Conformal dilaton gravity: Classical noninvariance gives rise to quantum invariance

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When quantizing conformal dilaton gravity, there is a conformal anomaly which starts at two-loop order. This anomaly stems from evanescent operators on the divergent parts of the effective action. The general form of the finite counterterm, which is necessary in order to insure cancellation of the Weyl anomaly to every order in perturbation theory, has been determined using only conformal invariance. Those finite counterterms do not have any inverse power of any mass scale in front of them (precisely because of conformal invariance), and then they are not negligible in the low-energy deep infrared limit. The general form of the ensuing modifications to the scalar field equation of motion has been determined, and some physical consequences have been extracted.

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I. INTRODUCTION

It is well known [1] that an *anomalous* term in a Ward identity only qualifies as a true anomaly when there is no local counterterm that can be added to the action in such a way that it cancels the putative anomalous piece of the identity. Put it in another way, there is a consistency condition, the Wess-Zumino consistency condition [2], which is a reflection on the gauge algebra acting on the effective action. If an anomalous variation of the action under a gauge transformation with parameters Λ^a appears,

$$\frac{\delta W}{\delta \Lambda_a(x)} \equiv G^a(x). \tag{1}$$

(Although this formalism has been developed with non-Abelian gauge anomalies in mind, it can easily be adapted to conformal anomalies as well [3]). Then, the consistency conditions read

$$\frac{\delta G^a(x)}{\delta \Lambda_b(y)} - \frac{\delta G^a(y)}{\delta \Lambda_b(x)} = f_{abc}\delta(x-y)G_c(x). \tag{2}$$

True anomalies are then solutions of the consistency equations which are not themselves variations, that is, that there is no local Lagrangian such that

$$G_a(x) = \frac{\delta C}{\delta \Lambda_a}. \tag{3}$$

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Defining the contraction of the anomaly with the ghosts

$$G^1 \equiv c^a G_a, \tag{4}$$

(the superindex as a reminder of the ghost number), the consistency relationships can be written in a sophisticated way as

$$sG^1 = d\alpha^2. \tag{5}$$

The appearance of a total differential on the second member is due to the fact that it is only necessary for it to vanish when integrated. The demand that the anomaly is not trivial reads in Becchi, Rouet, Stora and Tyutin (BRST) language

$$G^1 \neq sG^0 + d\beta^0. \tag{6}$$

In a recent paper [4] (the notation of which will be followed here), we have examined an apparently quasi-trivial theory, namely, what we have dubbed *conformal dilaton gravity* (CDG). In it, the Weyl parameter is upgraded to a Stückelberg field to compensate the Weyl transformation of the Einstein-Hilbert Lagrangian.

What we have found rests on the mild assumption that what was true at one loop (namely, that the on-shell counterterm is the Weyl transform of the Einstein-Hilbert counterterm) remains true to two loops, so that the two-loop counterterm will also be the Weyl transform of the Goroff-Sagnotti Goroff one. With this assumption, we found a two-loop Weyl anomaly in our *trivial* theory. This anomaly is, however, trivial in the sense that it can be eliminated by a counterterm, which is not strictly local, involving logarithms of the physical fields.

This is a very surprising result. It means that Weyl transformations are much less trivial than previously thought. It questions, in particular, the full equivalence of the Einstein frame and Jordan frame.

Of course it is possible to take the less rigid point of view that the counterterms are admissible in spite of them being nonlocal. Once this is taken for granted, then one quickly realizes that these counterterms are not suppressed by any mass scale (precisely because of conformal invariance). They could then legitimately also be included in the *classical* theory. Precisely our aim in the present paper is to follow the consequences of this viewpoint and speculate on some of the properties of a hypothetical conformal theory of gravity to all orders in perturbation theory.

Conformal invariance for us is *exactly* the same as Weyl invariance under

$$\begin{aligned} \tilde{g}_{\mu\nu} &\equiv \Omega^2(x) g_{\mu\nu}(x), \\ \tilde{\psi} &= \Omega^{-\lambda_\psi}\psi, \end{aligned} \tag{7}$$

where λ_ψ is by definition the conformal weight of the matter field ψ. To be specific, we will be interested in the four-dimensional action of CDG, that is,

$$S_{CDG} = -\int d^n x \sqrt{|g|}\left(-\frac{1}{12}\phi^2 R - \frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi + \frac{g}{4!}\phi^4\right), \tag{8}$$

which is classically Weyl invariant in a somewhat tautological way, provided the conformal weight of the field ϕ is $\lambda_\phi = 1$. This graviscalar field ϕ is none other than the Weyl parameter promoted to the range of a physical field. This is sometimes called a *Stückelberg field* or else a *compensating field*. For a mathematician, this is simply a *group averaging* of sorts.

