Recurrence Relations for Hermite Exceptional Orthogonal Polynomials

ARTICLE · JUNE 2015

Source: arXiv

4 AUTHORS:


David Gomez-Ullate
Complutense University of Madrid


Alex Kasman
College of Charleston


Arno Kuijlaars
University of Leuven


Robert Milson
Dalhousie University

RECURRENCE RELATIONS FOR EXCEPTIONAL HERMITE POLYNOMIALS

D. GÓMEZ-ULLATE, A. KASMAN, A.B.J. KUIJLAARS, AND R. MILSON

Abstract. The bispectral anti-isomorphism is applied to differential operators involving elements of the stabilizer ring to produce explicit formulas for all difference operators having any of the Hermite exceptional orthogonal polynomials as eigenfunctions with eigenvalues that are polynomials in x.

1. Introduction

The classical orthogonal polynomial families of Hermite, Laguerre and Jacobi [21] have three important properties: they are eigenfunctions of a differential operator in x having eigenvalues dependent on n, they satisfy a well-known three-term recurrence relation, and they form a basis of a given L^2 space.

In the past few years, the class of polynomials having the above properties has been extended to include exceptional orthogonal polynomials, which differ from the classical families in that there are a finite number of degrees for which no polynomial eigenfunction exists. The differential equation for exceptional polynomials contains rational instead of polynomial coefficients. If certain regularity assumptions are imposed then, like their classical counterparts, the exceptional orthogonal polynomials form the basis of a weighted L^2 space.

This paper is mostly concerned with the study of recurrence relations for exceptional Hermite polynomials. Some particular examples of these polynomials were investigated as early as [4], but a final classification appeared only recently in [9]. See that paper for a more detailed overview of the subject and for additional references. Exceptional Hermite polynomials were also derived from exceptional Charlier polynomials by taking a suitable limit [6].

Exceptional polynomials admit two different sets of recurrence relations. The first set of recurrence relations [9,16] have coefficients that are polynomial functions of x and n and consequently have no obvious bispectral interpretation; i.e., they cannot be interpreted as higher-order Jacobi operators. We shall not investigate these relations in this paper. The second set of recurrence relations have order $2N + 3$ where N is the codimension (number of missing degrees) of the exceptional family. Except for one term, the coefficients do not depend on x and are rational functions of n. This type of relations will be the main focus of our study.

The existence of this second type of recurrence relations of order $4N + 1$ was first established in [19]. Relations of order $2N + 3$ order for exceptional families obtained through a one-step Darboux transformation were recently given in [14], and later generalized in [17], but in both cases an explicit construction of the recurrence coefficients was missing. Somewhat closer to this goal is the recent work of Durán [7], which provides explicit relations for some examples of exceptional Hermite and Laguerre polynomials. His approach uses a bispectral correspondence between discrete Krall polynomials and exceptional Charlier polynomials and a limit procedure to obtain the recurrence relations for the exceptional Hermite.

The main result of this note (Theorem 5.8) is a straight-forward procedure for producing the general recurrence relations satisfied by the exceptional polynomials. We focus on the Hermite subclass but it is clear that the same techniques can be applied to the other families as well. In particular, by making use of a bispectral anti-isomorphism of rings of operators acting on the classical Hermite polynomials, the production of recurrence relations for the exceptional Hermite polynomials is reduced to elementary algebra.

The idea of the bispectral anti-isomorphism for creating "bispectral Darboux transformations" was pioneered by Kasman-Rothstein [13], further developed by Bakalov-Horozov-Yakimov [2], and first used in the context of orthogonal polynomials by Grünbaum-Yakimov [11]. The construction below does not depend on those prior results but is an application of the same ideas to the new context of exceptional orthogonal polynomials.

2. Background

2.1. **Hermite Polynomials.** Set $h(n,x) = H_n(x)$, where the latter is the nth degree Hermite polynomial as defined in Section 18.3 of [18]. These polynomials have associated operators with the following properties.

Letting

$$\Theta(f(n)) = f(n+1)$$

denote the elementary shift operator, define the following degree-raising and lower operators,

$$\Delta = \frac{1}{2}\Theta + n \circ \Theta^{-1}, \tag{2.1}$$

$$\Gamma = 2n \circ \Theta^{-1}, \tag{2.2}$$

where the n denotes the indicated multiplication operator. The action of Δ and Γ on the classical Hermite polynomials can also be expressed as operators in x, namely

$$xh(n,x) = \Delta h(n,x) \tag{2.3}$$

$$\partial h(n,x) = \Gamma h(n,x) \tag{2.4}$$

where $\partial = \frac{\partial}{\partial x}$ differentiates with respect to x. Note that (2.3) is just the 3-term recurrence relation for the classical Hermite polynomials. We will also need the second-order operator

$$T(y(x)) = y''(x) - 2xy'(x)$$

and express the Hermite differential equation as

$$T(h(n,x)) = -2nh(n,x). \tag{2.5}$$

Remark 2.1. To read and understand the remainder of the paper, it is helpful to keep in mind that operators in x only always commute with operators acting only in n. In general, we will try to write operators in x with Roman symbols (except for ∂) and operators in n with Greek ones to help the reader remember which variable an operator acts in. However, the map $\flat$ to be introduced later turns operators in x into operators in n. Moreover, $\flat(R)$ and R have the same effect on $h(n,x)$ for any $R \in \mathbb{C}[x,\partial]$. So, when either one of these expressions appears next to $h(n,x)$ in a formula it is possible to replace it by the other.

2.2. **Exceptional Hermite Polynomials.** This subsection quickly reviews the definitions related to the Exceptional Hermite Polynomials $\hat{h}(n,x)$ associated to the choice of a partition λ. For proofs of the claims and more details, see [9].

Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_\ell$ be a non-decreasing sequence of positive integers so that $\lambda = \{\lambda_1,\dots,\lambda_\ell\}$ is a partition of the positive integer $N = \sum \lambda_i$ into ℓ parts[1]. It is convenient to associate to the choice of λ the set $K = \{k_1,\dots,k_\ell\}$, where

$$k_i = \lambda_i + i - 1, \quad i = 1,\dots,\ell. \tag{2.6}$$

Using Wr to denote the Wronskian determinant with respect to x, to the selection of partition λ (or sequence K) we associate the polynomial $\eta(x)$ defined by

$$\eta(x) = \mathrm{Wr}\left[h(k_1,x),\dots,h(k_\ell,x)\right], \tag{2.7}$$

and define the order ℓ differential operator A by its action on an arbitrary function $y(x)$,

$$A(y(x)) = \mathrm{Wr}\left[h(k_1,x),\dots,h(k_\ell,x),y(x)\right]. \tag{2.8}$$

Now, a new sequence of polynomials is defined by

$$\hat{h}(n,x) := A(h(n+\ell-N,x)), \tag{2.9}$$

where it is clear from (2.8) that $\hat{h}(n,x)$ is either a degree n polynomial in x or zero.

