

ON FORMALITY OF SASAKIAN MANIFOLDS

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ABSTRACT. We investigate some topological properties, in particular formality, of compact Sasakian manifolds. Answering some questions raised by Boyer and Galicki, we prove that all higher (than three) Massey products on any compact Sasakian manifold vanish. Hence, higher Massey products do obstruct Sasakian structures. Using this we produce a method of constructing simply connected K -contact non-Sasakian manifolds.

On the other hand, for every $n \geq 3$, we exhibit the first examples of simply connected compact Sasakian manifolds of dimension $2n + 1$ which are non-formal. They are non-formal because they have a non-zero triple Massey product. We also prove that arithmetic lattices in some simple Lie groups cannot be the fundamental group of a compact Sasakian manifold.

1. INTRODUCTION

The present article deals with homotopical properties of Sasakian manifolds. Sasakian geometry has become an important and active subject, especially after the appearance of the fundamental treatise of Boyer and Galicki [5]. The Chapter 7 of this book contains an extended discussion on the topological problems in the theory of Sasakian, and, more generally, K -contact manifolds. In particular, there are several topological obstructions to the existence of the aforementioned structures on a compact manifold M of dimension $2n + 1$, for example:

- vanishing of the odd Stiefel-Whitney classes,
- the inequality $1 \leq \text{cup}(M) \leq 2n$ on the cup-length,
- the evenness of the p^{th} Betti number for p odd with $1 \leq p \leq n$,
- the estimate on the number of closed integral curves of the Reeb vector field (there should be at least $n + 1$),
- some torsion obstructions in dimension 5 discovered by Kollár.

Here we follow the above line of thinking. It is well-known that the theory of Sasakian manifolds is, in a sense, parallel to the theory of the Kähler manifolds. In fact, a Sasakian manifold is a Riemannian manifold (M, g) such that $M \times \mathbb{R}^+$ equipped with the cone metric $h = t^2g + dt^2$ is Kähler. In particular, M has odd dimension $2n + 1$, where $n + 1$ is the complex dimension of the Kähler cone. There is a deep theorem of Deligne, Griffiths, Morgan and Sullivan on the rational homotopy type of Kähler manifolds [10]. In the same spirit, rational homotopical properties of a manifold are related to the existence of suitable geometric structures on the manifold [14]. Therefore, it is important to build a version of such theory for compact Sasakian manifolds. It seems that not much known in this direction, although some partial results were obtained in [34].

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In [5, Chapter 7], the authors pose the following problems.

- (1) Are there obstructions to the existence of Sasakian structures expressed in terms of Massey products?
- (2) There are obstructions to the existence of Sasakian structures expressed in terms of Massey products, which depend on basic cohomology classes of the related K -contact structure. Can one obtain a topological characterization of them?
- (3) Do there exist simply connected K -contact non-Sasakian manifolds (open problem 7.4.1)?
- (4) Which finitely presented groups can be realized as fundamental groups of compact Sasakian manifolds?

The present paper deals with these problems. There are examples of non-simply connected compact Sasakian manifolds which are non-formal because they have a non-zero triple Massey product. Simply connected compact manifolds of dimension less than or equal to 6 are formal [15, 28]. In Section 3 we show that triple Massey products do not obstruct Sasakian structures neither in the non-simply connected case nor in the simply connected case. Indeed, in Theorem 3.2 we prove the following:

For every $n \geq 3$ there exists a simply connected compact regular Sasakian manifold M , of dimension $2n + 1$, with a non-zero triple Massey product.

The example that we construct is the total space of a non-trivial 3-sphere bundle over $(S^2)^{n-1} = S^2 \times \binom{n-1}{\dots} \times S^2$.

However, in Section 4, we fully answer the question about Massey product obstructions proving that *all higher order Massey products of Sasakian manifolds vanish* (Theorem 4.4 and Proposition 4.8). Using the latter we show a new method of constructing families of compact K -contact manifolds with no Sasakian structures. More precisely, we have the following result (Theorem 4.9):

Let M be a simply connected compact symplectic manifold of dimension $2k$ with an integral symplectic form ω , and a non-zero quadruple Massey product. Then there exists a sphere bundle $S^{2m+1} \rightarrow E \rightarrow M$, with $m + 1 > k$, such that the total space E is K -contact but has no Sasakian structure.

Note that the existence of simply connected K -contact non-Sasakian manifolds was proved by Hajduk and the fourth author in [19] using the evenness of the third Betti number of a compact Sasakian manifold.

Finally, in Section 5, we address the question (4) on the fundamental groups of K -contact and Sasakian manifolds. We show that every finitely presented group occurs as the fundamental group of some compact K -contact manifold (Theorem 5.2). In contrast, some arithmetic lattices in simple Lie groups cannot be realized as the fundamental group of some compact Sasakian manifold (Proposition 5.4).

2. FORMAL MANIFOLDS AND MASSEY PRODUCTS

In this section some definitions and results about minimal models and Massey products are reviewed [10, 13].

We work with *differential graded commutative algebras*, or DGAs, over the field \mathbb{R} of real numbers. The degree of an element a of a DGA is denoted by $|a|$. A DGA (\mathcal{A}, d) is *minimal* if:

- (1) \mathcal{A} is free as an algebra, that is, \mathcal{A} is the free algebra $\bigwedge V$ over a graded vector space $V = \bigoplus_i V^i$, and
- (2) there is a collection of generators $\{a_\tau\}_{\tau \in I}$ indexed by some well ordered set I , such that $|a_\mu| \leq |a_\tau|$ if $\mu < \tau$ and each da_τ is expressed in terms of preceding a_μ , $\mu < \tau$. This implies that da_τ does not have a linear part.

In our context, the main example of DGA is the de Rham complex $(\Omega^*(M), d)$ of a differentiable manifold M , where d is the exterior differential.

Given a differential graded commutative algebra (\mathcal{A}, d) , we denote its cohomology by $H^*(\mathcal{A})$. The cohomology of a differential graded algebra $H^*(\mathcal{A})$ is naturally a DGA with the product structure inherited from that on \mathcal{A} and with the differential being identically zero. The DGA (\mathcal{A}, d) is *connected* if $H^0(\mathcal{A}) = \mathbb{R}$, and \mathcal{A} is *1-connected* if, in addition, $H^1(\mathcal{A}) = 0$.