Conformal physics is not very intuitive in that it does not single out any scale. For example, we all are used to the idea that quantum gravity effects should decouple at energies much smaller than Planck mass, $M_p \equiv \frac{1}{\sqrt{16\pi G}}$, so that they can be safely ignored in particle physics except in exotic circumstances.

This fails to be true in a conformal theory, as has been explicitly shown in Ref. [4] (essentially because there is no preferred scale in them). Conformal theories are also bizarre in other aspects [5,6], notably in the absence of the usual concept of a *particle*.

The existence of a conformal invariant fundamental theory including the gravitational interaction is, however, one of the holy grails of theoretical physics. Such a hypothetical theory would be extremely interesting, were it only as a theoretical model [7]. We are still in the initial steps of this quest here.

When quantizing the theory in the background field gauge, as has been done in Ref. [4], there is a four-dimensional Ward identity stemming from conformal invariance, namely,

$$\begin{aligned} \mathcal{D}W[\bar{g},\bar{\phi}] &\equiv \frac{\delta}{\delta\Omega(x)}W[\bar{g},\bar{\phi}]\Big|_{\Omega=1} \\ &= \left(-2\bar{g}_{\mu\nu}\frac{\delta}{\delta\bar{g}_{\mu\nu}} + \bar{\phi}\frac{\delta}{\delta\bar{\phi}}\right)W[\bar{g},\bar{\phi}] = 0, \end{aligned} \tag{9}$$

where the on-shell free energy $W[\bar{g},\bar{\phi}]$ is the logarithm of the on-shell partition function given by the functional path integral

$$W[\bar{g},\bar{\phi}] \equiv -\log Z[\bar{g},\bar{\phi}]. \tag{10}$$

This is precisely the Ward identity we claim to be anomalous to two-loop order.

II. IMPLEMENTATION OF THE WARD IDENTITY IN PERTURBATION THEORY THROUGH COUNTERTERMS

It was shown in Ref. [4] that modulo wave-function renormalizations—-which are physically irrelevant—the one-loop UV divergence of conformal dilaton gravity is the appropriate Weyl transform of the corresponding one-loop UV divergence of general relativity found by Ref. [8], and thus it vanishes. The appropriate Weyl transformation in question reads

$$\bar{g}_{\mu\nu} \to \frac{1}{12M_p^2}\bar{\phi}^2\bar{g}_{\mu\nu}. \tag{11}$$

Furnished with this result and the fact that the classical action of conformal dilaton gravity is obtained from that of general relativity by applying the previous Weyl transformation, one is led to assume that modulo wave-function renormalizations the two-loop UV divergence of conformal dilaton gravity can be obtained from the two-loop counterterm [9] of general relativity by applying to the latter the Weyl transformation we have just mentioned. Hence, the resulting two-loop counterterm—-see Ref. [4], for further details—reads

$$W_\infty^{L=2}[\bar{\phi},\bar{g}] = \frac{1}{n-4}\frac{12}{(4\pi)^4}\frac{209}{2880}\int \sqrt{|g|}d^n x\bar{\phi}^{-2}W_6^{(n)}[\bar{g}], \tag{12}$$

in the n-dimensional space of dimensional regularization. $W_6^{(n)}[\bar{g}]$, which has conformal weight $\lambda = 6$, is defined in terms of the Weyl tensor $W^{(n)\alpha_1\alpha_2\alpha_2\alpha_4}$ in n dimensions as follows:

$$W_6^{(4)}[\bar{g}] = W^{(n)\alpha_1\alpha_2\alpha_2\alpha_4} W^{(n)}{}_{\alpha_3\alpha_4\alpha_5\alpha_6} W^{(n)\alpha_5\alpha_6}{}_{\alpha_1\alpha_2}. \tag{13}$$

The previous results make it natural to speculate that the L-loop divergence in conformal dilaton gravity will be of the form

$$W_\infty^L[\bar{\phi},\bar{g}] = \frac{1}{n-4}\int \sqrt{|g|} d^n x \bar{\phi}^{-2L+2} \sum_j g_j P_{L+1}^j[\bar{g}]. \tag{14}$$

The constants g_j are unknown but calculable coefficients, and $P^j_{(L+1)} j = 1 \ldots N$ stand for the set of purely gravitational terms with conformal weight $\lambda = 2L + 2$ (like the trace of the product of $L + 1$ Weyl tensors). These terms have mass dimension $2L + 2$, so that the full integrand is dimensionless. An example is the scalar made out of $L + 1$ Weyl tensors. The complete set of conformal tensors is not explicitly known, but this fact is not essential in our argument.