The degree sequence of the resulting non-zero polynomials is

$$\mathcal{I} = \{n \in \mathbb{N}_0 : n \geq N-\ell, n+\ell-N \notin K\}. \tag{2.10}$$

The degrees $0,1,\dots,N-\ell-1$ and the degrees $k_1+N-\ell,\dots,k_\ell+N-\ell$ are missing from $\mathcal{I}$, so that the polynomial sequence $\{\hat{h}(n,x) : n \in \mathcal{I}\}$ is missing a total of N degrees. Let us also denote by $\mathcal{U} \subset \mathbb{C}[x]$ the set of finite linear combinations of exceptional Hermite polynomials:

$$\mathcal{U} = \mathrm{span}\{\hat{h}(n,x) \colon n \in \mathcal{I}\} \tag{2.11}$$

The resulting families of polynomials generalize the classical Hermite family because they satisfy a Hermite-like differential equation

$$\hat{T}(\hat{h}(n,x)) = 2(N-n)\hat{h}(n,x), \qquad n \in \mathcal{I} \tag{2.12}$$

with

$$\hat{T}(y) := y'' - 2\left(x + \frac{\eta'}{\eta}\right)y' + \left(\frac{\eta''}{\eta} + 2x\frac{\eta'}{\eta}\right)y. \tag{2.13}$$

We will say that λ is an even partition if ℓ is even and if $\lambda_{2i-1} = \lambda_{2i}$ for every $i \leq \ell/2$. For an even partition λ, the exceptional Hermite polynomials $\hat{h}(n,x)$ satisfy the orthogonality relations[2]

$$\int_{\mathbb{R}} \hat{h}(n,x)\hat{h}(m,x)\hat{W}(x)dx = \delta_{m,n}\sqrt{\pi}\,2^{j+\ell}j!\prod_{i=1}^{\ell}(j-k_i), \qquad n,m \in \mathcal{I}$$

[1]Many of the objects in this paper depend on the selection of λ. However, to avoid cumbersome notation, we will consider λ to be selected and fixed and not specifically indicate the dependence of objects, such as $\hat{h}(n,x)$, $\mathcal{S}$, $\mathcal{U}$ and $\mathcal{I}$ on this choice.

[2]This formula corrects a misprint in equation (52) of [9].

where $j = n + \ell - N$, and the weight is defined as $\hat{W}(x) = e^{-x^2}/\eta(x)^2$. Moreover, if λ is an even partition then $\mathcal{U}$ is dense in $\mathrm{L}^2(\mathbb{R}, \hat{W}(x)dx)$.

Despite the importance of even partitions to the role of $\hat{h}(n,x)$ as orthogonal polynomials, the formulas for the recurrence relations that constitute the main result of this paper are valid for an arbitrary choice of λ so this distinction will not be further emphasized.

2.3. **Bispectrality.** The bispectral problem, as proposed by Grünbaum and initially studied by Duistermaat-Grünbaum [5] seeks to find functions depending on two variables that are joint eigenfunctions for operators in each of the two variables separately so that in each case the eigenvalue depends non-trivially on the other variable. Specifically, a *bispectral triple* (L, Λ, ψ) consists of a function $\psi(x,z)$, an operator L in x satisfying $L\psi = p(z)\psi$ and an operator Λ in z satisfying $\Lambda\psi = \pi(x)\psi$.

A great body of research by many authors has investigated bispectrality for various different kinds of operators, establishing theorems, classifying examples, finding connections to other areas of research such as soliton equations, or quantum and classical particle systems. (See, for instance, [12] and references therein.) Additionally, it has long been recognized that the classical orthogonal polynomials can be interpreted as exhibiting bispectrality if they are viewed as functions of two variables (one continuous and one discrete) rather than as a sequence of separate functions of x. The differential equation in x and the recurrence relation are then the two eigenvalue equations. (See, for example, [10, 20].)

3. The Anti-Isomorphism

The key observation leading to the main result is that the ring $\mathbb{C}[x,\partial]$ of polynomial coefficient differential operators and the ring $\mathbb{C}[\Delta,\Gamma]$ of difference operators generated by Δ and Γ are naturally anti-isomorphic.

Definition 3.1. Define the map $\flat$ by its action on monomials:

$$\begin{aligned} \flat : \mathbb{C}[x,\partial] &\to \mathbb{C}[\Delta,\Gamma] \\ x^i\partial^j &\mapsto \Gamma^j\Delta^i \end{aligned}$$

extended linearly to all of $\mathbb{C}[x,\partial]$.

Theorem 3.2. *The map* $\flat : \mathbb{C}[x,\partial] \to \mathbb{C}[\Delta,\Gamma]$ *satisfies*

$$Q(h(n,x)) = \flat(Q)h(n,x)\ \forall Q \in \mathbb{C}[x,\partial]$$

and

$$\flat(Q \circ R) = \flat(R) \circ \flat(Q)\ \forall Q, R \in \mathbb{C}[x,\partial].$$

In other words, $\flat$ *is an anti-isomorphism that has the property that an operator and its image always have exactly the same effect on the classical Hermite polynomials.*

Proof. Because $\flat$ extends linearly it is sufficient to check the first property on a monomial of the form $x^i\partial^j$. Then observe that because x and Γ commute one has

$$x^i\partial^j h(n,x) = x^i\Gamma^j h(n,x) = \Gamma^j x^i h(n,x) = \Gamma^j\Delta^i h(n,x) = \flat(x^i\partial^j)h(n,x).$$

