaction. As relevant applications, the expressions of these shifts for the Toda chain

$$\frac{d^2x_n}{dt^2}=\exp(x_{n+1}-x_n)-\exp(x_n-x_{n-1}) ,\quad -\infty<n<\infty \tag{1.1}$$

and the Langmuir lattice[6,7]

$$\frac{dc_n}{dt}=c_n(c_{n+1}-c_{n-1}),\quad -\infty<n<\infty \tag{1.2}$$

are obtained. Since the main ideas of our analysis can be easily described, to avoid loss of continuity we have relegated the discussion of several technical points to three appendixes at the end of this paper.

II. DISCRETE INTEGRABLE SYSTEMS

The linear problem in the IST method for solving the Toda lattice equation and its higher analogs is the difference equation[7]

$$c(n+1)^{1/2}f(k,n+1)+u(n)f(k,n)+c(n)^{1/2}f(k,n-1)=\left(k+\frac{1}{k}\right)f(k,n) , \tag{2.1}$$

where $c(n)$ and $u(n)$ are assumed to be real functions of the discrete "parameter" n, with $c(n)>0$ for all n and such that

$$U(n)=(c(n),u(n))\to(1,0),\quad n\to\pm\infty . \tag{2.2}$$

To describe the scattering data variables we will follow the same scheme as in the continuous case.[5] Thus we consider the Jost solution to (2.1) with asymptotic behavior

$$f(k,n)\to k^n,\quad n\to+\infty . \tag{2.3}$$

Under conditions (2.2), $f(k,n)$ exists for $|k|\le 1$ and its restriction to the unit circle determines two functions $a(k)$ and $b(k)$ through the asymptotic expression

$$f(k,n)\to a(k)k^n-b^*(k)k^{-n},\quad n\to-\infty,\quad |k|=1 . \tag{2.4}$$

They satisfy

$$a^*(k)=a\left(\frac{1}{k}\right),\quad b^*(k)=b\left(\frac{1}{k}\right),\quad |a(k)|^2-|b(k)|^2=1 . \tag{2.5}$$

Furthermore, $a(k)$ admits an analytic continuation to $|k|<1$ which takes the form (see Appendix B)

$$a(k)=\prod_{l=1}^{N}\epsilon_l\frac{k-k_l}{kk_l-1}\exp\left[\frac{i}{4\pi}\oint_\gamma \ln[1-|r(q)|^2]\frac{q+k}{q-k}\frac{dq}{q}\right],\quad \epsilon_l=\mathrm{sgn}k_l\ . \tag{2.6}$$

Here γ denotes the unit circle oriented in the positive sense, $r(k)=b(k)/a(k)$ is the reflection coefficient, and $\{k_l\}_1^N$ is a set of N different real numbers with $0<|k_l|<1$. For each k_l there exists a real number b_l such that

$$f(k_l,n)\to b_l k_l^{-n},\quad n\to-\infty\ . \tag{2.7}$$

The list of asymptotic properties of $f(k,n)$ with respect to the discrete variable n is completed by the relation

$$f(k,n)\to a(k)k^n,\quad n\to-\infty,\quad |k|<1,\quad a(k)\neq 0\ . \tag{2.8}$$

In what follows the function $a(k)$ and the numbers b_l will be referred to as the transition coefficient and the transition constants, respectively.

The inverse scattering transformation associated with (2.1) can be described in terms of the set of scattering data variables (see Appendix A)

$$S=\{k_l,r_l,r(k)\},\quad r_l=[b_l\dot a(k_l)]^{-1}\ . \tag{2.9}$$

Thus, by introducing the function

$$\Omega(n)=-\sum_l r_l k_l^n+\frac{1}{2\pi i}\oint_\gamma r(q)q^n dq\ , \tag{2.10}$$

and solving the GLM equation for $K(n,m)$,

$$K(n,m)+\Omega(n+m)+\sum_{l=n}^{\infty}\Omega(l+m+1)K(n,l)=0\ , \tag{2.11}$$

the functions $c(n)$ and $u(n)$ in (2.1) turn out to be given by

$$c(n)=\frac{1-K(n-1,n-2)}{1-K(n,n-1)}\ ,\quad u(n)=K(n,n)-K(n-1,n-1)\ . \tag{2.12}$$

