

# $m_\pi$ and $N_c$ dependence of resonances form Unitarized Chiral Perturbation Theory

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**Abstract.** We review our studies on the  $\rho$  and  $\sigma$  resonances properties derived from the Inverse Amplitude Method. In particular, we study the leading  $1/N_c$  behavior of the resonances masses and widths and their evolution with changing  $m_\pi$ . The  $1/N_c$  expansion gives a clear definition of  $\bar{q}q$  states, which is neatly satisfied by the  $\rho$  but not by the  $\sigma$ , showing that its *dominant component* is not  $\bar{q}q$ . The  $m_\pi$  dependence of the resonance properties is relevant to connect with lattice studies. We show that our predictions compare well with some lattice results and we find that the  $\rho\pi\pi$  coupling constant is  $m_\pi$  independent, in contrast with the  $\sigma\pi\pi$  coupling, that shows a strong  $m_\pi$  dependence.

**Keywords:** Scalar mesons, chiral lagrangians,  $1/N_c$  expansion

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Light hadron spectroscopy lies beyond the realm of perturbative QCD. At low energies, however, one can use the QCD low energy effective theory, named Chiral Perturbation Theory (ChPT) [1], to describe the dynamics of the lightest mesons. ChPT describes the interactions of the pseudo Nambu-Goldstone bosons of the QCD chiral symmetry breaking, namely, the pions, by means of a effective lagrangian compatible with all QCD symmetries involving only the pion field. The infinite tower of terms in this lagrangian is organized as a low energy expansion in powers of  $p^2/\Lambda_\chi^2$ , where  $p$  stands either for derivatives, momenta or masses, and  $\Lambda_\chi \simeq 4\pi f_\pi$ , where  $f_\pi$  denotes the pion decay constant. ChPT is renormalized order by order by absorbing loop divergences in the renormalization of higher order parameters, known as low energy constants (LECs), that parametrize the high energy QCD dynamics and *carry no energy or mass dependence*. They depend on a regularization scale  $\mu$  but after renormalization the observables are independent of this scale. The value of the LECs depend on the underlying QCD dynamics and are determined from experiment. Up to the desired order, the ChPT expansion provides a *systematic and model independent* description of how observables depend on some QCD parameters like the light quark mass  $\hat{m} = (m_u + m_d)/2$  or the number of colors,  $N_c$  [2].

The use of ChPT is limited to low energies and masses, nevertheless, combined with dispersion relations and elastic unitarity it leads to a successful description of meson dynamics up to energies around 1 GeV, generating resonant states not originally present in the lagrangian, without any a priori assumption on their existence or nature. In particular, we find the  $\rho$  and  $\sigma$  resonances as poles on the second Riemann sheet of  $\pi\pi$  elastic scattering amplitudes. With this approach we can then study some of these resonances properties, like their spectroscopic nature through their mass and width dependence on  $N_c$ , or their dependence on the pion mass in order to connect with lattice studies. In the following sections we review this “unitarized ChPT” approach, named the Inverse Amplitude Method (IAM) [3, 4, 5], and then apply it to study the leading  $1/N_c$  behavior and the chiral extrapolation of the  $\rho$  and  $\sigma$  mesons.

The  $\rho$  and  $\sigma$  resonances appear as poles on the second Riemann sheet of the  $(I, J) = (1, 1)$  and  $(I, J) = (0, 0)$   $\pi\pi$  scattering partial waves of definite isospin,  $I$  and angular momentum  $J$ , respectively. Elastic unitarity implies for these partial waves,  $t(s)$ , and physical values of  $s$  below inelastic thresholds, that

$$\text{Im}t(s) = \sigma(s)|t(s)|^2 \Rightarrow \text{Im} \frac{1}{t(s)} = -\sigma(s), \quad \text{with} \quad \sigma(s) = 2p/\sqrt{s}, \quad (1)$$

where  $s$  is the Mandelstam variable and  $p$  is the center of mass momentum. Consequently, the imaginary part of the inverse amplitude is known exactly. However, ChPT amplitudes, being an expansion  $t \simeq t_2 + t_4 + \dots$ , with  $t_k = O(p^k)$ , can only satisfy Eq. (1) perturbatively

$$\text{Im}t_2(s) = 0, \quad \text{Im}t_4(s) = \sigma(s)t_2^2(s) \dots \quad (2)$$

and cannot generate poles. Therefore the resonance region lies beyond the reach of standard ChPT. This region however, can be reached combining ChPT with dispersion theory through the IAM [3, 4, 5].