According to [10, page 249], an *elementary extension* of a differential graded commutative algebra (\mathcal{A}, d) is any DGA of the form

$$(\mathcal{B} = \mathcal{A} \otimes \bigwedge V, d_{\mathcal{B}}),$$

where $d_{\mathcal{B}}|_{\mathcal{A}} = d$, $d_{\mathcal{B}}(V) \subset \mathcal{A}$ and $\bigwedge V$ is the free algebra over a finite dimensional vector space V whose elements have all the same degree.

Morphisms between DGAs are required to preserve the degree and to commute with the differential.

We shall say that $(\bigwedge V, d)$ is a *minimal model* of the differential graded commutative algebra (\mathcal{A}, d) if $(\bigwedge V, d)$ is minimal and there exists a morphism of differential graded algebras

$$\rho : (\bigwedge V, d) \longrightarrow (\mathcal{A}, d)$$

inducing an isomorphism $\rho^* : H^*(\bigwedge V) \xrightarrow{\sim} H^*(\mathcal{A})$ of cohomologies.

In [20], Halperin proved that any connected differential graded algebra (\mathcal{A}, d) has a minimal model unique up to isomorphism. For 1-connected differential algebras, a similar result was proved earlier by Deligne, Griffiths, Morgan and Sullivan [10].

A *minimal model* of a connected differentiable manifold M is a minimal model $(\bigwedge V, d)$ for the de Rham complex $(\Omega^*(M), d)$ of differential forms on M . If M is a simply connected manifold, then the dual of the real homotopy vector space $\pi_i(M) \otimes \mathbb{R}$ is isomorphic to V^i for any i . This relation also holds when $i > 1$ and M is nilpotent, that is, the fundamental group $\pi_1(M)$ is nilpotent and its action on $\pi_j(M)$ is nilpotent for all $j > 1$ (see [10]).

Recall that a minimal algebra $(\bigwedge V, d)$ is called *formal* if there exists a morphism of differential algebras $\psi : (\bigwedge V, d) \longrightarrow (H^*(\bigwedge V), 0)$ inducing the identity map on cohomology. Also a differentiable manifold M is called *formal* if its minimal model is formal. Many examples of formal manifolds are known: spheres, projective spaces, compact Lie groups, homogeneous spaces, flag manifolds, and all compact Kähler manifolds.

The formality of a minimal algebra is characterized as follows.

Proposition 2.1 ([10]). *A minimal algebra $(\wedge V, d)$ is formal if and only if the space V can be decomposed into a direct sum $V = C \oplus N$ with $d(C) = 0$, and d injective on N , such that every closed element in the ideal $I(N) \subset \wedge V$ generated by N is exact.*

This characterization of formality can be weakened using the concept of s -formality introduced in [15].

Definition 2.2. A minimal algebra $(\wedge V, d)$ is s -formal ($s > 0$) if for each $i \leq s$ the space V^i of generators of degree i decomposes as a direct sum $V^i = C^i \oplus N^i$, where the spaces C^i and N^i satisfy the three following conditions:

- (1) $d(C^i) = 0$,
- (2) the differential map $d : N^i \rightarrow \wedge V$ is injective, and
- (3) any closed element in the ideal $I_s = I(\bigoplus_{i \leq s} N^i)$, generated by the space $\bigoplus_{i \leq s} N^i$ in the free algebra $\wedge(\bigoplus_{i \leq s} V^i)$, is exact in $\wedge V$.

A differentiable manifold M is s -formal if its minimal model is s -formal. Clearly, if M is formal then M is s -formal for all $s > 0$. The main result of [15] shows that sometimes the weaker condition of s -formality implies formality.

Theorem 2.3 ([15]). *Let M be a connected and orientable compact differentiable manifold of dimension $2n$ or $(2n - 1)$. Then M is formal if and only if it is $(n - 1)$ -formal.*

One can check that any simply connected compact manifold is 2-formal. Therefore, Theorem 2.3 implies that any simply connected compact manifold of dimension not more than 6 is formal.

In order to detect non-formality, instead of computing the minimal model, which usually is a lengthy process, one can use Massey products, which are known to be obstructions to formality. The simplest type of Massey product is the triple (also known as ordinary) Massey product, which we define next.

Let (\mathcal{A}, d) be a DGA (in particular, it can be the de Rham complex of differential forms on a differentiable manifold). Suppose that there are cohomology classes $[a_i] \in H^{p_i}(\mathcal{A})$, $p_i > 0$, $1 \leq i \leq 3$, such that $a_1 \cdot a_2$ and $a_2 \cdot a_3$ are exact. Write $a_1 \cdot a_2 = da_{1,2}$ and $a_2 \cdot a_3 = da_{2,3}$. The (triple) Massey product of the classes $[a_i]$ is defined to be

$$\begin{aligned} \langle [a_1], [a_2], [a_3] \rangle &= [a_1 \cdot a_{2,3} + (-1)^{p_1+1} a_{1,2} \cdot a_3] \\ &\in \frac{H^{p_1+p_2+p_3-1}(\mathcal{A})}{[a_1] \cdot H^{p_2+p_3-1}(\mathcal{A}) + [a_3] \cdot H^{p_1+p_2-1}(\mathcal{A})}. \end{aligned}$$

Now we move on to the definition of higher Massey products (see [35]). Given

$$[a_i] \in H^*(\mathcal{A}), \quad 1 \leq i \leq t, \quad t \geq 3,$$

the Massey product $\langle [a_1], [a_2], \dots, [a_t] \rangle$, is defined if there are elements $a_{i,j}$ on \mathcal{A} , with $1 \leq i \leq j \leq t$ and $(i, j) \neq (1, t)$, such that

$$\begin{aligned} a_{i,i} &= a_i, \\ da_{i,j} &= \sum_{k=i}^{j-1} (-1)^{|a_{i,k}|} a_{i,k} \cdot a_{k+1,j}. \end{aligned} \tag{2.1}$$

Then the *Massey product* $\langle [a_1], [a_2], \dots, [a_t] \rangle$ is the set of cohomology classes

$$\begin{aligned} &\langle [a_1], [a_2], \dots, [a_t] \rangle \\ &= \left\{ \left[\sum_{k=1}^{t-1} (-1)^{|a_{1,k}|} a_{1,k} \cdot a_{k+1,t} \right] \mid a_{i,j} \text{ as in (2.1)} \right\} \subset H^{|a_1|+\dots+|a_t|-(t-2)}(\mathcal{A}). \end{aligned}$$

We say that the Massey product is *zero* if

$$0 \in \langle [a_1], [a_2], \dots, [a_t] \rangle.$$

It should be mentioned that for $\langle a_1, a_2, \dots, a_t \rangle$ to be defined, it is necessary that all the lower order Massey products $\langle a_1, \dots, a_i \rangle$ and $\langle a_{i+1}, \dots, a_t \rangle$ with $2 < i < t - 2$ are defined and trivial.