The simplest nontrivial case in which (2.11) can be explicitly solved corresponds to a set of scattering data $S=\{k_1=\epsilon_1 e^{-\alpha},r_1,r(k)=0\}$ with only one k_l and an identically zero reflection coefficient, then (2.11) leads to

$$c(n+1)=1+\sinh^2\alpha\,\mathrm{sech}^2[\alpha(n-q)]\ ,\quad q=\frac{1}{2\alpha}\ln\left|\frac{r_1}{2\sinh\alpha}\right|=-\frac{1}{2}\left[\frac{1}{\alpha}\ln|b_1|+1\right]\ . \tag{2.13}$$

Another interesting situation is provided by $S=\{k_1=e^{-\alpha},\tilde{k}_1=-k_1;r_1,\tilde{r}_1=-r_1;r(k)=0\}$. Now (2.11) yields

$$c(n+1)=1+4\sinh^2\alpha\cosh\alpha\{2\cosh^2[\alpha(n-q)]+\cosh\alpha\}^{-1}\ ,\quad q=-\tfrac{1}{2}\left[\frac{1}{\alpha}\ln|b_1|+1\right],\quad u(n)=0\ . \tag{2.14}$$

There is an infinite family of evolution equations for the potential function $U(n)=(c(n),u(n))$ of (2.1) which can be solved by means of the IST method. Each member of this family, the Toda hierarchy, is associated with a function

$$\omega(k)=\left[k-\frac{1}{k}\right]\left[k+\frac{1}{k}\right]^M,\quad M=0,1,\ldots\ , \tag{2.15}$$

which determines the evolution law of the scattering data

$$k_l(t)=k_l,\quad r_l(t)=r_l e^{\omega(k_l)t}\ ,\quad r(t,k)=r(k)e^{\omega(k)t}\ . \tag{2.16}$$

Thus, by inserting the time evolution of r_1 into (2.13) we obtain the one-soliton solution of the Toda hierarchy.

For $\omega(k)=k-k^{-1}$ we get the evolution equation

$$\partial_t c(n)=c(n)[u(n)-u(n-1)]\ ,\quad \partial_t u(n)=u(n+1)-u(n)\ , \tag{2.17}$$

which reduces to the Toda lattice system (1.1) under the substitution $c(n)=\exp(x_n-x_{n-1})$, $u(n)=\dot{x}_n$. If the integer M in (2.15) is an odd one, then the constraint $u(n)=0$ turns out to be compatible with the corresponding member of the Toda hierarchy and one obtains an evolution equation for $c(n)$. The family of these equations is the Langmuir hierarchy. In particular,[7] for $M=1$ we get the Langmuir lattice system (1.2). Moreover, it follows that $u(n)=0$ leads to a set of scattering data of the form $S=\{k_l,\tilde{k}_l=-k_l;r_l,\tilde{r}_l=-r_l;r(k)=r(-k):l=1,\ldots,N\}$. The one-soliton solution of the Langmuir hierarchy obtains by inserting the evolution law

$$b_1(t)=b_1\exp[-\omega(k_1)t]$$

into (2.14).

III. SOLITON-RADIATION INTERACTIONS

From Eqs. (2.11) and (2.12) it is clear that the values of $c(n)$ and $u(n)$ to the right of a certain integer n_0 depend only on the form of $\Omega(n)$ for $n\geq 2n_0-3$. This fact, together with the form of the evolution laws (2.15), implies that to each equation of the Toda hierarchy, there corresponds a spectrum of propagation velocities for the scattering data. To show this, let $U(t,n)=(c(t,n),u(t,n))$ be a solution to a member of the hierarchy and let $S(t)=\{k_l=\epsilon_l e^{-\alpha_l},r_l(t),r(t,k):l=1,\ldots,N\}$ be its corresponding trajectory in the space of scattering data variables. The value of $U(t,n)$ at a freely moving point $n(t)=vt$ depends only on the values $\Omega(t,2n(t)+m)$ with $m\geq-3$. According to (2.10) and (2.16) the contribution of a discrete mode $(k_l,r_l(t))$ to $\Omega(t,2n(t)+m)$ is given by

$$-r_l(t)k_l^{2n(t)+m}=-r_l\exp[2\alpha_l(v_l-v)t]\ , \tag{3.1}$$

where

$$v_l=\frac{\omega(k_l)}{2\alpha_1} \tag{3.2}$$

while the contribution of the continuous modes to $\Omega(t,2n(t)+m)$ is

$$\frac{1}{2\pi i}\oint r(t,q)q^{2n(t)+m}dq$$
$$=\frac{1}{2\pi}\int_{-\pi}^{\pi} r(e^{i\theta})\exp\{[2i\theta v+\omega(e^{i\theta})]t\}d\theta \ . \tag{3.3}$$