As a consequence, we have for any Q and R in $\mathbb{C}[x,\partial]$ that

$$\flat(Q \circ R)h(n,x) = (Q \circ R)h(n,x).$$

But, note that

$$\flat(R)\circ\flat(Q)h(n,x)=\flat(R)(Qh)=Q(\flat(R)h(n,x))=Q(R(h(n,x)))=(Q\circ R)h(n,x)$$

also because Q commutes with $\flat(R)$. Then $\flat(Q\circ R)-\flat(R)\circ\flat(Q)$ is an operator in $\mathbb{C}[\Delta,\Gamma]$ which has the function $h(n,x)$ in its kernel for all n. This implies that it is the zero operator. □

4. Factorization Lemma and Eigenvalue Equations

Define another differential operator B of order ℓ by stating that for an arbitrary function $y(x)$,

$$B(y(x))=(-1)^{\lfloor \ell/2\rfloor}e^{x^2}\eta^{-\ell}\,\mathrm{Wr}\left[\tilde{h}_1(x),\dots,\tilde{h}_\ell(x),e^{-x^2}y(x)\right] \tag{4.1}$$

where

$$\tilde{h}_j(x):=\mathrm{Wr}\left[h(k_1,x),\dots\widehat{h(k_j,x)},\dots,h(k_\ell,x)\right] \tag{4.2}$$

with the hat decoration indicating omission from the sequence[3]. The main result of this section is Corollary 4.4 providing eigenvalue equations for the products $A\circ B$ and $B\circ A$. Unfortunately, the introduction of several technical definitions and lemmas is necessary to obtain it.

For $0\le j\le k$, let $\lambda^{(j)}$ denote the truncated partition $\lambda^{(j)}=(\lambda_1\le\dots\le\lambda_j)$ and $K^{(j)}=\{k_1,\dots,k_j\}$ the corresponding truncated set. Let

$$\eta_j(x)=\mathrm{Wr}\left[h(k_1,x),\dots,h(k_j,x)\right], \tag{4.3}$$

denote the corresponding Hermite Wronskians. For $1\le j\le\ell$, define the first-order differential operators

$$A_j(y):=\frac{\eta_j}{\eta_{j-1}}\left(y'-\frac{\eta_j'}{\eta_j}y\right)=\frac{\mathrm{Wr}\left[\eta_j,y\right]}{\eta_{j-1}}, \tag{4.4}$$

$$B_j(y):=\frac{\eta_{j-1}}{\eta_j}\left(y'-\left(2x+\frac{\eta_{j-1}'}{\eta_{j-1}}\right)y\right)=\frac{\mathrm{Wr}\left[e^{x^2}\eta_{j-1},y\right]}{e^{x^2}\eta_j}, \tag{4.5}$$

where we define $\eta_0=1$.

Proposition 4.1. *The above operators are linear factors of the order ℓ operators A and B defined in* (2.8) *and* (4.1)*. Indeed, the following factorizations hold:*

$$A=A_\ell\circ\dots\circ A_2\circ A_1 \tag{4.6}$$

$$B=B_1\circ B_2\circ\dots\circ B_\ell \tag{4.7}$$

Proof. The proof makes use of the following classical identities [15]. Throughout, the arguments are sufficiently differentiable functions of one variable.

$$\mathrm{Wr}\left[gf_1,gf_2,\dots,gf_n\right]=g^n\,\mathrm{Wr}\left[f_1,f_2,\dots,f_n\right], \tag{4.8}$$

$$\mathrm{Wr}\left[\mathrm{Wr}\left[f_1,\dots,f_n,g\right],\mathrm{Wr}\left[f_1,\dots,f_n,h\right]\right]=\mathrm{Wr}\left[f_1,\dots,f_n\right]\mathrm{Wr}\left[f_1,\dots,f_n,g,h\right], \tag{4.9}$$

$$\mathrm{Wr}\left[\tilde{f}_1,\dots,\tilde{f}_n\right]=(-1)^{\lfloor n/2\rfloor}\,\mathrm{Wr}\left[f_1,\dots,f_n\right]^{n-1}, \tag{4.10}$$

[3]This formula corrects a misprint in equation (27) of [9].

where

$$\tilde{f}_j = \mathrm{Wr}\left[f_1, \dots, \hat{f}_j, \dots, f_n\right].$$

Let us prove (4.6). Since A_j annihilates η_j and $A_{j-1} \circ \cdots \circ A_1 h(k_j, x) = \eta_j(x)$ (which can be shown using (4.3), (4.4) and (4.9)), the product $A_\ell \circ \cdots \circ A_1$ annihilates all $h(k_j, x)$, $j = 1, \dots, \ell$. By construction, the ℓth order operator A has the same property and so must be a multiple of the product. Equality is established by considering the coefficient of the highest order derivative $y^{(\ell)}(x)$.

We prove (4.7) by induction on ℓ. For $\ell = 1$ the equality of (4.1) and (4.5) with $j = 1$ follows from (4.8). Suppose that the identity (4.7) holds for $\ell - 1$. For $1 \le i < j \le \ell$, let $\tilde{h}_{i,j}$ denote the $\ell - 2$ order Wronskian like (4.2), but with $h(k_i, x)$ and $h(k_j, x)$ omitted. Since

$$\eta_{\ell-1} = \tilde{h}_\ell,$$

we have by (4.5), (4.8) and (4.9),

$$B_\ell(e^{x^2}\tilde{h}_j) = \frac{e^{x^2}}{\eta_\ell} \mathrm{Wr}\left[\tilde{h}_\ell, \tilde{h}_j\right] = -e^{x^2}\tilde{h}_{j,\ell}.$$

By the inductive hypothesis,

$$\begin{aligned}(B_1 \circ \cdots \circ B_{\ell-1}) &\left(e^{x^2}\tilde{h}_{j,\ell}\right) \\ &= (-1)^{[(\ell-1)/2]} e^{x^2} \eta_{\ell-1}^{-(\ell-1)} \mathrm{Wr}\left[\tilde{h}_{1,\ell}, \cdots, \tilde{h}_{\ell-1,\ell}, \tilde{h}_{j,\ell}\right] = 0, \qquad j = 1, \dots, \ell - 1,\end{aligned}$$

and hence

$$(B_1 \circ \cdots \circ B_{\ell-1} \circ B_\ell)\left(e^{x^2}\tilde{h}_j\right) = 0, \qquad j = 1, \dots, \ell.$$

The last identity also holds for $j = \ell$, since $B_\ell(e^{x^2}\tilde{h}_\ell) = 0$.