Massey products are related to formality by the following well-known result.

Theorem 2.4 ([10, 35]). *A DGA which has a non-zero Massey product is not formal.*

Another obstruction to the formality is given by the *a-Massey products* introduced in [7], that generalize the triple Massey products. They have the advantage of being simpler to compute than the higher order Massey products. The *a-Massey products* are defined as follows.

Let (\mathcal{A}, d) be a DGA, and let $a, b_1, \dots, b_m \in \mathcal{A}$ be closed elements such that the degree $|a|$ of a is even and $a \cdot b_i$ is exact for all i . Let ξ_i be any element such that $d\xi_i = a \cdot b_i$. Then, the m^{th} order *a-Massey product* of the b_i is the subset

$$\begin{aligned} &\langle a; b_1, \dots, b_m \rangle \\ &:= \left\{ \left[\sum_i (-1)^{|\xi_1|+\dots+|\xi_{i-1}|} \xi_1 \cdot \dots \cdot \xi_{i-1} \cdot b_i \cdot \xi_{i+1} \cdot \dots \cdot \xi_m \right] \mid d\xi_i = a \cdot b_i \right\} \subset H^q(\mathcal{A}), \end{aligned}$$

where $q = (m - 1) \deg a + (\sum_{i=1}^m \deg b_i) - m + 1$. We say that the *a-Massey product* is *zero* if $0 \in \langle a; b_1, \dots, b_m \rangle$.

Lemma 2.5 ([7]). *The a-Massey product $\langle a; b_1, \dots, b_m \rangle$ depends only on the cohomology classes $[a]$, $\{[b_i]\}_{i=1}^m$, and not on the choices of elements representing these cohomology classes.*

Theorem 2.6 ([7]). *A DGA which has a non-zero a-Massey product is not formal.*

The concept of formality is also defined for nilpotent CW-complexes, and all the above discussions can be extended to the nilpotent CW-complexes by using the DGA of piecewise polynomial differential forms $\mathcal{A}_{PL}(X)$ on a CW-complex X (instead of using differential forms) [13, 18].

3. NON-FORMAL SIMPLY CONNECTED REGULAR SASAKIAN MANIFOLDS

In this section we produce examples of simply connected compact regular Sasakian manifolds, of dimension ≥ 7 , which are non-formal. As mentioned before, Theorem 2.3 gives that simply connected compact manifolds of dimension at most 6 are all formal [15, 28].

First, we recall some definitions and results on Sasakian manifolds (see [5] for more details).

Let M be a $(2n + 1)$ -dimensional manifold. An *almost contact metric structure* on M consists of a quadruplet (η, ξ, ϕ, g) , where η is a differential 1-form, ξ is a nowhere vanishing vector field (known as the *Reeb* vector field), ϕ is a C^∞ section of $\text{End}(TM)$ and g is a Riemannian metric on M , satisfying the following conditions

$$\eta(\xi) = 1, \quad \phi^2 = -\text{Id} + \xi \otimes \eta, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad (3.1)$$

for all vector fields X, Y on M . Thus, the kernel of η defines a codimension one distribution $\mathcal{D} = \ker(\eta)$, and there is the orthogonal decomposition of the tangent bundle TM of M

$$TM = \mathcal{D} \oplus \mathcal{L},$$

where \mathcal{L} is the trivial line subbundle of TM generated by ξ . Note that conditions in (3.1) imply that

$$\phi(\xi) = 0, \quad \eta \circ \phi = 0. \quad (3.2)$$

For an almost contact metric structure (η, ξ, ϕ, g) on M , the fundamental 2-form F on M is defined by

$$F(X, Y) = g(\phi X, Y),$$

where X and Y are vector fields on M . Hence,

$$F(\phi X, \phi Y) = F(X, Y),$$

that is F is compatible with ϕ , and $\eta \wedge F^n \neq 0$ everywhere.

An almost contact metric structure (η, ξ, ϕ, g) on M is said to be a *contact metric* if

$$g(\phi X, Y) = d\eta(X, Y).$$

In this case η is a *contact form*, meaning

$$\eta \wedge (d\eta)^n \neq 0$$

at every point of M . If (η, ξ, ϕ, g) is a contact metric structure such that ξ is a Killing vector field for g , meaning $\mathcal{L}_\xi g = 0$, where \mathcal{L}_ξ denotes the Lie derivative, then (η, ξ, ϕ, g) is called a *K-contact* structure. A manifold with a K-contact structure is called a *K-contact manifold*.

Just as in the case of an almost Hermitian structure, there is the notion of integrability of an almost contact metric structure. More precisely, an almost contact metric structure (η, ξ, ϕ, g) is called *normal* if the Nijenhuis tensor N_ϕ associated to the tensor field ϕ , defined by

$$N_\phi(X, Y) := \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y], \quad (3.3)$$

satisfies the equation

$$N_\phi = -d\eta \otimes \xi.$$

This last equation is equivalent to the condition that the almost complex structure J on $M \times \mathbb{R}$ given by

$$J \left(X, f \frac{\partial}{\partial t} \right) = \left(\phi X - f\xi, \eta(X) \frac{\partial}{\partial t} \right) \quad (3.4)$$

is integrable, where f is a smooth function on $M \times \mathbb{R}$ and t is the standard coordinate on \mathbb{R} (see [32]). In other words, ϕ defines a complex structure on the kernel $\ker(\eta)$ compatible with $d\eta$.

A *Sasakian structure* is a normal contact metric structure, in other words, an almost contact metric structure (η, ξ, ϕ, g) such that

$$N_{\Phi} = -d\eta \otimes \xi, \quad d\eta = F.$$

If (η, ξ, ϕ, g) is a Sasakian structure on M , then (M, η, ξ, ϕ, g) is called a *Sasakian manifold*.