From (2.15) we have

$$\omega(e^{i\theta})=i2^{M+1}\sin\theta(\cos\theta)^M \ , \tag{3.4}$$

which takes purely imaginary values only. Hence, the integral (3.3) is a superposition of Fourier modes with group velocity

$$v(q)=-2^M(\cos\theta)^{M-1}[(M+1)\cos^2\theta-M], \quad q=e^{i\theta} \ . \tag{3.5}$$

Equations (3.2) and (3.5) define the propagation velocities for the contributions of the scattering data to the asymptotic form of $U(t,n)$ as $t\to\pm\infty$. Indeed, Eqs. (3.1) and (3.3) show that as $t\to+\infty$ $(t\to-\infty)$ the scattering data with propagation velocity smaller (greater) than v do not contribute to $U(t,n)$ at $n=vt$. Hence, if $v\neq v_l$ for all $l=1,\ldots,N$, we conclude that as $t\to\pm\infty$ the form of $U(t,n)$ to the right of $n=vt$ is determined through the GLM equation corresponding to the following sets of scattering data (see Appendix C):

$$S_\pm(v;t)=\{k_l,r_l(t),l \text{ such that } \pm(v_l-v)>0;\theta(\pm[v(k)-v])r(t,k)\} \ , \tag{3.6}$$

where θ denotes the Heaviside step function. In other words, as $t\to\pm\infty$ the truncated solutions[8]

$$U_\pm(v;t,n)=\begin{cases}U(t,n), & \pm(n-vt)>0,\\ (1,0) & \text{otherwise},\end{cases} \tag{3.7}$$

are associated with the sets $S_\pm(v;t)$ of scattering data. Observe that Eqs. (2.6) and (3.6) imply the following expressions for the transition coefficients associated with $U_\pm(v,t)$:

$$a_\pm(v,k)=\prod_{\pm(v_l-v)>0}\epsilon_l\frac{k-k_l}{kk_l-1}\exp\left[\frac{i}{4\pi}\oint_\gamma\theta(\pm[v(q)-v])\ln[1-|r(q)|^2]\frac{q+k}{q-k}\frac{dq}{q}\right] \ . \tag{3.8}$$

We can now analyze the asymptotic trajectories of the solitons arising in $U(t,n)$. Consider the limits of the truncated solutions (3.7) as $v\to v_j\pm0$, with v_j being one of the velocities (3.2). If $v_l\neq v_j$ for all $l\neq j$ (Ref. 9), then

$$S_+(v_j-0;t)=S_+(v_j+0;t)U\{k_j,r_j(t)\} \ , \tag{3.9a}$$

$$S_-(v_j+0;t)=S_-(v_j-0;t)U\{k_j,r_j(t)\} \ . \tag{3.9b}$$

Since $U_\pm(v,t)$ represent the components of the solution propagating to the right of $n=vt$ as $t\to\pm\infty$, and in view of (3.9), it is natural to assume

$$U_+(v_j-0;t)=U_+(v_j+0;t)+(j\text{th soliton}), \quad t\to+\infty \ , \tag{3.10a}$$

$$U_-(v_j+0;t)=U_-(v_j-0;t)+(j\text{th soliton}), \quad t\to-\infty \ , \tag{3.10b}$$

where by the jth soliton we mean the one propagating with velocity v_j. Now, let us consider the scattering problem (2.1) corresponding to $U_+(v_j-0;t)$; as a consequence of (2.7) and (3.9a), the form of the Jost solution $f(k_j,t,n)$ to the left of the potential support will be

$$f(k_j,t,n)=[r_j(t)\dot{a}_+(v_j-0;k_j)]^{-1}k_j^{-n} \ . \tag{3.11}$$

On the other hand, from (3.10a) the same result must be found if we first continue the Jost solution from $n=+\infty$ to the left of $U_+(v_j+0,t)$, and then we continue the result to the left of the jth soliton. This sequence gives the outcome