By (4.1) we also have

$$B\left(e^{x^2}\tilde{h}_j\right) = 0, \qquad j = 1, \dots, \ell.$$

Thus $B_1 \circ \cdots B_\ell$ and B are two ℓ-th order linear differential operators that agree for ℓ linearly independent functions and hence they agree up to a multiplicative left factor. Directly from the definition (4.5), we have

$$(B_1 \circ \cdots \circ B_\ell)(y) = \frac{1}{\eta_\ell} y^{(\ell)} + \text{ terms with lower order derivatives.}$$

By (4.1), (4.8) and (4.10), we have

$$\begin{aligned} B(y) &= (-1)^{\lfloor \ell/2 \rfloor} \frac{\mathrm{Wr}\left[e^{x^2}\tilde{h}_1, \dots, e^{x^2}\tilde{h}_\ell, y\right]}{\left(e^{x^2}\eta_\ell\right)^\ell} \\ &= (-1)^{\lfloor \ell/2 \rfloor} \frac{\mathrm{Wr}\left[\tilde{h}_1, \dots, \tilde{h}_\ell\right]}{(\eta_\ell)^\ell} y^{(\ell)} + \text{ terms with lower order derivatives} \\ &= \frac{1}{\eta_\ell} y^{(\ell)} + \text{ terms with lower order derivatives.} \end{aligned}$$

This establishes (4.7). □

Next, let

$$T_j(y) := y'' - 2\left(x + \frac{\eta_j'}{\eta_j}\right) y' + \left(\frac{\eta_j''}{\eta_j} + 2x\frac{\eta_j'}{\eta_j}\right) y,$$

denote the exceptional operators corresponding to the truncated partition $\lambda^{(j)}$. Thus $T_0 = T$ is the classical Hermite operator, while $T_\ell = \hat{T}$, see (2.13).

Proposition 4.2. *With A_j, B_j, T_j defined as above, the following intertwining relations hold:*

$$A_j \circ T_{j-1} = (T_j - 2) \circ A_j, \tag{4.11}$$

$$T_{j-1} \circ B_j = B_j \circ (T_j - 2). \tag{4.12}$$

Proof. For $j = 1, \ldots, \ell$, set $\psi_j(x) = e^{-x^2/2} h(k_j, x)$ and observe that $\mathrm{Wr}\,[\psi_1(x), \ldots, \psi_j(x)] = e^{-jx^2/2}\eta_j(x)$. Next, set $\mathcal{H}_j = -\partial_x^2 + U_j$, where

$$U_j = x^2 - 2\partial_x^2 \log \eta_j(x) + 2j.$$

The operator $\mathcal{H}_j$ is such that

$$\left(\frac{e^{-x^2/2}}{\eta_j}\right)^{-1} \circ \mathcal{H}_j \circ \left(\frac{e^{-x^2/2}}{\eta_j}\right) = -T_j + 2j + 3. \tag{4.13}$$

Since

$$(-\partial_x^2 + x^2)\psi_j = (2k_j + 1)\psi_j,$$

Crum's Theorem [3] implies that

$$\mathcal{H}_{j-1}\left(\frac{\mathrm{Wr}\,[\psi_1, \ldots, \psi_j]}{\mathrm{Wr}\,[\psi_1, \ldots, \psi_{j-1}]}\right) = (2k_j + 1)\frac{\mathrm{Wr}\,[\psi_1, \ldots, \psi_j]}{\mathrm{Wr}\,[\psi_1, \ldots, \psi_{j-1}]}.$$

Equivalently, in view of (4.3),

$$\mathcal{H}_{j-1}\left(\frac{e^{-x^2/2}\eta_j}{\eta_{j-1}}\right) = (2k_j + 1)\frac{e^{-x^2/2}\eta_j}{\eta_{j-1}}.$$

This gives the factorizations

$$\mathcal{H}_{j-1} = \left(-\partial_x + x - \frac{\eta_j'}{\eta_j} + \frac{\eta_{j-1}'}{\eta_{j-1}}\right) \circ \left(\partial_x + x - \frac{\eta_j'}{\eta_j} + \frac{\eta_{j-1}'}{\eta_{j-1}}\right) + 2k_j + 1,$$

$$\mathcal{H}_j = \left(\partial_x + x - \frac{\eta_j'}{\eta_j} + \frac{\eta_{j-1}'}{\eta_{j-1}}\right) \circ \left(-\partial_x + x - \frac{\eta_j'}{\eta_j} + \frac{\eta_{j-1}'}{\eta_{j-1}}\right) + 2k_j + 1.$$

Hence $\mathcal{H}_j$ is obtained from $\mathcal{H}_{j-1}$ by a Darboux transformation. Straightforward calculations show that

$$\begin{aligned} A_j &= \frac{\eta_j}{\eta_{j-1}}\left(\partial_x - \frac{\eta_j'}{\eta_j}\right) \\ &= \left(\frac{e^{-x^2/2}}{\eta_j}\right)^{-1} \circ \left(\partial_x + x - \frac{\eta_j'}{\eta_j} + \frac{\eta_{j-1}'}{\eta_{j-1}}\right) \circ \left(\frac{e^{-x^2/2}}{\eta_{j-1}}\right), \\ -B_j &= \frac{\eta_{j-1}}{\eta_j}\left(-\partial_x + 2x + \frac{\eta_{j-1}'}{\eta_{j-1}}\right) \\ &= \left(\frac{e^{-x^2/2}}{\eta_{j-1}}\right)^{-1} \circ \left(-\partial_x + x - \frac{\eta_j'}{\eta_j} + \frac{\eta_{j-1}'}{\eta_{j-1}}\right) \circ \left(\frac{e^{-x^2/2}}{\eta_j}\right). \end{aligned}$$

Thus, for $1 \leq j \leq \ell$,

$$T_{j-1} = B_j \circ A_j + 2(j - k_j - 1), \tag{4.14}$$

$$T_j = A_j \circ B_j + 2(j - k_j). \tag{4.15}$$

Multiplying (4.14) on the left and (4.15) on the right by A_j and equating the terms equal to $A_j \circ B_j \circ A_j$ yields the desired equalities. ☐

The operators $A \circ B$ and $B \circ A$ are polynomials in the operators $\hat{T}$ and T respectively:

Lemma 4.3. *The products of the operators A and B defined in* (2.8) *and* (4.1) *factor as:*

$$B \circ A = (T + 2k_1) \circ \cdots \circ (T + 2k_\ell), \tag{4.16}$$

$$A \circ B = (\hat{T} - 2\ell + 2k_1) \circ \cdots \circ (\hat{T} - 2\ell + 2k_\ell). \tag{4.17}$$