Riemannian manifolds with a Sasakian structure can also be characterized in terms of the Riemannian cone over the manifold. A Riemannian manifold (M, g) admits a compatible Sasakian structure if and only if $M \times \mathbb{R}^+$ equipped with the cone metric $h = t^2g + dt \otimes dt$ is Kähler [5]. Furthermore, in this case the Reeb vector field is Killing, and the covariant derivative of ϕ with respect to the Levi-Civita connection of g is given by

$$(\nabla_X \phi)(Y) = g(\xi, Y)X - g(X, Y)\xi,$$

where X and Y are vector fields on M .

In the study of Sasakian manifolds, the *basic cohomology* plays an important role. Let (M, η, ξ, ϕ, g) be a Sasakian manifold of dimension $2n + 1$. A differential form α on M is called *basic* if

$$\iota_{\xi} \alpha = 0 = \iota_{\xi} d\alpha,$$

where ι_{ξ} denotes the contraction of differential forms by ξ . We denote by $\Omega_B^k(M)$ the space of all basic k -forms on M . Clearly, the de Rham exterior differential d of M takes basic forms to basic forms. Denote by d_B the restriction of the de Rham differential d to $\Omega_B^*(M)$. The cohomology of the differential complex $(\Omega_B^*(M), d_B)$ is called the *basic cohomology*. These basic cohomology groups are denoted by $H_B^*(M)$. When M is compact, the dimensions of the de Rham cohomology groups $H^*(M)$ and the basic cohomology groups $H_B^*(M)$ are related as follows:

Theorem 3.1 ([5, Theorem 7.4.14]). *Let (M, η, ξ, ϕ, g) be a compact Sasakian manifold of dimension $2n + 1$. Then, the Betti number $b_r(M)$ and the basic Betti number $b_r^B(M)$ are related by*

$$b_r^B(M) = b_{r-2}^B(M) + b_r(M),$$

for $0 \leq r \leq n$, In particular, if r is odd and $r \leq n$, then $b_r(M) = b_r^B(M)$.

A Sasakian structure on M is called *quasi-regular* if there is a positive integer δ satisfying the condition that each point of M has a foliated coordinate chart (U, t) with respect to ξ (the coordinate t is in the direction of ξ) such that each leaf for ξ passes through U at most δ times. If $\delta = 1$, then the Sasakian structure is called *regular*. (See [5, p. 188].)

A result proved in [29] says that if M admits a Sasakian structure, then it also admits a quasi-regular Sasakian structure. It should be clarified that the word ‘‘fibration’’ in [29, Theorem 1.11] means ‘‘orbi-bundle’’.

If N is a compact Kähler manifold whose Kähler form ω defines an integral cohomology class, then the total space of the circle bundle

$$S^1 \hookrightarrow M \xrightarrow{\pi} N \quad (3.5)$$

with Euler class $[\omega] \in H^2(M, \mathbb{Z})$ is a regular Sasakian manifold with a contact form η that satisfies the equation $d\eta = \pi^*(\omega)$, where π is the projection in (3.5).

3.1. A non-simply connected non-formal Sasakian manifold. Recall from [9] that the real Heisenberg group H^{2n+1} admits a homogeneous regular Sasakian structure with its standard 1-form $\eta = dz - \sum_{i=1}^n y_i dx_i$. As a manifold H^{2n+1} is just \mathbb{R}^{2n+1} which can be realized in terms of $(n+2) \times (n+2)$ nilpotent matrices of the form

$$A = \begin{pmatrix} 1 & a_1 & \cdots & a_n & c \\ 0 & 1 & 0 & \cdots & b_1 \\ \vdots & & \ddots & \cdots & \vdots \\ 0 & \cdots & 0 & 1 & b_n \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix}, \quad (3.6)$$

where $a_i, b_i, c \in \mathbb{R}$, $i = 1, \dots, n$. Then a global system of coordinates x_i, y_i, z for H^{2n+1} is defined by $x_i(A) = a_i$, $y_i(A) = b_i$, $z(A) = c$. A standard calculation shows that we have a basis for the left invariant 1-forms on H^{2n+1} which consists of

$$\{dx_i, dy_i, dz - \sum_{i=1}^n x_i dy_i\}.$$

Consider the discrete subgroup Γ of H^{2n+1} defined by the matrices of the form given in (3.6) with integer entries. The quotient manifold

$$M := \Gamma \backslash H^{2n+1}$$

is compact. The 1-forms dx_i , dy_i and $dz - \sum_{i=1}^n x_i dy_i$ descend to 1-forms α_i , β_i and γ respectively on M . We note that $\{\alpha_i, \beta_i, \gamma\}$ is a basis for the 1-forms on M . Let $\{X_i, Y_i, Z\}$ be the basis of vector fields on M that is dual to the basis $\{\alpha_i, \beta_i, \gamma\}$. Define the almost contact metric structure (η, ξ, ϕ, g) on M by

$$\begin{aligned} \eta &= \gamma, & \xi &= Z, & \phi(X_i) &= Y_i, & \phi(Y_i) &= -X_i \\ \phi(\xi) &= 0, & g &= \gamma^2 + \sum_{i=1}^n ((\alpha_i)^2 + (\beta_i)^2). \end{aligned}$$

Then one can check that (η, ξ, ϕ, g) is a regular Sasakian structure on M . In fact, the manifold M can be also defined as a circle bundle over a torus T^{2n} . Moreover, M is non-formal since it is not 1-formal in the sense of Definition 2.2.

3.2. A non-simply connected formal Sasakian manifold. In order to construct an example of a formal compact non-simply connected regular Sasakian manifold, we consider the simply connected, solvable non-nilpotent Lie group L^3 of dimension 3 consisting of matrices of the form

$$A = \begin{pmatrix} \cos 2\pi c & \sin 2\pi c & 0 & a \\ -\sin 2\pi c & \cos 2\pi c & 0 & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (3.7)$$

where $a, b, c \in \mathbb{R}$. Then a global system of coordinates x, y, z for L^3 is defined by $x(A) = a$, $y(A) = b$, $z(A) = c$, and a standard calculation shows that a basis for the right invariant 1-forms on L^3 consists of

$$\{dx - 2\pi y dz, dy + 2\pi x dz, dz\}.$$

Notice that the solvable Lie group L^3 is not completely solvable. Let D be a discrete subgroup of L^3 such that the quotient space L^3/D is compact (such a subgroup D exists; see for example [1] or [35]). The forms $dx - 2\pi y dz$, $dy + 2\pi x dz$ and dz descend to 1-forms α , β and γ respectively on L^3/D .