$$f(k_j,t,n)\to a_+(v_j+0;k_j)k_j^n$$
$$\to b_j^+(t)a_+(v_j+0;k_j)k_j^{-n} \ , \tag{3.12}$$

where $b_j^+(t)$ is the transition constant of the jth soliton as $t\to+\infty$. Therefore, taking into account that

$$a_+(v_j-0;k)=\epsilon_j\frac{k-k_j}{kk_j-1}a_+(v_j+0;k) \ , \tag{3.13}$$

we get

$$b_j^+(t)=\epsilon_j(k_j^2-1)[r_j(t)a_+^2(v_j+0;k_j)]^{-1} \ . \tag{3.14}$$

Similarly, as $t\to-\infty$ the transition constant of the jth soliton may be found through (3.9b) and (3.10b) and turns out to be given by

$$b_j^-(t)=\epsilon_j(k_j^2-1)[r_j(t)a_-^2(v_j-0;k_j)]^{-1} \ . \tag{3.15}$$

Hence, in virtue of the expression (2.13) for the one-soliton solution, we conclude that as $t\to\pm\infty$ the center of the jth soliton moves along the free trajectories $q_j^\pm(t)=q_j^\pm+v_jt$ with

$$\Delta_j=q_j^+-q_j^-=\frac{1}{\alpha_j}\ln\left|\frac{a_+(v_j+0,k_j)}{a_-(v_j-0,k_j)}\right| \ ; \tag{3.16}$$

then, by using (3.8) we find

$$\Delta_j=\frac{1}{\alpha_j}\sum_{l\neq j}\operatorname{sgn}(v_l-v_j)\ln\left|\frac{k_l-k_j}{k_lk_j-1}\right|+\frac{1}{4\pi\alpha_j}\operatorname{Re}\left[i\oint_\gamma \operatorname{sgn}[v(q)-v_j]\ln[1-|r(q)|^2]\frac{q+k_j}{q-k_j}\frac{dq}{q}\right]. \qquad (3.17)$$

This expression exhibits the contributions to the position shift of a soliton due to the presence of the other solitons and the radiation component. Obviously, the integral term describes the effect of the radiation. For the Toda lattice the spectrum of propagation velocities is

$$v_l=-\epsilon_l\alpha_l^{-1}\sinh\alpha_l,\quad v(q)=-\cos\theta\ , \qquad (3.18)$$

so that $\operatorname{sgn}[v(q)-v_j]=\epsilon_j$. Then, taking into account that

$$k_j=\epsilon_je^{-\alpha_j},\quad \operatorname{Re}\left[\frac{e^{i\theta}+k_j}{e^{i\theta}-k_j}\right]=\frac{\epsilon_j\sinh\alpha_j}{\epsilon_j\cosh\alpha_j-\cos\theta}\ ,$$

one gets the following representation for the position shift due to the soliton-radiation interaction:

$$\Delta_j^{\text{rad}}=-\frac{1}{\alpha_j}\int_0^\pi J(\theta)\frac{\sinh\alpha_j}{\epsilon_j\cosh\alpha_j-\cos\theta}d\theta\ , \qquad (3.19)$$

where

$$J(\theta)=\frac{1}{2\pi}\ln[1-|r(e^{i\theta})|^2]\ . \qquad (3.20)$$

We notice that (3.19) is in agreement with the soliton-phonon spatial shift found by Theodorakopoulos and Mertens[10] through a different method.

We now discuss briefly the applications to the Langmuir hierarchy. Suppose given a solution $c(t,n)$ to an equation of the hierarchy, and let

$$S(t)=\{k_l=e^{-\alpha_l},\tilde{k}_l=-k_l;r_l(t),\tilde{r}_l(t)=-r_l(t)\ ;\ r(t,k)=r(t,-k)\}$$

be its associated scattering data. The same analysis as for the Toda family applies here, but now the analogs of Eqs. (3.9) and (3.13)–(3.15) are

$$S_\pm(v_j\mp0;t)=S_\pm(v_j\pm0;t)U\{k_j,\tilde{k}_j,r_j(t),\tilde{r}_j(t)\}\ , \qquad (3.21)$$

$$a_\pm(v_j\mp0;k)=\frac{k^2-k_j^2}{k^2k_j^2-1}a_\pm(v_j\pm0;k)\ , \qquad (3.22)$$