Proof. It follows from (4.14) that

$$B_1 \circ A_1 = T_0 - 2k_1 = T - 2k_1.$$

Now, suppose that for some $1 \leq n < \ell$ that

$$B_1 \circ B_2 \circ \cdots \circ B_n \circ A_n \circ A_{n-1} \circ \cdots \circ A_1 = (T + 2k_1) \circ (T + 2k_2) \circ \cdots \circ (T + 2k_n). \tag{4.18}$$

Consider the left-hand side of (4.18) with n replaced by $n+1$. Using (4.14) once to replace $B_{n+1} \circ A_{n+1}$ and following repeated use of (4.11) to move the term involving T_i all the way to the right we obtain

$$\begin{aligned}
&B_1 \circ \cdots \circ B_n \circ (B_{n+1} \circ A_{n+1}) \circ A_n \circ \cdots \circ A_1 \\
&= B_1 \circ \cdots \circ B_n \circ (T_n - 2(n - k_{n+1})) \circ A_n \circ \cdots \circ A_1 \\
&= B_1 \circ \cdots \circ B_n \circ A_n \circ (T_{n-1} - 2(n - k_{n+1} - 1)) \circ \cdots \circ A_1 \\
&\qquad \vdots \\
&= B_1 \circ \cdots \circ B_n \circ A_n \circ \cdots \circ A_1 \circ (T_0 + 2k_{n+1}) \\
&= (T + 2k_1) \circ (T + 2k_2) \circ \cdots \circ (T + 2k_{n+1})
\end{aligned}$$

where the last equality is obtained by making use of the assumption (4.18) and the fact that $T_0 = T$.

Then by induction we find that (4.18) is true when $n = \ell$. Since the left-hand side in that case is equal to $B \circ A$ by Proposition 4.1, this proves (4.16).

Factorization (4.17) can be similarly obtained by repeated use of (4.12) and (4.15). ☐

Combining Lemma 4.3 with the eigenvalue equations (2.5) and (2.12) leads immediately to:

Corollary 4.4. *The operators $A \circ B$ and $B \circ A$ satisfy the eigenvalue equations:*

$$(B \circ A)h(n, x) = \pi(n)h(n, x), \tag{4.19}$$

$$(A \circ B)\hat{h}(n, x) = \pi(n - N + \ell)\hat{h}(n, x). \tag{4.20}$$

where

$$\pi(n) = \prod_{i=1}^{\ell}(-2n + 2k_i). \tag{4.21}$$

5. The Stabilizer and Recurrence Relations

Definition 5.1. Let us define the stabilizer ring $\mathcal{S} \subset \mathbb{C}[x]$ by

$$\mathcal{S} = \left\{ f \in \mathbb{C}[x] \ : f(x)\hat{h}(n,x) \in \mathcal{U} \text{ for all } n \right\}.$$

Remark 5.2. By the orthogonality properties of the polynomials $\hat{h}(n,x)$ it is possible to show that if $f \in \mathcal{S}$, then for all $n \in \mathcal{I}$, necessarily

$$f(x)\hat{h}(n,x) = \sum_{j=-\deg f}^{\deg f} \gamma_{j,n} \hat{h}(n+j,x)$$

for some $\gamma_{j,n} \in \mathbb{C}$.

It was already observed in [19] that $\eta^2(x) \in \mathcal{S}$ and more recently in [7, 14, 17] that $\int^x \eta \in \mathcal{S}$ as well. The following proposition provides a characterization of $\mathcal{S}$.

Proposition 5.3. *If $f(x) \in \mathbb{C}[x]$ and $f'(x)$ is divisible by $\eta(x)$, then $f \in \mathcal{S}$. The converse is also true, provided $\eta(x)$ has only simple roots.*

Proof. We first show that $p \in \mathbb{C}[x]$ belongs to $\mathcal{U}$ if and only if

$$2\eta'(x)(xp(x) - p'(x)) + \eta''(x)p(x) \tag{5.1}$$

is divisible by η. Indeed, the operator (2.13) transforms every element $p \in \mathcal{U}$ into a polynomial, so the numerator of the singular part of $\hat{T}(p)$ must be divisible by η: this is precisely (5.1). Conversely, divisibility of (5.1) by η imposes precisely $\deg \eta = N$ independent linear conditions on $p \in \mathbb{C}[x]$. Since N is also the codimension of $\mathcal{U}$ it follows by dimensional exhaustion that a polynomial $p \in \mathbb{C}[x]$ that satisfies these conditions is necessarily an element of $\mathcal{U}$.

Now suppose that $f'(x)$ is divisible by $\eta(x)$. Let $p \in \mathcal{U}$ and set $q = fp$. Then,

$$\begin{aligned}&2\eta'(x)(xq(x) - q'(x)) + \eta''(x)q(x) = \\ &\qquad f(x)(2\eta'(x)(xp(x) - p'(x)) + \eta''(x)p(x)) - 2\eta'(x)p(x)f'(x)\end{aligned}$$

is divisible by η, and hence is also in $\mathcal{U}$.

Conversely, suppose that $\eta(x)$ has simple zeros, and that the above expression is divisible by $\eta(x)$ for all $p \in \mathcal{U}$. Also, $p(x)f'(x)$ is divisible by $\eta(x)$ for all $p \in \mathcal{U}$. By Proposition 5.5 in [9], the polynomials in $\mathcal{U}$ do not have a common root, which implies that $f'(x)$ must be divisible by $\eta(x)$. □

Note that as a consequence of Proposition 5.3 the ring $\mathcal{S}$ is not only non-empty but moreover guaranteed to have elements of every sufficiently high degree. Hence if $\eta(x)$ has simple zeros, then $\int^x \eta \in \mathcal{S}$ is the element in $\mathcal{S}$ of minimal degree.[4]

The elements of $\mathcal{S}$ can also be characterized as precisely those polynomials in $\mathbb{C}[x]$ that act as linear endomorphisms on $\mathcal{U}$. From [9] we have

Lemma 5.4. *The operators A, B defined in* (2.8) *and* (4.1) *act as linear transformations $A : \mathbb{C}[x] \to \mathcal{U}$ and $B : \mathcal{U} \to \mathbb{C}[x]$, respectively.*

An immediate consequence is the following.