Let us recall that $\mathcal{D}$ be given by

$$\mathcal{D} = -2\bar{g}_{\mu\nu}\frac{\delta}{\delta \bar{g}_{\mu\nu}} + \bar{\phi}\frac{\delta}{\delta\bar{\phi}}. \tag{15}$$

Then, in the n-dimensional space of dimensional regularization,

$$\mathcal{D}\left(d(\mathrm{vol})\bar{\phi}^{-2L+2}\sum_j g_j P_{L+1}^j\right) = -(n-4)\left(d(\mathrm{vol})\bar{\phi}^{-2L+2}\sum_j g_j P_{L+1}^j[\bar{g}]\right). \tag{16}$$

This means that the integrand of $W_\infty^L[\bar{\phi},\bar{g}]$ is an evanescent operator which yields a putative anomaly,

$$\mathcal{A}^L[\bar{\phi},\bar{g}] \equiv -\int d(\mathrm{vol})\bar{\phi}^{-2L+2}\sum g_j P_{L+1}^j[\bar{g}]. \tag{17}$$

This anomaly-to-be can actually be cancelled by a *finite* counterterm,

$$\Delta W^L[\bar{\phi},\bar{g}] = \int d(\mathrm{vol})\bar{\phi}^{-2L+2}\log\bar{\phi}\sum g_j P_{L+1}^j[\bar{g}], \tag{18}$$

(just because $\mathcal{D}\log\bar{\phi} = 1$). It is to be remarked that the integrand of the above expression is again dimensionless, so that there is no room for any dimensionful coupling constant in front.

The full modified action of CDG will be of the form

$$W_{CDG}[\bar{\phi},\bar{g}] = S_{CDG}[\bar{\phi},\bar{g}] + \sum_{L=1}^{\infty}(W_R^L[\bar{\phi},\bar{g}] + \Delta W^L[\bar{\phi},\bar{g}]). \tag{19}$$

This modified action obeys

$$\mathcal{D}W_{CDG}[\bar{\phi},\bar{g}] = 0, \tag{20}$$

so it qualifies as a conformally invariant one.

The finite part of this action (which we propose to reconsider as a classical action of sorts) reads

$$W_{CDG}^{\mathrm{class}}[\bar{\phi},\bar{g}] \equiv -\int d(\mathrm{vol})\Bigg[-\frac{1}{12}\bar{\phi}^2\bar{R} - \frac{1}{2}\nabla_\mu\bar{\phi}\nabla^\mu\bar{\phi} + \lambda\phi^4 + \sum_{L=1}^{\infty}\phi^{-(2L-2)}\log\bar{\phi}\sum_j g_j P_{L+1}^j[\bar{g}_{\mu\nu}]\Bigg] \tag{21}$$

and does obey instead

$$\mathcal{D}W_{CDG}^{\mathrm{class}}[\bar{\phi},\bar{g}] = \sum_L \mathcal{A}^L[\bar{\phi},\bar{g}]. \tag{22}$$

III. CLASSICAL EFFECTS OF THE FINITE QUANTUM COUNTERTERMS

The counterterms needed to cancel the putative anomalies enjoy two main properties. First of all, they are finite. Besides, and more importantly, there is no *small* coupling constant (i.e., no κ^2, because there is no κ in the original Lagrangian) in front of them; there is no reason why they should be negligible compared with the classical Lagrangian. This justifies a consideration of those terms already at the classical level. The modified scalar equation of motion (EM) reads (suppressing hats on the fields from now on)

$$\Box\phi - \frac{1}{6}R\phi + 4\lambda\phi^3 - \sum_{L=1}^{\infty}(1-(2L-2)\log\phi)\phi^{-2L+1} \times \sum_j g_j P_{L+1}^j[\bar{g}_{\mu\nu}] = 0. \tag{23}$$