$$b_j^\pm(t)=(k_j^4-1)[2k_jr_j(t)a_\pm^2(v_j\pm0;k_j)]^{-1}\ . \qquad (3.23)$$

Substituting (3.23) into formula (2.14) for the soliton position, and keeping in mind the symmetry properties of the scattering data one obtains

$$\Delta_j=\frac{1}{\alpha_j}\sum_{l\neq j}\operatorname{sgn}(v_l-v_j)\ln\left|\frac{k_l^2-k_j^2}{k_l^2k_j^2-1}\right|+\frac{1}{4\pi\alpha_j}\operatorname{Re}\left[i\oint_\gamma \operatorname{sgn}[v(q)-v_j]\ln[1-|r(q)|^2]\frac{q^2+k^2}{q^2-k^2}\frac{dq}{q}\right]. \qquad (3.24)$$

For example, the Langmuir lattice has a spectrum of velocities given by

$$v_l=-\alpha_l^{-1}\sinh2\alpha_l,\quad v(q)=-2\cos2\theta\ . \qquad (3.25)$$

Hence $\operatorname{sgn}[v(q)-v_j]=1$, and therefore the integral term in (3.24) reduces to

$$\Delta_j^{\text{rad}}=-\frac{1}{\alpha_j}\int_0^\pi J(\theta)\frac{\sinh2\alpha_j}{\cosh2\alpha_j-\cos2\theta}d\theta\ . \qquad (3.26)$$

ACKNOWLEDGMENTS

It is a pleasure to thank Professor P. C. Sabatier and the group of the Laboratoire de Physique Mathématique of Montpellier University for warm hospitality while this work was in progress. Partial financial support from the Comisión Asesora de Investigación Científica y Técnica, Spain is also acknowledged.

APPENDIX A

In this appendix we indicate a simple way of deriving the GLM equations (2.10)–(2.12) starting from the singular integral equation[11]

$$F(k,n)=1-\sum_l\frac{k_l^{2n}}{k_l-k^{-1}}r_lF(k_l,n)+\frac{1}{2\pi i}\oint_\gamma\frac{q^{2n}}{q-k^{-1}}r(q)F(q,n)dq,\quad |k|<1\ , \qquad (\text{A1})$$

satisfied by the modified Jost solution

$$F(k,n)=\left[\prod_{m>n}c(m)^{1/2}\right]k^{-n}f(k,n)\ . \qquad (\text{A2})$$

First of all, it is convenient to introduce the following distribution:

$$R(q)=\pi\sum_l r_l\delta(q-k_l)+\frac{i}{2}r(q)\delta_\gamma(q)\ , \qquad (\text{A3})$$

which acts on functions defined on the unit disk $D=\{q=q_1+iq_2:|q|\le1\}$, where $\delta(q-k_l)$ are Dirac deltas supported at k_l $(l=1,\ldots,N)$ and $r(q)\delta_\gamma(q)$ is defined by

$$\int\int_D r(q)\delta_\gamma(q)\phi(q)dq_1dq_2=\oint_\gamma r(q)\phi(q)dq\ .$$

Thus, Eq. (A1) takes the simpler form

$$F(k,n)=1-\frac{1}{\pi}\int\int_D \frac{q^{2n}}{q-k^{-1}}R(q)F(q,n)dq_1dq_2\ ,\quad |k|<1\ . \quad (A4)$$

It is known[7] that the potentials in the eigenvalue problem (2.1) can be written as

$$c(n)=\frac{B(n-1)}{B(n)},\quad u(n)=A(n)-A(n-1)\ , \quad (A5)$$

where $A(n)$ and $B(n)$ are obtained from $F(k,n)$ through the asymptotic behavior

$$F(k,n)\to 1+kA(n)+O(k^2),\quad k\to 0 \quad (A6)$$

and the integral representation

$$B(n)=1-\frac{1}{\pi}\int\int_D q^{2n-1}R(q)F(q,n)dq_1dq_2\ . \quad (A7)$$

On the support of $R(q)$ we may expand

$$(q-k^{-1})^{-1}=-\sum_{m=0}^{\infty}(qk)_k^m$$

for all $|k|<1$, so that (A4) can be written as

$$F(k,n)=1+\sum_{m=0}^{\infty}K(n,n+m)k^{m+1}\ , \quad (A8)$$

where

$$K(n,m)=-\frac{1}{\pi}\int\int_D q^{n+m}R(q)F(q,n)dq_1dq_2\ . \quad (A9)$$

Inserting the expression (A8) for F into (A9) immediately yields the GLM equation (2.11). On the other hand, Eq. (2.12) is a simple consequence of (A5)–(A9).