[4]The minimal degree of an element of $\mathcal{S}$ and thus the minimal order of the recurrence relation for the exceptional Hermite polynomials was conjectured to be $2N+3$ in the previous works [7,14,17]. We see that this conjecture on the minimal order relies on Veselov's conjecture on the simplicity of the zeros of the Wronskian of Hermite polynomials [8].

Lemma 5.5. *For $f \in \mathcal{S}$, $B \circ f \circ A$ is a differential operator in $\mathbb{C}[x, \partial]$.*

Proof. By definition of $\mathcal{S}$ and the preceding lemma, applying $B \circ f \circ A$ to any polynomial results in a polynomial. Applying it to 1 then reveals that the order zero term is a polynomial. The claim then follows by induction since applying the operator to x^j reveals that the coefficient of ∂^j is a linear combination of powers of x multiplied by the lower order coefficients and the result, all of which are polynomial by assumption. □

The reason we need the previous lemma is so that we can apply the anti-isomorphism $\flat$ to $B \circ f \circ A$:

Definition 5.6. For $f \in \mathcal{S}$, let

$$\hat{\Delta}_f = \Theta^{\ell-N} \circ \flat(B \circ f \circ A) \circ \frac{1}{\pi(n)} \circ \Theta^{N-\ell} \tag{5.2}$$

Lemma 5.7. *The linear transformation $B : \mathcal{U} \to \mathbb{C}[x]$ is injective.*

Proof. By (4.20),

$$(A \circ B)\hat{h}(n,x) = 2^\ell \left(\prod_{i=1}^{\ell} (k_i + N - \ell - n) \right) \hat{h}(n,x).$$

By (2.10), if $n \in \mathcal{I}$ is one of the permitted degrees, then the right side is not zero. Hence, none of the basis elements of $\mathcal{U}$ can be in the kernel of B. □

Our main result is that the differential operator $\hat{\Delta}_f$ has $\hat{h}(n,x)$ as eigenfunction with eigenvalue f:

Theorem 5.8. *For all $f \in \mathcal{S}$, $\hat{\Delta}_f \hat{h}(n,x) = f(x)\hat{h}(n,x)$.*

Proof. Let $f \in \mathcal{S}$ and set

$$\tilde{\Delta}_f = \flat(B \circ f \circ A) \circ \frac{1}{\pi(n)}.$$

Hence, applying $\tilde{\Delta}_f \circ \pi(n)$ to $h(n,x)$ and invoking Corollary 4.4 gives

$$\begin{aligned}(\tilde{\Delta}_f \circ \pi(n))(h(n,x)) &= (\tilde{\Delta}_f \circ B \circ A)(h(n,x)) \\ &= (B \circ \tilde{\Delta}_f \circ A)(h(n,x)) \\ &= (B \circ \tilde{\Delta}_f)(\hat{h}(n+N-\ell,x)) \end{aligned} \tag{5.3}$$

Furthermore,

$$\begin{aligned}\flat(B \circ f \circ A)(h(n,x)) &= (B \circ f \circ A)(h(n,x)) \\ &= (B \circ f)(\hat{h}(n+N-\ell,x)) \end{aligned} \tag{5.4}$$

Since $\tilde{\Delta}_f \circ \pi(n) = \flat(B \circ f \circ A)$, the expressions (5.3) and (5.4) are equal and so the operator B annihilates the polynomial

$$\tilde{\Delta}_f(\hat{h}(n+N-\ell,x)) - f(x)\hat{h}(n+N-\ell,x).$$

For each fixed n, the first term is a linear combination of polynomials in $\mathcal{U}$ and the second term is in $\mathcal{U}$, by assumption. Since no nonzero elements of $\mathcal{U}$ are in the kernel of B (see Lemma 5.7), this must be the zero polynomial. Finally, note that

$$\begin{aligned}\hat{\Delta}_f \hat{h}(n,x) &= (\Theta^{\ell-N} \circ \tilde{\Delta}_f \circ \Theta^{N-\ell})\hat{h}(n,x) = \Theta^{\ell-N}(\tilde{\Delta}_f \hat{h}(n+N-\ell,x)) \\ &= \Theta^{\ell-N}(f(x)\hat{h}(n+N-\ell,x)) = f(x)\hat{h}(n,x),\end{aligned}$$

as claimed in Theorem 5.8. □

6. Examples

In this section we illustrate Theorem 5.8 by exhibiting a closed form description of the recurrence relation including some specific examples as well as some very general ones.

6.1. **One-step examples.** Let us first consider the case of $\ell = 1$ and write $K = \{k\}$ where $k \geq 1$ is an integer. By Proposition 5.3 and the fact that $H'_{k+1} = 2(k+1)H_k$, we have that $h(k+1,x) \in \mathcal{S}$, which leads us to the following identity.

Proposition 6.1. *Fix $k \geq 1$ and write*

$$\hat{h}(n,x) = \mathrm{Wr}\left[h(k,x), h(n-k+1,x)\right], \quad n \geq k-1,\ n \neq 2k-1$$

for the corresponding 1-step exceptional Hermite polynomial of degree n. Then,

$$\begin{aligned}(6.1)\quad & h(k+1,x)\hat{h}(n,x) \\ &= (n-2k+1)\sum_{j=0}^{k+1} 2^j \binom{k+1}{j} (n+3-k-j)_{j-1}\hat{h}(n+k+1-2j,x)\end{aligned}$$

where

$$(X)_n = X(X+1)\cdots(X+n-1)$$

denotes the usual Pochhammer symbol.

The proof of Proposition 6.1 requires the following two lemmas.

Lemma 6.2. *Setting $f(x) = h(k+1,x)$ and*

$$\begin{aligned}B(y(x)) &= (y' - 2xy)/h(k,x) \\ A(y(x)) &= \mathrm{Wr}\left[h(k,x), y(x)\right]\end{aligned}$$

as per the definitions above, we have

$$(B \circ f \circ A)[y] = h(k+1,x)y'' - h(k+2,x)y' + 2k(h(k+1,x) - 2(k+1)h(k-1,x))y$$

Proof. A direct calculation suffices to establish this. □

Lemma 6.3. *For integers $k, n \geq 0$ we have*

$$h(k,\Delta) = \sum_{j=0}^{k} 2^j \binom{k}{j} (n-j+1)_j \circ \Theta^{k-2j},$$

where $(n-j+1)_j$ is the multiplication operator by the indicated polynomial in n.