We define the product manifold $B^4 = (L^3/D) \times S^1$. Then, there are 1-forms $\alpha, \beta, \gamma, \mu$ on B^4 such that

$$d\alpha = -\beta \wedge \gamma, \quad d\beta = \alpha \wedge \gamma, \quad d\gamma = d\mu = 0,$$

and at each point of B^4 , the cotangent vectors $\{\alpha, \beta, \gamma, \mu\}$ form a basis for the cotangent fiber. Moreover, using Hattori's theorem [22] it is straightforward to compute the real cohomology groups of B^4 , which are:

$$\begin{aligned} H^0(B^4) &= \langle 1 \rangle, \\ H^1(B^4) &= \langle [\gamma], [\mu] \rangle, \\ H^2(B^4) &= \langle [\alpha \wedge \beta], [\gamma \wedge \mu] \rangle, \\ H^3(B^4) &= \langle [\alpha \wedge \beta \wedge \gamma], [\alpha \wedge \beta \wedge \mu] \rangle, \\ H^4(B^4) &= \langle [\alpha \wedge \beta \wedge \gamma \wedge \mu] \rangle. \end{aligned}$$

A Kähler structure (g, ω) on B^4 is given by the Kähler metric $g = \alpha^2 + \beta^2 + \gamma^2 + \mu^2$ and the Kähler form

$$\omega = \alpha \wedge \beta + \gamma \wedge \mu.$$

According to [21], the complex manifold B^4 is a finite quotient of a compact complex torus.

We can suppose that the Kähler form ω on B^4 defines an integral cohomology class. Therefore, the total space M^5 of the circle bundle over B^4 with Euler class $[\omega]$ has a regular Sasakian structure.

Clearly, B^4 is formal as it is a compact Kähler manifold. In order to prove that M^5 is formal, we first observe that the minimal model of B^4 must be a differential graded algebra of the form (\mathcal{M}, d) , where $\mathcal{M} = \bigwedge(a_1, a_2, b, c)$ is a free algebra such that the generators a_i have degree 1, the generator b has degree 2 and the generator c has degree 3. The differential d is given by $da_i = db = 0$ and $dc = b^2$. A homomorphism

$$\rho: \mathcal{M} \longrightarrow \Omega(B^4)$$

that induces an isomorphism on cohomology is defined by $\rho(a_1) = \gamma$, $\rho(a_2) = \mu$, $\rho(b) = \alpha \wedge \beta$, and $\rho(c) = 0$.

Now, according to [31], a (non-minimal) model of

$$S^1 \hookrightarrow M^5 \longrightarrow B^4$$

with Euler class $[\omega] \in H^2(B^4, \mathbb{Z})$ is given by $\bigwedge(a_1, a_2, b, c) \otimes \bigwedge(a_3)$, where $\bigwedge(a_1, a_2, b, c)$ is the above minimal model for B^4 , the generator a_3 has degree 1 and its differential is

given by $da_3 = a_1a_2 + b$. Then, the minimal model associated to this model is

$$(\widetilde{\mathcal{M}}, \widetilde{d}) = (\bigwedge(a_1, a_2, x), \widetilde{d}),$$

where the generators a_i have degree 1 and the generator x has degree 3 while the differential \widetilde{d} is given by $\widetilde{d}a_i = \widetilde{d}x = 0$. Therefore, we get

$$C^1 = \langle a_1, a_2 \rangle, \quad N^1 = 0 = C^2 = N^2.$$

Now Theorem 2.3 implies that M^5 is formal because it is 2-formal.

3.3. A simply connected non-formal Sasakian manifold. The most basic example of a simply connected compact regular Sasakian manifold is the odd-dimensional sphere S^{2n+1} considered as the total space of the Hopf fibration $S^{2n+1} \hookrightarrow \mathbb{C}\mathbb{P}^n$. It is well-known that S^{2n+1} is formal. In the next theorem we give examples of non-formal simply connected compact regular Sasakian manifolds.

Theorem 3.2. *For every $n \geq 3$, there exists a simply connected compact regular Sasakian manifold M^{2n+1} , of dimension $2n + 1$, which is non-formal. More precisely, there is a non-trivial 3-sphere bundle over $(S^2)^{n-1}$ which is a non-formal simply connected compact regular Sasakian manifold.*

Proof. First take $n = 3$. We will determine a minimal model of the 7-manifold M^7 . A minimal model of the 3-sphere S^3 is the differential algebra $(\bigwedge(z), d)$, where the generator z has degree 3 and $dz = 0$. A minimal model of $S^2 \times S^2$ is the differential algebra $(\bigwedge(a, b, x, y), d)$, where a, b have degree 2, while x, y have degree 3, and the differential d is given by the following: $da = db = 0$, $dx = a^2$ and $dy = b^2$. Therefore, a minimal model of the total space of a fiber bundle

$$S^3 \hookrightarrow M^7 \longrightarrow S^2 \times S^2$$

is the differential algebra over the vector space V generated by the elements a, b of degree 2 and x, y, z of degree 3, and the differential d is given by

$$da = db = 0, \quad dx = a^2, \quad dy = b^2, \quad dz = eab,$$

where $e[ab] \in H^4(S^2 \times S^2, \mathbb{Z})$ is the Euler class of the S^3 -bundle with $e \in \mathbb{Z}$.

Let us assume that $e \neq 0$, so the S^3 -bundle is non-trivial. For $1 \leq i \leq 4$, the subspace $V^i \subset V$ of degree i decomposes as $V^i = C^i \oplus N^i$, where $C^1 = N^1 = 0$, $C^2 = \langle a, b \rangle$, $N^2 = 0$, $C^3 = 0$, $N^3 = \langle x, y, z \rangle$ and $C^4 = N^4 = 0$. Therefore, $(\bigwedge V, d)$ is 2-formal because $N^1 = N^2 = 0$. However the minimal model $(\bigwedge V, d)$ is not 3-formal because the element $\nu = az - xb$ lies in the ideal $N^{\leq 3} \bigwedge(V^{\leq 3})$ generated by $N^{\leq 3}$ in $\bigwedge(V^{\leq 3})$, and ν is closed but non-exact in $(\bigwedge V, d)$. This proves that M^7 is non-formal because it is not 3-formal. In terms of Massey products, we have $a^2 = dx$, $eab = dz$, which implies that the triple Massey product $\langle a, a, b \rangle$ is defined and it is non-zero.