The analytic structure of the  $\pi\pi$  scattering amplitude  $t(s)$ , consisting on a right cut extending from  $s_{th} = 4m_\pi^2$  to  $\infty$ , and a left cut from  $-\infty$  to 0, allows to write a dispersion relation for the auxiliary function  $G(s) \equiv t_2^2(s)/t(s)$

$$G(s) = G(0) + G'(0)s + \frac{1}{2}G''(0)s^2 + \frac{s^3}{\pi} \int_{s_{th}}^{\infty} ds' \frac{\text{Im} G(s')}{s'^3(s' - s - i\epsilon)} + LC(G) + PC, \quad (3)$$

where the integral over the left cut has been abbreviated as  $LC(G)$  and  $PC$  stands for possible pole contributions corresponding to zeros of  $t$ . The different terms in Eq.(3) can be evaluated using unitarity and ChPT as follows: The right cut can be *exactly* evaluated taking into account the elastic unitarity conditions Eqs. (1), (2),  $\text{Im} G(s') = -\sigma(s')t_2^2(s') = -\text{Im} t_4(s')$ , for  $s' \in (4m_\pi^2, \infty)$ . The subtraction constants only involve the amplitude and its derivatives evaluated at  $s = 0$ , so they can be safely approximated with ChPT:  $G(0) \simeq t_2(0) - t_4(0)$ ,  $G'(0) \simeq t_2'(0) - t_4'(0)$ ,  $G''(0) \simeq -t_4''(0)$ . The left cut, which is suppressed by  $1/s'^3(s' - s)$ , is *weighted at low energies*, so it is appropriate to approximate it with ChPT:  $LC(G) \simeq -LC(t_4)$ . The pole contribution only appears in the scalar wave, which vanishes at the so called Adler zero. It counts  $O(p^6)$  and has been calculated explicitly [6] and is not just formally suppressed, but numerically negligible except near the Adler zero, away from the physical region.

Neglecting  $PC$  for the moment, and taking into account that  $t_2(s)$  is just a first order polynomial in  $s$ , and that a dispersion relation can be also written for  $t_4$ , we can write Eq.(3) as

$$G(s) \equiv \frac{t_2^2(s)}{t(s)} \simeq t_2(0) - t_2'(0)s - t_4(0) - t_4'(0)s - \frac{1}{2}t_4''(0)s^2 - \frac{s^3}{\pi} \int_{s_{th}}^{\infty} ds' \frac{\text{Im} t_4(s')}{s'^3(s' - s - i\epsilon)} - LC(t_4) = t_2(s) - t_4(s), \quad (4)$$

which immediately leads to the IAM formula  $t^{IAM}(s) = \frac{t_2^2(s)}{t_2(s) - t_4(s)}$ . The IAM formula satisfies exact elastic unitarity and, when reexpanded at low energies, reproduces the ChPT expansion up to the order used to approximate the subtraction constants and the left cut. Here we have presented an  $O(p^4)$  IAM but it can be generalized to higher chiral orders. Note that in the IAM derivation ChPT has been always used at low energies, to evaluate parts of a dispersion relation whose elastic unitarity cut has been taken into account exactly. Thus, there are no additional model dependencies in the approach, which is reliable up to energies where inelasticities become important. Taking the pole contribution into account leads to a modified IAM formula [6] which is almost indistinguishable from the ordinary one except in the Adler zero region, where it fixes some problems of the ordinary IAM with the Adler zero [6]. Actually, this modified IAM formula is the one used in this work since, as it will be shown below, one amplitude pole gets near the Adler zero region.