Proof. This is just the restatement of the linearization formula for the product of Hermite polynomials [1, p. 42] □

Proof of Proposition 6.1. By the first lemma,

$$\flat(B\circ f\circ A)=\Gamma^2\circ h(k+1,\Delta)-\Gamma\circ h(k+2,\Delta)+2k(h(k+1,\Delta)-2(k+1)h(k-1,\Delta))$$

Using the second lemma and elementary algebraic manipulations gives

$$\flat(B\circ f\circ A)=(k-n)\sum_{j=0}^{k+1}2^{j+1}(n-2j+1)\binom{k+1}{j}(n-j+2)_{j-1}\Theta^{k+1-2j}$$

Now

$$\pi(n)=2(k-n).$$

Applying the Definition 5.6 amounts to division of the preceding expression by $-2(n-2j+1)$ and gives the desired formula. □

6.2. **Multi-step examples.** Next, we rederive the Miki-Tsujimoto-Durán relation [7,14]. In this case, $\lambda=(1,1)$ and $K=\{1,2\}$, and

$$\pi(n)=4(n-1)(n-2). \tag{6.2}$$

We have

$$\eta(x)=\mathrm{Wr}\,[h(1,x),h(2,x)]=4(1+2x^2) \tag{6.3}$$

$$\hat h(n,x)=\mathrm{Wr}\,[h(1,x),h(2,x),h(n,x)]\,. \tag{6.4}$$

In order to avoid the use of fractions, we take

$$f(x)=\frac{3}{4}\int_0^x\eta(s)ds=x(3+2x^2)\in\mathcal{S}. \tag{6.5}$$

Applying (2.8) and (4.1) gives

$$A(y)=4((1+2x^2)y''-4xy'+4y)$$
$$B(y)=\frac{1}{4(1+2x^2)}\,y''-\frac{2x(1+x^2)}{(1+2x^2)^2}\,y'+\frac{1}{2}\,y.$$

Further calculations using this and (6.5) then show that

$$\begin{aligned}(B\circ f\circ A)(y)&=x(3+2x^2)\,y''''+(6-8x^4)\,y'''\\&\quad-x(6+8x^2-8x^4)\,y''+x^2(24-16x^2)\,y'-x(24-16x^2)\,y\end{aligned} \tag{6.6}$$

which has indeed only polynomial coefficients, as claimed in Lemma 5.5.

Then applying $\flat$ according to our Definition 3.1, we find

$$\begin{aligned}\flat(B\circ f\circ A)&=\Gamma^4\Delta(3+2\Delta^2)+\Gamma^3(6-8\Delta^4)\\&\quad-\Gamma^2\Delta(6\Delta+8\Delta^2-8\Delta^4)+8\Gamma\Delta^2(24-16\Delta^2)-\Delta(24-16\Delta^2).\end{aligned}$$

We rewrite this in terms of powers of the shift operator Θ by means of the identities (2.1)–(2.2). This is a lengthy calculation (which is easier to do with the help of symbolic software) and it results in

(6.7)
$$\flat(B\circ f\circ A)=(n-2)_2\Theta^3+6(n-2)_3\Theta+12(n-3)_4\Theta^{-1}+8(n-5)_2(n-2)_3\Theta^{-3}.$$

Note that we use Pochhammer symbols.

By Definition 5.6 and (6.2) we have (since $N = \ell = 2$ in this example)

$$4\hat{\Delta}_f = \flat(B \circ f \circ A) \circ \frac{1}{(n-2)_2} \tag{6.8}$$

$$= \frac{(n-2)_2}{(n+1)_2}\Theta^3 + 6(n-2)\Theta + 12(n-1)_2\Theta^{-1} + 8(n-2)_3\Theta^{-3} \tag{6.9}$$

where we used $\Theta^k \circ a(n) = a(n+k)\,\Theta^k$ and simplifications of the Pochhammer symbols. We therefore recover the desired recurrence relation

$$\begin{aligned} 4x(3+2x^2)\hat{h}(n,x) &= 4\hat{\Delta}_f\hat{h}(n,x) \\ &= \frac{(n-2)_2}{(n+1)_2}\hat{h}(n+3,x) + 6(n-2)\hat{h}(n+1,x) \\ &\qquad + 12(n-1)_2\hat{h}(n-1,x) + 8(n-2)_3\hat{h}(n-3,x). \end{aligned} \tag{6.10}$$

The above procedure is entirely algorithmic and can be easily implemented using a computer algebra system. Here, for example, is the 2-step relation corresponding to $\lambda = (2,2)$. Let

$$\hat{h}(n,x) = \mathrm{Wr}\,[h(2,x), h(3,x), h(n-2,x)]$$

be the corresponding exceptional Hermite polynomial of degree n. Then

$$\begin{aligned} \left(24x + \frac{32}{5}x^5\right)\hat{h}(n,x) &= \frac{1}{5}\frac{(n-5)_2}{(n)_2}\hat{h}(n+5,x) + 2\frac{(n-5)_2}{n-1}\hat{h}(n+3,x) \\ &+ 8(n-5)(n-3)\hat{h}(n+1,x) + 16(n-2)^2(n-4)\hat{h}(n-1,x) \\ &+ 16(n-5)_4\hat{h}(n-3,x) + \frac{32}{5}(n-6)_5\hat{h}(n-5,x) \end{aligned}$$

Here is a 4-step example with $\lambda = (1,1,2,2)$. Let

$$\hat{h}(n,x) = \mathrm{Wr}\,[h(1,x), h(2,x), h(4,x), h(5,x), h(n-2,x)]$$

be the corresponding degree n exceptional polynomial. Then,

$$\begin{aligned} &(105x + 70x^3 + 84x^5 + 24x^7)\hat{h}(n,x) = \\ &\quad \frac{3}{16}\frac{(n-7)_2(n-4)_2}{(n)_2(n+3)_2}\hat{h}(n+7,x) + \frac{21}{8}\frac{(n-7)_2(n-4)_2}{(n-1)(n+1)_2}\hat{h}(n+5,x) \\ &\quad + \frac{7}{4}\frac{(n-7)_2(80-57n+9n^2)}{(n-1)_2}\hat{h}(n+3,x) + \frac{105}{2}(n-7)_2(n-4)\hat{h}(n+1,x) \\ &\quad + 105(n-7)_2(n-3)_2\hat{h}(n-1,x) + 14(n-4)_3(332-111n+9n^2)\hat{h}(n-3,x) \\ &\quad + 84(n-7)_6\hat{h}(n-5,x) + 24(n-8)_7\hat{h}(n-7,x) \end{aligned}$$

7. Closing Remarks

Connections have already been recognized between the well-established theory of bispectrality and the more recent subject of exceptional orthogonal polynomials. However, as this note demonstrates, these connections have not been fully utilized. Although progress had been made recently on the question of recurrence relations for the exceptional orthogonal polynomials, their existence and precise form remained a difficult problem. The use of the bispectral anti-isomorphism reduces the problem to very simple (nearly trivial) algebraic manipulation.