Now, to complete the proof for $n = 3$ we need to show that M^7 is a regular Sasakian manifold for suitable $e \neq 0$. We will first show that M can be considered as the circle bundle over $S^2 \times S^2 \times S^2$ with Euler class the Kähler form on $S^2 \times S^2 \times S^2$. Indeed, let a_1, a_2, a_3 be the generators of the cohomology of each of the S^2 -factors of $S^2 \times S^2 \times S^2$. Then the Kähler form ω has cohomology class $[\omega] = a_1 + a_2 + a_3$. Consider the principal S^1 -bundle

$$S^1 \longrightarrow N \longrightarrow S^2 \times S^2 \times S^2$$

with first Chern class equal to $a_1 + a_2 + a_3$. Then the Gysin sequence gives that

$$\begin{aligned} H^0(N, \mathbb{Z}) &= H^7(N, \mathbb{Z}) = \mathbb{Z}, \\ H^1(N, \mathbb{Z}) &= H^3(N, \mathbb{Z}) = H^6(N, \mathbb{Z}) = 0, \\ H^2(N, \mathbb{Z}) &= H^5(N, \mathbb{Z}) = \mathbb{Z}^2, \\ H^4(N, \mathbb{Z}) &= \mathbb{Z}\langle a_1a_2, a_1a_3, a_2a_3 \rangle / \langle a_1a_2 + a_1a_3, a_2a_1 + a_2a_3, a_3a_1 + a_3a_2 \rangle = \mathbb{Z}_2. \end{aligned}$$

If we restrict to each $\{(x, y)\} \times S^2$, then this circle bundle has first Chern class equal to a_3 , the generator of $H^2(S^2, \mathbb{Z})$. So this is the Hopf bundle

$$S^1 \longrightarrow S^3 \longrightarrow S^2.$$

Varying over all $(x, y) \in S^2 \times S^2$, we have an S^3 -bundle

$$S^3 \longrightarrow N \longrightarrow B = S^2 \times S^2.$$

Let $e a_1a_2 \in H^4(B, \mathbb{Z})$ be its Euler class, where $e \in \mathbb{Z}$. The Gysin sequence gives

$$\begin{aligned} H^0(N, \mathbb{Z}) &= H^7(N, \mathbb{Z}) = \mathbb{Z}, \\ H^1(N, \mathbb{Z}) &= H^3(N, \mathbb{Z}) = H^6(N, \mathbb{Z}) = 0, \\ H^2(N, \mathbb{Z}) &= H^5(N, \mathbb{Z}) = \mathbb{Z}^2, \\ H^4(N, \mathbb{Z}) &= \mathbb{Z}_e. \end{aligned}$$

Hence taking $e = 2$, we have that $N \cong M$. Therefore, M^7 admits a regular Sasakian structure for $e = 2$.

The case $n > 3$ is similar; it is deduced as follows. Consider $B = S^2 \times \binom{n}{!} \times S^2$. Let $a_1, \dots, a_n \in H^2(B)$ be the cohomology classes of degree two given by each of the S^2 -factors. Then the Kähler class is given by $[\omega] = a_1 + \dots + a_n$. Consider the circle bundle

$$S^1 \longrightarrow N \longrightarrow B$$

with first Chern class equal to $[\omega]$. As above, this is an S^3 -bundle over $B' = S^2 \times \binom{n-1}{!} \times S^2$. The Euler class is

$$\sum_{1 \leq i < j \leq n-1} e_{ij} a_i a_j \in H^4(B'),$$

where $e_{ij} \in \mathbb{Z}$. Restricting to each $S^2 \times S^2$ embedded in the i -th and j -th factors, we see that $e_{ij} = 2$ for every $i < j$, by the computations in the case $n = 3$.

The minimal model of such manifold N is worked out as before. It is

$$\bigwedge (a_1, \dots, a_{n-1}, x_1, \dots, x_{n-1}, y),$$

where $|a_i| = 2$, $|x_i| = 3$, $|y| = 3$, $dx_i = a_i^2$, $dy = 2 \sum_{i < j} a_i a_j$.

We will show that N^{2n+1} is not 3-formal. For that, firstly, $C^1 = N^1 = 0$, $C^2 = \langle a_1, \dots, a_{n-1} \rangle$, $N^2 = 0$, $C^3 = 0$ and $N^3 = \langle x_1, \dots, x_{n-1}, y \rangle$. As $H^{2n+1}(N) = \mathbb{Z}$, there is an element $\nu \in \bigwedge^{2n+1} V$ with $[\nu] \in H^{2n+1}(N)$ the generator. However, as C is generated by elements of even degree, it must be $\nu \in I(N)$ (actually, $\nu = a_1 \cdots a_{n-1} (y - \sum x_k a_k)$). So N^{2n+1} is not 3-formal. We now conclude that it is non-formal because it is not 3-formal.

Therefore, N^{2n+1} is non-formal and has a regular Sasakian structure. \square

4. SIMPLY CONNECTED COMPACT K -CONTACT MANIFOLDS WITH NO SASAKIAN STRUCTURE

In this section, we show that the higher Massey products rule out the possibility of existence of Sasakian structures on a compact manifold. Moreover, using such an obstruction, we exhibit a new method to construct simply connected K -contact non-Sasakian manifolds.

We will now recall the notions of symplectic and contact fatness developed by Sternberg and Weinstein in the symplectic setting, [33], [36], and by Lerman in the contact case [25], [26]. Let

$$G \longrightarrow P \longrightarrow B$$

be a principal G -bundle on B equipped with a connection. Let θ and Θ respectively be the connection one-form and the corresponding curvature 2-form on P . Both forms have values in the Lie algebra \mathfrak{g} of the group G . Denote the natural pairing between \mathfrak{g} and its dual \mathfrak{g}^* by $\langle \cdot, \cdot \rangle$. By definition, a vector $u \in \mathfrak{g}^*$ is *fat* if the 2-form

$$(X, Y) \longrightarrow \langle \Theta(X, Y), u \rangle$$

is nondegenerate for all horizontal vectors X, Y . Note that if u is fat, then each element of the coadjoint orbit of u is fat.

Let (M, η) be a contact co-oriented manifold endowed with a contact action of a Lie group G . Define a *contact moment map* by the formula

$$\mu_\eta : M \longrightarrow \mathfrak{g}^*, \quad \langle \mu_\eta(x), X \rangle = \eta_x(X_x^*),$$

for any $x \in M$ and any $X \in \mathfrak{g}$, where X^* denotes the fundamental vector field on M generated by $X \in \mathfrak{g}$ using the action of G on M . Note that the moment map depends on the contact form. The theorem below is due to Lerman.