APPENDIX B

We prove here the important formula (2.6) which provides an explicit representation of the transition coefficient in terms of the scattering data. The function $a(k)$ is analytic in $|k|<1$ and continuous in $|k|\leq 1$, it has a finite number of simple zeros k_l which are real and such that $0<|k_l|<1$. The reduced function

$$\tilde{a}(k)=\prod_l \frac{kk_l-1}{k-k_l}a(k)\ , \quad (B1)$$

is zero-free in the closed disk $|k|\leq 1$; therefore, there exist logarithmic branches $\ln_1\tilde{a}(k)$ and $\ln_2\tilde{a}(k^{-1})$ analytic (continuous) in $|k|<1$ and $|k|>1$ ($|k|\leq 1$ and $|k|\geq 1$), respectively. Since $\tilde{a}^*(q)=\tilde{a}(q^{-1})$ for $|q|=1$, we can choose these branches so that

$$\ln_1\tilde{a}(q)+\ln_2\tilde{a}(q^{-1})=\ln|\tilde{a}(q)|^2=\ln|a(q)|^2\ ,\quad |q|=1\ , \quad (B2)$$

where ln denotes the principal branch of the logarithm. Then, one readily finds

$$\frac{1}{2\pi i}\oint_\gamma \ln|a(q)|^2\frac{q+k}{q-k}\frac{dq}{q} =2\ln_1\tilde{a}(k)-\ln_1\tilde{a}(0)+\ln_2\tilde{a}(0)\ , \quad (B3)$$

which together with (B1) and (2.5) implies

$$a(k)=\epsilon\prod_l\frac{k-k_l}{kk_l-1}\times\exp\left[\frac{i}{4\pi}\oint_\gamma \ln[1-|r(q)|^2]\frac{q+k}{q-k}\frac{dq}{q}\right]\ , \quad (B4)$$

where $\epsilon=\frac{1}{2}[\ln_2\tilde{a}(0)-\ln_1\tilde{a}(0)]$ is a square root of unity. Furthermore, analysis of (2.1) shows that[7]

$$a(0)=\left[\prod_n c(n)\right]^{-1/2} \quad (B5)$$

then $a(0)>0$, and therefore (B4) implies $\epsilon=\prod_l \operatorname{sgn}k_l$.

APPENDIX C

The main step in our asymptotic analysis of the GLM equation is the description of the truncated asymptotic solutions (3.7) in terms of the sets of scattering data (3.6). This result is analogous to what is found in the context of continuous integrable models,[5,8] and, as we are going to see now, it can also be derived by a method quite similar to the one used in Appendix A of Ref. 5. The method uses the asymptotic properties of the modified Jost solution (A2), which according to (2.4), (2.7), (2.8), and (B5) verifies

$$F(k,n)\underset{n\to-\infty}{\sim}\begin{cases}[a(k)-b^*(k)k^{-2n}]a(0)^{-1}, & |k|=1,\\ a(k)a(0)^{-1}, & |k|<1,\ a(k)\neq 0,\\ b_lk_l^{-2n}a(0)^{-1}, & k=k_l,\ a(k_l)=0.\end{cases} \quad (C1)$$

Suppose we are given a solution $U(t,n)=(c(t,n),u(t,n))$ to an evolution equation of the Toda hierarchy. If we want to calculate $F(t,k,n)$ as $t\to+\infty$ at a point $n=vt$ with v different from the discrete velocities v_l, it is reasonable to expect that it will behave as if the potentials in (2.1) were the trivial ones. This is so because the solitons arising in $U(t,n)$ as $t\to+\infty$ move with the velocities v_l ($l=1,\ldots,N$), so that $c(t,vt)\to 1$ and $u(t,vt)\to 0$ as $t\to+\infty$. Further, as $t\to+\infty$ only those solitons with velocity $v_l>v$ are to the right of $n=vt$. These considerations and (C1) suggest the following ansatz in the soliton-free regions $n=vt$ as $t\to+\infty$:

$$F(t,k,n)=\begin{cases}[\tilde{a}(k)-\tilde{b}^*(k)k^{-2n}e^{\omega(k)t}]\tilde{a}(0)^{-1}, & |k|=1,\\ \tilde{a}(k)\tilde{a}(0)^{-1}, & |k|<1,\ \tilde{a}(k)\neq 0,\\ \tilde{b}_le^{-\omega(k_l)t}k_l^{-2n}\tilde{a}(0)^{-1}, & k=k_l,\ v_l>v.\end{cases} \quad (C2)$$