Under a Weyl rescaling, the conformal wave operator $\Box_c \equiv \Box - \frac{1}{6}R$ behaves as

$$\Box_c \to \Omega^{-3}\Box_c\Omega. \tag{24}$$

It is then possible to write the Weyl transform of the scalar EM,

$$\Omega^{-3}\left(\Box - \frac{1}{6}R\right)\phi + 4\Omega^{-3}\lambda\phi^3 - \Omega^{-3} \times \sum_{L=1}^{\infty}\left(1-(2L-2)\log\frac{\phi}{\Omega}\right)\phi^{-2L+1} \times \sum_j g_j P^j_{L+1}[\bar{g}_{\mu\nu}] = 0. \quad (25)$$

A. One-dimensional toy model

As the general form of the gravitational terms is not known, a one-dimensional toy model that captures some features of the general setting can be studied. To do that, let us assume that $\sum_j g_j P^j_{L+1}[\bar{g}_{\mu\nu}] \equiv C$ is just a constant independent of L. Then, the sums over the loop order can be exactly done,

$$\sum_{L=1}^{\infty}\phi^{-2L+1} = \frac{\phi}{\phi^2-1}, \quad (26)$$

$$\sum_{L=1}^{\infty}(2L-2)\phi^{-2L+1} = \frac{2\phi}{(\phi^2-1)^2}, \quad (27)$$

which are convergent only when

$$\left|\frac{1}{\phi}\right| < 1, \quad (28)$$

and are extended to the whole complex plane by analytic continuation. In that sense, $\phi = 0$ is still a solution. Specific analyses are necessary in other situations.

In this case, the one-dimensional model then would read

$$\frac{d^2f(x)}{dx^2} - Af(x) + Bf(x)^3 - C\left(\frac{f(x)}{f(x)^2-1} - \frac{2f(x)}{(f(x)^2-1)^2}\log f(x)\right) = 0. \quad (29)$$

Consider, besides, A and B as arbitrary constants (where A is proportional to the Ricci scalar and B to the self-coupling g).

As the noninvariance of the theory comes from the logarithm, the (toy version of the) conformal case can be recovered in the case $C = 0$. In this case, the equation can be solved easily by transforming $f(x) \to \lambda f(x)$; then, it reads

$$\lambda\frac{d^2f(x)}{dx^2} - A\lambda f(x) + \lambda^3 Bf(x)^3 = 0. \quad (30)$$

Setting $\lambda^2 B = 2$ (which can be done as long as $B \neq 0$) and dividing by λ, we find now

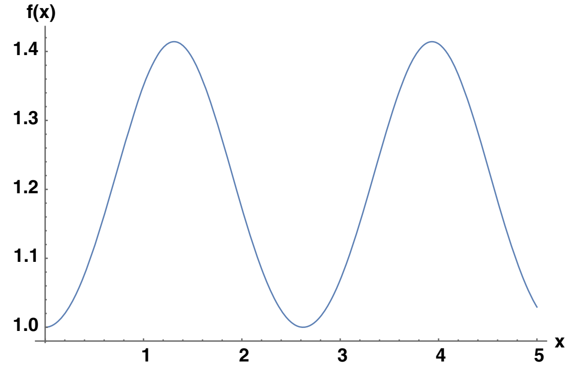

FIG. 1. Conformal case.

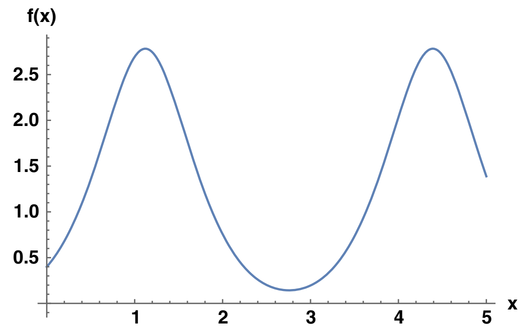

FIG. 2. Non conformal case for $C < 0$.

$$\frac{d^2f(x)}{dx^2} - Af(x) + 2f(x)^3 = 0; \quad (31)$$

finally, calling $m = 2 - A$, the previous equation now reads

$$\frac{d^2f(x)}{dx^2} - (2-m)f(x) + 2f(x)^3 = 0, \quad (32)$$

which is solved by the Jacobi elliptic function $\mathrm{dn}(x|m)$ Fig. 1. This function is defined in terms of the elliptic integral

$$u = \int_0^{\phi}\frac{d\theta}{\sqrt{1-m\sin^2\theta}}, \quad (33)$$

where m is called the *parameter*. Then the function we are dealing with is defined as $\mathrm{sn}(u|m) = \sin\phi$.