Certainly, there are more opportunities for a fruitful exchange of ideas between these two closely related fields of research that have not yet been realized. In future publications, we hope to address the structure of $\{\hat{\Delta}_f : f \in \mathcal{S}\}$ as a commutative ring of difference operators and its corresponding spectral curve (cf. [22]) as well as the implications of the ad-nilpotency of its elements (cf. [5]).

Acknowledgements

The authors are grateful to the organizers of the conference NEEDS 2015 in Sardinia, Italy without which we might never have recognized the potential application of these tools to this problem. Likewise, the authors are grateful to Yves Grandati and Satoshi Tsujimoto for helpful conversations and their presentations on related topics at that conference.

D.G.U. has been supported in part by Spanish MINECO Grants MTM2012-31714 and FIS2012-38949-C03-01 and by the ICMAT-Severo Ochoa grant SEV-2011-0087. A.B.J.K. is supported by KU Leuven Research Grant OT/12/073, the Belgian Interuniversity Attraction Pole P07/18, and FWO Flanders projects G.0641.11 and G.0934.13. R.M. is supported by NSERC grant RGPIN-228057-2004.

References

[1] R. Askey. *Orthogonal Polynomials and Special Functions*, volume 21 of *Regional Conference Series in Applied Mathematics*. SIAM, 1975.

[2] B. Bakalov, E. Horozov, and M. Yakimov. General methods for constructing bispectral operators. *Physics Letters A*, 222(1-2):59–66, 1996.

[3] M.M. Crum, Associated Sturm-Liouville systems, *Quart. J. Math. Oxford Ser. (2)* **6** (1955), 121–127.

[4] S. Yu. Dubov, V. M. Eleonskii, and N. E. Kulagin. Equidistant spectra of anharmonic oscillators. *Chaos*, 4(1):47–53, 1994.

[5] J. J. Duistermaat and F. A. Grünbaum. Differential equations in the spectral parameter. *Communications in Mathematical Physics*, 103:177–240, 1986.

[6] A. J. Durán. Exceptional Charlier and Hermite orthogonal polynomials, *Journal of Approximation Theory*, 182:29–58, 2014.

[7] A. J. Durán. Higher order recurrence relation for exceptional Charlier, Meixner, Hermite and Laguerre orthogonal polynomials. *Integral Transforms and Special Functions*, 26(5):357–376, 2015.

[8] G. Felder, A.D. Hemery, and A.P. Veselov. Zeros of Wronskians of Hermite polynomials and Young diagrams, *Physica D*, 241(23-24):2131–2137, 2012.

[9] D. Gómez-Ullate, Y. Grandati, and R. Milson. Rational extensions of the quantum harmonic oscillator and exceptional Hermite polynomials. *Journal of Physics A: Mathematical and Theoretical*, 47(1):015203, 2014.

[10] F. A. Grünbaum, L. Haine, and E. Horozov. Some functions that generalize the Krall-Laguerre polynomials. *Journal of Computational and Applied Mathematics*, 106(2):271–297, 1999.

[11] F. A. Grünbaum and M. Yakimov. Discrete bispectral Darboux transformations from Jacobi operators. *Pacific Journal of Mathematics*, 204(2):395–431, 2002.

[12] J. Harnad and A. Kasman (eds.). *The bispectral problem*, volume 14 of *CRM Proceedings & Lecture Notes*. American Mathematical Society, 1998.

[13] A. Kasman and M. Rothstein. Bispectral Darboux transformations: The generalized Airy case. *Physica D: Nonlinear Phenomena*, 102(3-4):159–176, 1997.

[14] H. Miki and S. Tsujimoto. A new recurrence formula for generic exceptional orthogonal polynomials. *Journal of Mathematical Physics*, 56:033502, 2015.

[15] T. Muir, *A treatise on the theory of determinants, Rev. and enlarged by W. H. Metzler*. Dover, 1960.

[16] S. Odake. Recurrence relations of the multi-indexed orthogonal polynomials. *Journal of Mathematical Physics*, 54(8):083506, 2013.

[17] S. Odake. Recurrence relations of the multi-indexed orthogonal polynomials. II. *Journal of Mathematical Physics*, 56(5):053506, 2015.
[18] F. W. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark. *NIST Handbook of Mathematical Functions*. Cambridge University Press, 2010.
[19] R. Sasaki, S. Tsujimoto, and A. Zhedanov. Exceptional Laguerre and Jacobi polynomials and the corresponding potentials through Darboux-Crum transformations. *Journal of Physics A: Mathematical and Theoretical*, 43(31):315204, 2010.
[20] V. Spiridonov, L. Vinet, and A. Zhedanov. Bispectrality and Darboux transformations in the theory of orthogonal polynomials. *The Bispectral Problem, CRM. Proc. and Lecture Notes*, 14:111–122, 1998.
[21] G. Szegő. *Orthogonal Polynomials*, volume 23 of *Colloquium Publications*. American Mathematical Society, 1939.
[22] G. Wilson. Bispectral commutative ordinary differential operators. *Journal für die reine und angewandte Mathematik*, 442:177–204, 1993.

Departamento de Física Teórica II, Universidad Complutense de Madrid, 28040 Madrid, Spain.

Instituto de Ciencias Matemáticas (CSIC-UAM-UC3M-UCM), C/ Nicolas Cabrera 15, 28049 Madrid, Spain.

Department of Mathematics, College of Charleston, Charleston SC, USA

Department of Mathematics, KU Leuven, Belgium

Department of Mathematics, Dalhousie University, Halifax, Nova Scotia, Canada