Here the zeros of $\tilde{a}(k)$ are assumed to be those k_l such that $v_l>v$. The function $F(t,k,n)$ must satisfy the integral equation (A1) with the appropriate time dependence of the scattering data, and this is just an equation for the values of F at $k=k_l$ ($l=1,\ldots,N$) and on the unit circle γ. By inserting the ansatz (C2) into (A1) and using the asymptotic formula

$$\frac{1}{2\pi i}\oint_\gamma \frac{\exp[t\omega(q)]}{q-k\exp(+0)}\phi(q)dq \underset{t\to+\infty}{\sim} -\theta(-k\dot\omega(k))\exp[t\omega(k)]\phi(k), \quad |k|=1 , \tag{C3}$$

we are led to

$$\tilde a(k)=\tilde a(0)+\sum_{v_l>v}\frac{\tilde b_l r_l}{k^{-1}-k_l} -\frac{1}{2\pi i}\oint_\gamma \frac{r(q)\tilde b^*(q)}{q-k^{-1}}dq, \quad |k|<1 , \tag{C4}$$

$$\tilde b(k)=\theta(v(k)-v)r(k)\tilde a(k), \quad |k|=1 . \tag{C5}$$

Now if we take $\tilde a(k)=a_+(v;k)$ and use the relation $r(q)\tilde b^*(q)=\tilde a(q)^{-1}-\tilde a(q)$, then it is readily shown that (C4) holds provided

$$\tilde b_l=[r_l\dot{\tilde a}(k_l)]^{-1} . \tag{C6}$$

According to (C2) the set of scattering data for the truncated solution $U_+(v;t,n)$ [see Eq. (3.7)] is given by $\{k_l,\tilde r_l=[\tilde b_l\dot{\tilde a}(k_l)]^{-1}$: l such that $v_l>v;r(t,k)=\tilde b(k)\times e^{\omega(k)t}/\tilde a(k)\}$, but what (C5) and (C6) prove is that this set coincides with $S_+(v;t)$. A similar analysis can be applied to the truncated solution $U_-(v;t)$.

[1]M. Toda, *Theory of Nonlinear Lattices* (Springer, Berlin, 1981).
[2]M. J. Ablowitz and H. Segur, *Solitons and the Inverse Scattering Transform* (SIAM, Philadelphia, 1981).
[3]B. A. Kupershmidt, *Discrete Lax Equations and Differential-Difference Calculus, Astérisque 123* (Société Mathématique de France, Paris, 1985).
[4]N. Theodorakopoulos, in *Dynamical Problems in Soliton Systems,* proceedings of the Seventh Kyoto Summer Institute, edited by S. Takeno (Springer, Berlin, 1985).
[5]L. Martínez Alonso, Phys. Rev. D **32**, 1459 (1985).
[6]S. V. Manakov, Zh. Eksp. Teor. Fiz. **67**, 543 (1974) [Sov. Phys. JETP **40**, 269 (1975)].
[7]V. E. Zakharov, S. V. Manakov, S. P. Novikov, and L. P. Pitaevskii, *Theory of Solitons: The Inverse Scattering Method* (Plenum, New York, 1984).
[8]For a treatment of truncated asymptotic solutions and their applications to a continuous system, see L. Martínez Alonso, Phys. Lett. **112A**, 361 (1985).
[9]This condition is always satisfied by those members of the Toda hierarchy for which the integer M in Eq. (2.15) is even. In particular, it holds for the Toda lattice.
[10]N. Theodorakopoulos and F. G. Mertens, Phys. Rev. B **28**, 3512 (1983).
[11]This equation is obtained by setting $z=k^{-1}$ in Eq. (2.5) of Ref. 7, p. 52.