To see explicitly the properties of this solution, it is useful to define the quarter-periods K and iK', in the following way. Let m_1 (the *complementary parameter*) be such that $m + m_1 = 1$; then,

$$K = \int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1-m\sin^2\theta}} = \frac{\pi}{2}\sum_{n=0}^{\infty}\left(\frac{(2n)!}{2^{2n}(n!)^2}\right)^2 m^n, \quad (34)$$

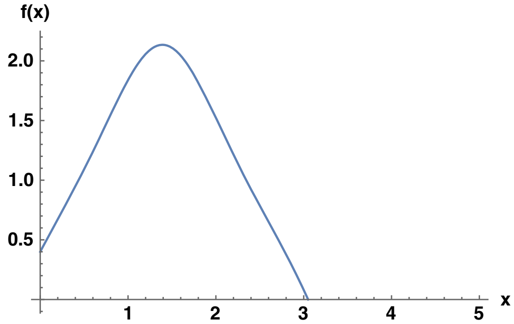

FIG. 3. $f'(0) > 0$.

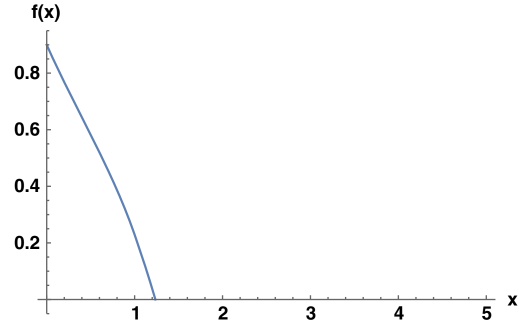

FIG. 4. $f'(0) < 0$.

$$iK' = i\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-m_1\sin^2\theta}} = i\frac{\pi}{2}\sum_{n=0}^{\infty}\left(\frac{(2n)!}{2^{2n}(n!)^2}\right)^2(1-m)^n. \tag{35}$$

In terms of these, the function has periods $2K$, $4K+4iK'$, and $4iK'$. As long as we have a real solution here, we are only concerned by $2K$, which can be expressed (to the lowest order) in terms of A as

$$2K = \frac{3\pi}{4} - \frac{\pi}{8}A, \tag{36}$$

and thus its period is smaller as the curvature gets bigger.

We consider now the nonconformal case (i.e., $C \neq 0$) given by Eq. (30). The first thing to notice is that, due to the presence of the logarithm, $f(x) = 0$ is no longer a solution. In fact, this is expected as we are working in the broken phase ($\phi \neq 0$). This is the main difference between the conformal and nonconformal cases, which is also expected in a more complicated model.

Concerning the constants, neither the sign of A (i.e., the curvature) nor C changes the shape of the function. Depending on the initial conditions, there are two possibilities:

(1) In the case of $C > 0$ (Fig. 2), there is an interval where the solution is again periodic but and looks as the conformal case. (Although the ordinary differential equation (ODE) diverges for $f(x) = 1$, numeric integration yields a finite answer even in this case.)

(2) In any other case (Figs. 3 and 4), the solution does not oscillate and goes to zero.

To summarize, the inclusion of the nonconformal part does not greatly change the solution. Depending on the initial values, it can be periodic as in the conformal one or go to zero and vanish.

There is another difference between both theories. In the conformal case, in addition to the trivial solution, the constant values $f(x) = \pm\sqrt{\frac{A}{B}}$ are solutions of the equations of motion. However, this does not happen in the nonconformal model, in which there are not any constant solutions.

Although our toy model is one dimensional, it is likely that it embodies some of the characteristics of the full-fledged four-dimensional situation.

IV. SYMMETRIC PHASE OF CDG

It is not obvious how the symmetric phase of CDG (which corresponds in the background field language to $\bar{\phi} = 0$) should be understood. The first problem is that there is no propagator to damp the gravitational fluctuations in the loop approximation. This has been emphasized in particular by 't Hooft [10]. Nevertheless, there are some observations that can be made on general grounds. The full partition function can be written as

$$Z[\bar{g}_{\mu\nu}] \equiv \int \mathcal{D}\phi\mathcal{D}h_{\mu\nu}e^{-\int d(\mathrm{vol})[\phi(\bar{\nabla}^2-\frac{1}{6}\bar{R})\phi+O(\phi^2h,h^3)+O(\phi^2h^2)]}. \tag{37}$$

Please note that the integral over the ghosts and auxiliary fields are implicitly included in the measure $\mathcal{D}h_{\alpha\beta}\mathcal{D}\phi$. Also note that the gauge-fixing conditions that suit our analysis should contain no monomial linear in the fields $h_{\mu\nu}$ and ϕ. This way, the gauge-fixing terms will contain three or more quantum fields.