Theorem 4.1 ([26]). *Let (F, η) be a contact manifold equipped with an action of G that preserves η , and let ν be a contact moment map on F . Let*

$$G \longrightarrow P \longrightarrow M$$

be a principal G -bundle endowed with a connection such that the image $\nu(F) \subset \mathfrak{g}^$ consists of fat vectors. Then there exists a fiberwise contact structure on the total space of the associated bundle*

$$F \longrightarrow P \times^G F \longrightarrow M.$$

If the fiber (F, η) is K -contact, and G preserves the K -contact structure, then the total space $P \times^G F$ is also K -contact.

The second part of Theorem 4.1 yields an explicit construction of a fibered K -contact structure on a fiber bundle and it will be our tool to prove Theorem 4.9.

Let (M, ω) be a symplectic manifold such that the cohomology class $[\omega]$ is integral. Consider the principal S^1 -bundle

$$\pi : P \longrightarrow M$$

given by the cohomology class $[\omega] \in H^2(M, \mathbb{Z})$. Fibrations of this kind were first considered in [4] and are called *Boothby-Wang fibrations*. By [23], the total space P carries an S^1 -invariant contact form θ such that θ is a connection form whose curvature is $\pi^*\omega$. This

implies that the moment map is constant and nonzero. Moreover, by [36], a principal S^1 -bundle is fat if and only if it is a Boothby-Wang fibration. Therefore, we have the following:

Theorem 4.2. *Let*

$$S^1 \longrightarrow P \longrightarrow M$$

be a Boothby-Wang fibration. Let (F, η) be a contact manifold endowed with an S^1 -action preserving the contact form η . Then the associated fiber bundle

$$F \longrightarrow P \times^{S^1} F \longrightarrow M$$

admits a fiberwise contact form. If (F, η) is K -contact, then the same is valid for the fiberwise contact structure on $P \times^{S^1} F$.

In order to prove that the higher order Massey products are zero for any compact Sasakian manifold, we proceed as follows. Let (M, η, ξ, ϕ, g) be a compact Sasakian manifold. As in Section 3, we denote by $\Omega_B^k(M)$ the space of all basic k -forms on M , and by d_B the restriction of the de Rham differential d to $\Omega_B^*(M)$. As before, $H_B^*(M)$ denotes the basic cohomology of M . Let $[d\eta]_B \in H_B^2(M)$ be the basic cohomology class of $d\eta$, where η is the above contact form (note that $d\eta$ is d_B -closed but not d_B -exact). Tievsky in [34], motivated by the approach in [10] and using the basic cohomology of M , determines a model for M , that is, a DGA with the same minimal model as the manifold M .

Theorem 4.3 ([34]). *Let (M, η, ξ, ϕ, g) be a compact Sasakian manifold. Then a model for M is given by the DGA $(H_B^*(M) \otimes \Lambda(x), D)$, where $|x| = 1$, $D(H_B^*(M)) = 0$ and $Dx = [d\eta]_B$. Therefore, $(H_B^*(M) \otimes \Lambda(x), D)$ is an elementary extension of the DGA $(H_B^*(M), 0)$ (the differential is zero).*

Theorem 4.4. *Let M be a compact Sasakian manifold. Then, all the higher order Massey products for M are zero.*

Before proving Theorem 4.4, we recall the definition of the *hard Lefschetz property* and prove Proposition 4.5 below.

Let $(\mathcal{A} = \bigoplus_{i=0}^{2n} A^i, 0)$ be a graded differential algebra with a non-zero element $\omega \in A^2$. For every $0 \leq k \leq n$, define the Lefschetz map

$$L_\omega : A^{n-k} \longrightarrow A^{n+k}, \quad \beta \longmapsto \beta \cdot \omega^{n-k}.$$

We say that \mathcal{A} satisfies the hard Lefschetz property if L_ω is an isomorphism for every $0 \leq k \leq n$.

Proposition 4.5. *Let $(\mathcal{A} = \bigoplus_{i=0}^{2n} A^i, 0)$ be a differential graded commutative algebra, and let $\omega \in A^2$ be a nondegenerate element, such that the hard Lefschetz property with respect to ω holds. Consider the elementary extension $(\mathcal{A} \otimes \Lambda(y), d)$ of $(\mathcal{A}, 0)$, where $\deg(y) = 1$ and $dy = \omega$. Then the higher order Massey products $\langle a_1, \dots, a_m \rangle$, where $a_i \in H^*(\mathcal{A} \otimes \Lambda(y), d)$ and $m \geq 4$, all vanish.*

Proof. Since $\mathcal{A} = \bigoplus_{i=0}^{2n} A^i$ satisfies the hard Lefschetz property with respect to ω , the map

$$L_\omega^{n-p} : A^p \longrightarrow A^{2n-p}, \quad \alpha \longmapsto \alpha \cdot \omega^{n-p}$$

is an isomorphism for all $p \leq n - 1$. Let $a \in H^k(\mathcal{A} \otimes \bigwedge(y), d)$. Then

$$a = [\alpha + \beta \cdot y],$$

where $\alpha, \beta \in \mathcal{A}$ and $d(\alpha + \beta \cdot y) = \bar{\beta} \cdot \omega = 0$. Here, $\bar{\beta}$ stands for $(-1)^{\deg \beta} \beta$, and so on. If $\deg a \leq n$, then $\deg \beta \leq n - 1$. By the injectivity of the map $L_\omega : A^p \rightarrow A^{p+2}$ for $p \leq n - 1$, we have $\beta = 0$. Thus we have $a = [\alpha]$. However, if $\deg a \geq n + 1$, then $\deg \alpha \geq n + 1$. By the surjectivity of

$$L_\omega : A^{q-2} \rightarrow A^q$$

for $q \geq n + 1$, we have that $\alpha = \gamma \cdot \omega = d(\bar{\gamma} \cdot y)$. Hence $a = [\beta \cdot y]$.

Let us consider a higher order Massey product

$$\langle a_1, \dots, a_m \rangle,$$

where $a_i \in H^*(\mathcal{A} \otimes \bigwedge(y), d)$ and $m \geq 4$. The degree of the Massey product $\langle a_1, \dots, a_m \rangle$ is $(\sum_{i=1}^m \deg a_i) - m + 2$. As

$$\left(\sum_{i=1}^m \deg a_i \right) - m + 2 \leq 2n + 1,$$

it follows that there is at most one a_i with $\deg a_i \geq n + 1$.