This simple IAM formula is able to reproduce  $\pi\pi$  scattering phase shift data up to roughly 1 GeV and generates the poles associated to the  $\rho$  and  $\sigma$  resonances with values of the LECs compatible with standard ChPT [5]. The  $1/N_c$  expansion is implemented in ChPT through the LECs, whose leading  $1/N_c$  scaling is known from QCD. Also, the quark mass dependence implemented in the IAM agrees with that of ChPT up to the order used. Hence, it is straightforward to study the leading  $1/N_c$  behavior and the  $\hat{m}$  dependence of the resonances generated with the IAM, which we proceed to expose in the following sections.

## NATURE OF RESONANCES FROM THEIR LEADING $1/N_c$ BEHAVIOR

The QCD  $1/N_c$  expansion [2] provides a clear definition of  $\bar{q}q$  bound states: their masses and widths scale as  $O(1)$  and  $O(1/N_c)$  respectively. The QCD leading  $1/N_c$  behavior of the ChPT parameters ( $f_\pi$ ,  $m_\pi$  and the LECs) is well known. Hence, by scaling with  $N_c$  the ChPT parameters in the IAM, the  $N_c$  dependence of the  $\rho$  and  $\sigma$  mesons mass and width has been determined [7, 8]. They are defined from the pole positions as  $\sqrt{s_{pole}} = M - i\Gamma$ . Note that we should not take too large  $N_c$  values, since the  $N_c \rightarrow \infty$  is a weakly interacting limit, where the IAM approach is less reliable [9]. Also, for very large  $N_c$  values, even a tiny admixture of  $\bar{q}q$  in the physical state would become dominant, but this does not give any information about the dominant component of the  $N_c = 3$  physical state.

Fig. 1 (left) shows the  $\rho$  and  $\sigma$  mass and width  $N_c$  scaling. It can be clearly seen that the  $\rho$  follows remarkably well the expected behavior of a  $\bar{q}q$  state, confirming that the method can obtain the correct  $N_c$  behavior of well known  $\bar{q}q$  states. In contrast, the  $\sigma$  does not that  $\bar{q}q$  pattern, allowing us to conclude that *the  $\sigma$  dominant component is not  $\bar{q}q$* .

Loop contributions play an important role in determining the  $\sigma$  pole position. Since they are  $1/N_c$  suppressed compared to tree level terms, it may happen that for larger  $N_c$  values they become comparable to tree level  $O(p^6)$  contributions, which are subdominant in the chiral expansion, but not  $N_c$  suppressed. Thus we checked the  $O(p^4)$  results with an  $O(p^6)$  IAM calculation [8]. We defined a  $\chi^2$ -like function to measure how close a resonance is from a

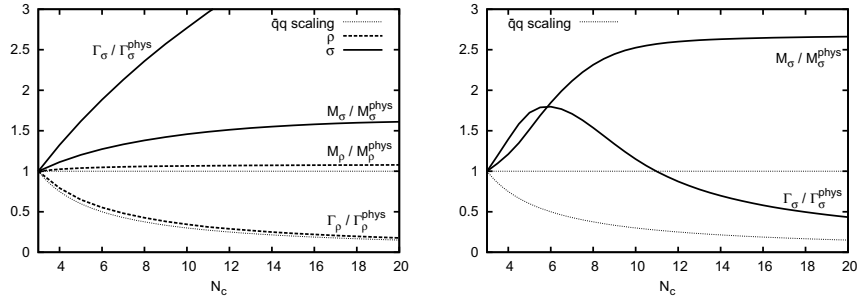


FIGURE 1. **Left:**  $\rho$  and  $\sigma$   $1/N_c$  scaling  $O(p^4)$ . **Right:**  $\sigma$   $1/N_c$  scaling  $O(p^6)$ .