The one-loop contribution only involves the quantum fluctuations of the dilaton. Its divergent part can be easily computed:

$$\begin{aligned} Z_\infty^{(L=1)}[\bar{g}_{\mu\nu}] &= \exp\left(-\frac{1}{16\pi^2}\frac{1}{n-4}\int d(\mathrm{vol})\left(\frac{1}{180}(\bar{R}^2_{\mu\nu\rho\sigma} - \bar{R}^2_{\mu\nu} + \bar{\Box}\bar{R})\right)\right) \\ &= \exp\left(-\frac{1}{16\pi^2}\frac{1}{n-4}\int d(\mathrm{vol})\frac{1}{180}\left(\frac{3}{2}\bar{W}_4 - \frac{1}{2}\bar{E}_4 + \bar{\Box}\bar{R}\right)\right). \end{aligned} \tag{38}$$

We have represented by W_4 the square of the Weyl tensor

$$W_4 \equiv R^2_{\mu\nu\rho\sigma} - 2R^2_{\mu\nu} + \frac{1}{3}R^2, \tag{39}$$

and by E_4 the four-dimensional Pfaffian (which upon integration yields Euler's characteristic up to a constant)

$$E_4 \equiv R^2_{\mu\nu\rho\sigma} - 4R^2_{\mu\nu} + R^2. \tag{40}$$

We are not able to perform the loop integrals over the gravitational fluctuations because there is no propagator for the gravitational field. Assuming (and this is an explicit hypothesis) that this integral makes sense (for example, by discretizing the system), then the partition function can be defined as given by the expression

$$Z_\infty = \sum_{L=1}^{\infty} Z^L_\infty[\bar{g}_{\mu\nu}], \tag{41}$$

which is the sum of all higher-loop divergent pieces $Z^L_\infty[\bar{g}_{\mu\nu}]$, each of which is a conformal invariant functional of $\bar{g}_{\mu\nu}$. There is only one of those in four dimensions, namely,

$$Z_\infty[\bar{g}_{\mu\nu}] \equiv e^{-g\int d(vol)\bar{W}_4}. \tag{42}$$

All loop contributions are of the same form, so that we can represent by g the coefficient of the whole sum. This procedure is formal in more than one way; there is no reason in particular to expect the loop expansion to converge or even to be asymptotic.

V. CONCLUSIONS

The general form of the finite counterterms which is necessary to insure the cancellation of the Weyl anomaly to every order in perturbation theory has been determined—under a sensible assumption, using only conformal invariance. They involve logarithms of the physical scalar, so that they are not local terms *sensu stricto*.

We found it interesting to examine the most broad-minded hypothesis in which they are indeed acceptable counterterms. Then, two facts immediately came to our attention. First of all, and in spite of their being loop effects (and so carrying powers of $\hbar$, so to speak), those finite counterterms do not have any inverse power of any mass scale in front of them (precisely because of conformal invariance), and then they are not negligible in the low-energy deep infrared limit. This might be identified in some sense with the *classical limit*.

It is then of interest to consider the classical effects of those terms. The most important is that the status of the trivial conformal invariant solution

$$\phi = 0 \tag{43}$$

changes slightly. This solution is the only one compatible with the symmetric phase of conformal symmetry.

When the space-time is of Petrov type O (that is, Weyl flat), then the symmetric configuration is still a solution. When the Weyl tensor does not vanish, then the analysis is more involved.

Consider the oversimplified situation in which the purely gravitational contribution is L independent,

$$\sum_j g_j P^j_{L+1}[\bar{g}_{\mu\nu}] \equiv G(x). \tag{44}$$

Then, the scalar equation of motion reduces to

$$\left(\Box - \frac{1}{6}R + \frac{g}{6}\phi^2\right)\phi + \left(2\frac{\phi}{(\phi^2-1)^2}\log\phi - \frac{\phi}{\phi^2-1}\right)G(x) = 0. \tag{45}$$

In that sense, $\phi = 0$ is still a solution. Specific analyses are necessary in other situations. To get an idea of how to do that, a one-dimensional toy model has been studied. A comparison was made between results, without taking into account the counterterms (this a conformally invariant situation which can be exactly solved in terms of Jacobian elliptic functions) and results including terms in our toy model that mimic the said counterterms.

It is to be stressed that, in spite of the above, those are *not* the classical equations of motion to be used in the context of the background field gauge technique to express physical results on shell. Those correspond to the $\hbar = 0$ sector only, that is, without including the corrections studied in the present paper. The fact that the solution corresponding to the symmetric phase is not always admissible in the present setting has to be interpreted as the fact that in those cases the counterterms are necessarily singular in the symmetric phase.

Finally, we conjecture that the form of the symmetric phase of conformal dilaton gravity ought to be proportional to the Weyl squared theory.

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APPENDIX: CONFORMAL INVARIANTS

Let us summarize here some known facts about conformal (Weyl) invariants. The Schouten tensor is defined as

$$A_{\alpha\beta} \equiv \frac{1}{n-2}\left(R_{\alpha\beta} - \frac{1}{2(n-1)} R g_{\alpha\beta}\right). \tag{A1}$$

It is invariant under rigid Weyl rescaling, that is, it transforms under $\Omega \equiv e^{\sigma}$ as

$$\tilde{A}_{\alpha\beta} = A_{\alpha\beta} - \sigma_{\alpha\beta} - \frac{1}{2}(\nabla\sigma)^2 g_{\alpha\beta}. \tag{A2}$$

The Weyl tensor reads

$$W_{\alpha\beta\mu\nu} \equiv R_{\alpha\beta\mu\nu} + (A_{\beta\mu}g_{\alpha\nu} + A_{\alpha\nu}g_{\beta\mu} - A_{\beta\nu}g_{\alpha\mu} - A_{\alpha\mu}g_{\beta\nu}). \tag{A3}$$

It transforms as a conformal tensor of weight $\lambda = -1$:

$$\tilde{W}_{\alpha\beta\mu\nu} \equiv e^{2\sigma} W_{\alpha\beta\mu\nu}. \tag{A4}$$

Its square has got scale dimension $\lambda = 2$,

$$\tilde{W}_{\alpha\beta\mu\nu}\tilde{W}^{\alpha\beta\mu\nu} = e^{-4\sigma} W_{\alpha\beta\mu\nu}W^{\alpha\beta\mu\nu}, \tag{A5}$$

in such a way that

$$|g|^{2/n} W^2 \tag{A6}$$

is pointwise invariant (but behaves as a true scalar in four dimensions only).

The Weyl tensor vanishes identically in low-dimension $n = 2$ and $n = 3$. A space with $n \geq 4$ is conformally flat iff $W = 0$.

The Cotton tensor reads

$$C_{\alpha\beta\gamma} \equiv \nabla_\alpha A_{\beta\gamma} - \nabla_\beta A_{\alpha\gamma} \tag{A7}$$

and is a conformal invariant of scaling dimension $\lambda = 0$ in $n = 3$ dimensions (and only there).

It is traceless in any dimension,

$$g^{\mu\nu}C_{\alpha\mu\nu} = \nabla_\alpha A - g^{\mu\nu}\nabla_\mu A_{\alpha\nu} = \frac{1}{2(n-1)}\nabla_\alpha R - \frac{1}{2(n-1)}\nabla_\alpha R = 0. \tag{A8}$$

The Bach tensor reads

$$B_{\mu\nu} \equiv \nabla^\rho C_{\rho\mu\nu} + A^{\alpha\beta}W_{\alpha\mu\nu\beta} = \nabla^\alpha\nabla_\delta W_{\alpha\mu\nu\delta} - \frac{1}{2}R^{\alpha\delta}W_{\alpha\mu\nu\delta} \tag{A9}$$

(this fact stems from the second Bianchi identity).

The Bach tensor is transverse,

$$\nabla_\beta B^{\alpha\beta} = 0, \tag{A10}$$

and it inherits its tracelessness from the same property for Weyl and Cotton tensors,

$$g^{\mu\nu}B_{\mu\nu} = 0. \tag{A11}$$

It is a conformal invariant of scaling dimension $\lambda = 1$ in four dimensions only.

The variation of the four-dimensional Weyl-squared action yields precisely the Bach tensor

$$\delta\int |W|^2 d(\mathrm{vol}) = \int B_{\mu\nu}\delta g^{\mu\nu} d(\mathrm{vol}). \tag{A12}$$

It is of course well known that that there is an extension of the Laplacian,

$$\Box_c \equiv \Box - \frac{n-2}{4(n-1)}R, \tag{A13}$$

that is such that

$$\tilde{\Box}_c = \Omega^{-\frac{n+2}{2}}\Box_c\Omega^{\frac{n-2}{2}}. \tag{A14}$$