$\bar{q}q$  behavior. First, we used it at  $O(p^4)$  to show that it is not possible for the  $\sigma$  to behave predominantly as a  $\bar{q}q$  state describing simultaneously the data and the  $\rho$   $\bar{q}q$  behavior. Next, we obtained an  $O(p^6)$  data fit where the  $\rho$   $\bar{q}q$  behavior was imposed. Figure 1 (right) shows the  $M_\sigma$  and  $\Gamma_\sigma$   $N_c$  scaling obtained from that fit. Note that both  $M_\sigma$  and  $\Gamma_\sigma$  grow near  $N_c = 3$ , confirming the  $O(p^4)$  result of a non  $\bar{q}q$  dominant component. However, for  $N_c$  between 8 and 15, where we still trust the IAM,  $M_\sigma$  becomes constant and  $\Gamma_\sigma$  starts decreasing. This may hint to a *subdominant  $\bar{q}q$  component*, arising as loops become suppressed as  $N_c$  grows. Finally, by forcing the  $\sigma$  to behave as a  $\bar{q}q$ , we found that in the best case this subdominant component could become dominant around  $N_c > 6 - 8$ , but always with an  $N_c \rightarrow \infty$  mass above 1 GeV instead of its physical  $\sim 450$  MeV value. This supports the emerging picture of two low energy scalar nonets, one of exotic nature below 1 GeV and another of ordinary  $\bar{q}q$  nature above 1 GeV.

## CHIRAL EXTRAPOLATION OF THE $\rho$ AND $\sigma$ RESONANCES

ChPT also provides an expansion of  $m_\pi$  in terms of  $\hat{m}$  (at leading order  $m_\pi^2 \sim \hat{m}$ ). Thus, by changing  $m_\pi$  in the amplitudes we see how the IAM poles depend on  $\hat{m}$ . We report here our analysis of the  $\rho$  and  $\sigma$  properties dependence on  $m_\pi$  [10].

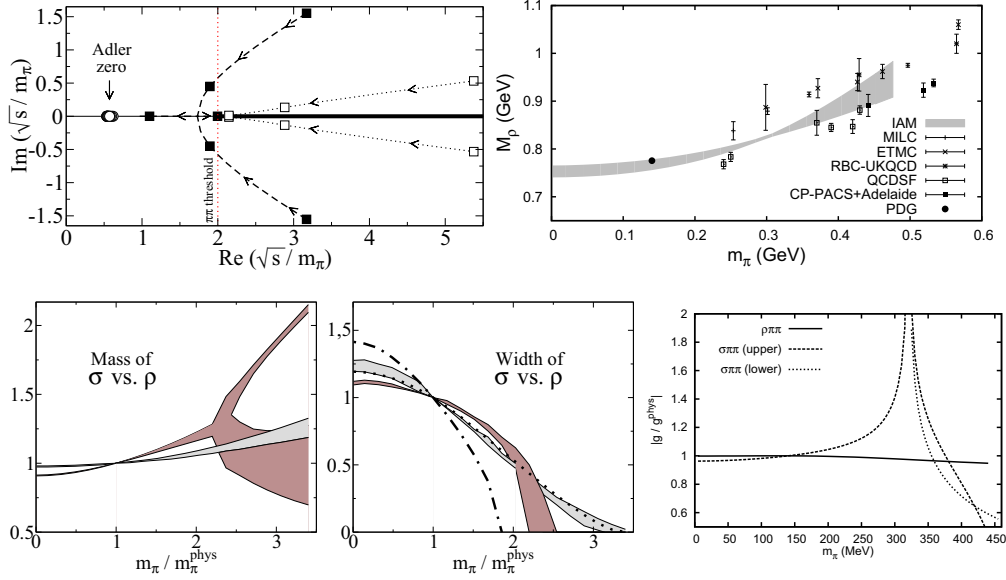
The values of  $m_\pi$  considered should fall within the ChPT applicability range and allow for some elastic regime below  $K\bar{K}$ , that would almost disappear if  $m_\pi > 500$ , which would be the most optimistic applicability range. We expect higher order corrections to be more relevant as  $m_\pi$  increases. Thus, our results become less reliable as  $m_\pi$  grows.

Fig. 2 (top left) shows the evolution of the  $\sigma$  and  $\rho$  pole positions as  $m_\pi$  is increased. In order to see the pole movements relative to the  $\pi\pi$  threshold, which is also increasing, we use units of  $m_\pi$ , so the threshold is fixed at  $\sqrt{s} = 2$ . Both poles move closer to threshold and they approach the real axis. The  $\rho$  poles reach the real axis at the same time that they cross threshold. One of them jumps into the first sheet and becomes a bound state, while its conjugate partner remains on the second sheet practically at the very same position as that in the first. In contrast, the  $\sigma$  poles go below threshold with a finite imaginary part before they meet in the real axis, still on the second sheet, becoming virtual states. As  $m_\pi$  increases, one pole moves toward threshold and jumps through the branch point to the first sheet staying in the real axis below threshold, very close to it as  $m_\pi$  keeps growing. The other  $\sigma$  pole moves down in energies away from threshold and remains on the second sheet. Similar movements were found within quark models [12] and a finite density analysis [13].

Fig. 2 (top right) shows our results for the  $\rho$  mass dependence on  $m_\pi$  compared with some lattice results [14], and the PDG value for the  $\rho$  mass. Now  $M_\rho$  is defined as the point where the phase shift crosses  $\pi/2$ , except for those  $m_\pi$  values where the  $\rho$  becomes a bound state, where it is defined from the pole position. In view of the incompatibilities between different lattice collaborations, we find a qualitative good agreement with lattice results. The  $M_\rho$  dependence on  $m_\pi$  agrees also with estimations for the two first coefficients of its chiral expansion [15].

In Fig. 2 (bottom left) we compare the  $m_\pi$  dependence of  $M_\rho$  and  $M_\sigma$ , normalized to their physical values. The bands cover the LECs uncertainties. Both masses grow with  $m_\pi$ , but  $M_\sigma$  grows faster than  $M_\rho$ . Above  $2.4m_\pi^{\text{phys}}$ , we show two bands since the two  $\sigma$  poles lie on the real axis with two different masses.

In the bottom center panel of Fig. 2 we compare the  $m_\pi$  dependence of  $\Gamma_\rho$  and  $\Gamma_\sigma$  normalized to their physical values: note that both widths become smaller. We compare this decrease with the expected phase space reduction as



**FIGURE 2.** **Top Left:** Movement of the  $\sigma$  (dashed lines) and  $\rho$  (dotted lines) poles for increasing  $m_\pi$  (direction indicated by the arrows) on the second sheet. The filled (open) boxes denote the pole positions for the  $\sigma$  ( $\rho$ ) at pion masses  $m_\pi = 1, 2,$  and  $3 \times m_\pi^{\text{phys}}$ , respectively. **Top Right:** Comparison the IAM  $M_\rho$  dependence on  $m_\pi$  with some recent lattice results[14]. **Bottom Left:** Comparison of the  $\rho$  (light) and  $\sigma$  (dark) mass dependence on  $m_\pi$ . **Bottom Center:** Comparison of the  $\rho$  (light) and  $\sigma$  (dark) width dependence on  $m_\pi$ . The dotted ( $\rho$ ) and dot-dashed ( $\sigma$ ) lines show the decrease due to only phase space assuming a constant coupling to  $\pi\pi$ . **Bottom Right:**  $\rho$  and  $\sigma$  couplings calculated from the pole residue. In all panels, the bands cover the LECs uncertainty.

resonances approach the  $\pi\pi$  threshold. We find that  $\Gamma_\rho$  follows very well this expected behavior, which implies that the  $\rho\pi\pi$  coupling is almost  $m_\pi$  independent. In contrast,  $\Gamma_\sigma$  deviates from the phase space reduction expectation. This suggests a strong  $m_\pi$  dependence of the  $\sigma$  coupling to two pions, which we confirm with an explicit calculation of the resonances couplings from the pole residues as shown in the bottom left panel.

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