On the other hand, the operator (which is a total derivative)

$$\Box_2 \equiv \sqrt{-g}\Box \tag{A15}$$

transforms as

$$\tilde{\Box}_2 = \partial_\mu(\Omega^{-2}g^{\mu\nu}\partial_\nu). \tag{A16}$$

The quartic Paneitz operator in an arbitrary dimension,

$$Q(g) \equiv \Box^2 + \nabla_\nu\left(-\frac{4}{n-2}R^{\mu\nu} + \frac{n^2-4n+8}{2(n-1)(n-2)}Rg^{\mu\nu}\right)\partial_\mu, \tag{A17}$$

is conformal invariant in the same sense as the Laplacian. That is, under

$$\tilde{g}_{\alpha\beta} \equiv \Omega^2 g_{\alpha\beta} \tag{A18}$$

transforms as

$$\tilde{Q} = \Omega^{-\frac{n+4}{2}} Q \Omega^{\frac{n-4}{2}}. \tag{A19}$$

In four dimensions, this gives

$$\Delta_P \equiv \sqrt{-g}\left(\Delta^2 + 2\nabla_\mu \left(R^{\mu\nu} - \frac{1}{3} R g^{\mu\nu}\right)\nabla_\nu\right). \tag{A20}$$

The Fefferman-Graham (FG) *obstruction tensor* $O_{\mu\nu}$ [11] is a trace-free symmetric 2-tensor which has got scaling dimension

$$\lambda = \frac{n-2}{2} \tag{A21}$$

and is divergenceless,

$$\nabla^\lambda O_{\mu\lambda} = 0, \tag{A22}$$

and vanishes for conformally Einstein metrics. It is the Dirichlet obstruction to the existence of a formal power series solution for the ambient metric associated to a given conformal structure. For example, the equation

$$R_{\alpha\beta}[g_+] + n g^+_{\mu\nu} = O(x^{n-1}\log x) \tag{A23}$$

admits a solution of the form

$$g^+_{\mu\nu} = \frac{1}{x^2}(dx^2 + g^x_{\mu\nu}), \tag{A24}$$

where

$$g^x_{\mu\nu} = h^x_{\mu\nu} + r^x_{\mu\nu} x^n \log x. \tag{A25}$$

Then,

$$n c_n r^0_{\mu\nu} = O_{\mu\nu}, \tag{A26}$$

where

$$c_n \equiv \frac{2^{n-2}(n/2-1)!^2}{n-2} \tag{A27}$$

In higher dimensions $n \geq 6$, Graham and Hirachi [12] have shown that the Weyl tensor and the FG obstruction are the basic building blocks of conformal invariants. Although explicit formulas are not known, it can be shown that

$$\begin{aligned} O_{\mu\nu} &= \Delta^{n/2-2}(\Delta P_{\mu\nu} - \nabla_\nu \nabla_\mu P^\lambda_\lambda) + \text{lots} \\ &= \frac{1}{3-n}\Delta^{n/2-2}\nabla^\rho \nabla^\sigma W_{\sigma\mu\nu\rho} + \text{lots}. \end{aligned} \tag{A28}$$

There is also an analog of the four-dimensional Weyl action [13], namely, the Q curvature [14], which is not a pointwise conformal invariant but yields nevertheless a conformal invariant under integration on a compact manifold; in fact, [15] $\int Q$ is a combination of the Euler characteristic and the integral of a pointwise conformal invariant.

It is related to the conformally invariant nth power of the Laplacian P_n and under Weyl $\tilde{g} = e^\sigma g$,

$$e^{\frac{n\sigma}{2}}\tilde{Q} = Q + P_n \frac{\sigma}{2}. \tag{A29}$$

Given the fact that P_n is self-adjoint and annihilated constants, the preceding result follows.

Consider an asymptotic expansion of the volume

$$\begin{aligned} \text{Vol}_{g_+}(\epsilon < x < \epsilon_0) &= c_0\epsilon^{-n} + c_2\epsilon^{-n+2} + \cdots + c_{n-2}\epsilon^{-2} \\ &\quad + \text{L}\log\frac{1}{\epsilon} + O(1). \end{aligned} \tag{A30}$$

The logarithmic term is related to the integral of the Q curvature:

$$\int Q dv = (-1)^{n/2} n(n-2) c_n L \tag{A31}$$

$$\delta\int_M Q\sqrt{|g|}d^n x = (-1)^{\frac{n}{2}}\frac{n-2}{2}\int_M \sqrt{|g|}d^n x O_{\mu\nu}\delta g^{\mu\nu}. \tag{A32}$$

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