Suppose that the higher Massey product $\langle a_1, \dots, a_m \rangle$ is defined, and all $\deg a_i \leq n$. Set $a_i = [\alpha_i]$. According to Section 2, any element $b \in \langle a_1, \dots, a_m \rangle$ is a cohomology class in $H^*(\mathcal{A} \otimes \bigwedge(y), d)$, which is obtained by taking

$$\gamma_{ij} = \alpha_{ij} + \beta_{ij} \cdot y \in \mathcal{A} \otimes \bigwedge(y),$$

where $1 \leq i \leq j \leq m$ and $(i, j) \neq (1, m)$, such that $\gamma_{ii} = \alpha_{ii} = \alpha_i$ and

$$d\gamma_{ij} = \sum_{i \leq k \leq j-1} \bar{\gamma}_{ik} \cdot \gamma_{k+1, j}. \quad (4.1)$$

(Here it is used that we can fix the representatives α_i of the cohomology classes a_i [2].) Then, we have

$$b = \left[\sum_{1 \leq k \leq m-1} \bar{\gamma}_{1k} \cdot \gamma_{k+1, m} \right].$$

Note that the equations in (4.1) are equivalent to the following

$$\bar{\beta}_{ij} \cdot \omega = \sum_{i \leq k \leq j-1} \bar{\alpha}_{ik} \cdot \alpha_{k+1, j} \quad \text{and} \quad 0 = \sum_{i \leq k \leq j-1} (\bar{\alpha}_{ik} \cdot \beta_{k+1, j} - \bar{\beta}_{ik} \cdot \bar{\alpha}_{k+1, j}). \quad (4.2)$$

Now, we choose

$$\tilde{\gamma}_{ij} = \tilde{\alpha}_{ij} + \tilde{\beta}_{ij} \cdot y \in \mathcal{A} \otimes \bigwedge(y), \quad 1 \leq i \leq j \leq m, \quad (i, j) \neq (1, m),$$

as follows:

$$\tilde{\gamma}_{ii} = \tilde{\alpha}_{ii} = \alpha_i, \quad \tilde{\gamma}_{ij} = \tilde{\beta}_{ij} \cdot y = \beta_{ij} \cdot y \quad \text{for } j = i + 1, \quad \text{and } \tilde{\gamma}_{ij} = 0 \quad \text{for } j \geq i + 2.$$

Then, using (4.2), one can check that

$$\bar{\tilde{\beta}}_{ij} \cdot \omega = \sum_{i \leq k \leq j-1} \bar{\tilde{\alpha}}_{ik} \cdot \tilde{\alpha}_{k+1, j} \quad \text{and} \quad 0 = \sum_{i \leq k \leq j-1} (\bar{\tilde{\alpha}}_{ik} \cdot \tilde{\beta}_{k+1, j} - \bar{\tilde{\beta}}_{ik} \cdot \bar{\tilde{\alpha}}_{k+1, j}).$$

This means that the elements $\tilde{\gamma}_{ij}$ satisfy

$$d\tilde{\gamma}_{ij} = \sum_{i \leq k \leq j-1} \overline{\tilde{\gamma}_{ik}} \cdot \tilde{\gamma}_{k+1,j},$$

and so they define an element $\tilde{b} \in \langle a_1, \dots, a_m \rangle$ given by

$$\tilde{b} = \left[\sum_{1 \leq k \leq m-1} \overline{\tilde{\gamma}_{1k}} \cdot \tilde{\gamma}_{k+1,m} \right],$$

which is the zero cohomology class because every term of $\sum_{1 \leq k \leq m-1} \overline{\tilde{\gamma}_{1k}} \cdot \tilde{\gamma}_{k+1,m}$ vanishes.

To complete the proof of the proposition, suppose that the higher Massey product $\langle a_1, \dots, a_m \rangle$ is defined but there is one cohomology class a_t with $1 \leq t \leq m$ and $\deg a_t \geq n+1$. Then $a_i = [\alpha_i]$ for $i \neq t$, and $a_t = [\beta_t \cdot y]$. So a cohomology class $b \in \langle a_1, \dots, a_m \rangle$ is given by taking $\gamma_{ij} = \alpha_{ij} + \beta_{ij} \cdot y$, where $1 \leq i \leq j \leq m$ and $(i, j) \neq (1, m)$, such that

$$\gamma_{ii} = \alpha_{ii} = \alpha_i, \quad \text{for } i \neq t, \quad \gamma_{tt} = \beta_{tt} \cdot y = \beta_t \cdot y, \quad \text{and} \quad d\gamma_{ij} = \sum_{i \leq k \leq j-1} \overline{\gamma_{ik}} \cdot \gamma_{k+1,j}.$$

Thus, we have $b = \left[\sum_{1 \leq k \leq m-1} \overline{\gamma_{1k}} \cdot \gamma_{k+1,m} \right]$. Again,

$$d\gamma_{ij} = \sum_{i \leq k \leq j-1} \overline{\gamma_{ik}} \cdot \gamma_{k+1,j}$$

are equivalent to

$$\overline{\beta_{ij}} \cdot \omega = \sum_{i \leq k \leq j-1} \overline{\alpha_{ik}} \cdot \alpha_{k+1,j} \quad \text{and} \quad 0 = \sum_{i \leq k \leq j-1} (\overline{\alpha_{ik}} \cdot \beta_{k+1,j} - \overline{\beta_{ik}} \cdot \overline{\alpha_{k+1,j}}).$$

In particular,

$$\beta_{t-1,t} \cdot \omega = \beta_{t,t+1} \cdot \omega = 0 \quad \text{and} \quad \alpha_{t-1} \cdot \beta_t = 0,$$

since $\alpha_t = 0 = \beta_{t-1}$.

Now, we consider the elements $\tilde{\gamma}_{ij} = \tilde{\alpha}_{ij} + \tilde{\beta}_{ij} \cdot y \in \mathcal{A} \otimes \Lambda(y)$, $1 \leq i \leq j \leq m$ and $(i, j) \neq (1, m)$, given by

$$\begin{aligned} \tilde{\gamma}_{ii} &= \tilde{\alpha}_{ii} = \alpha_i, \quad \text{for } i \neq t, & \tilde{\gamma}_{tt} &= \tilde{\beta}_{tt} \cdot y = \beta_t \cdot y, \\ \tilde{\gamma}_{ij} &= \tilde{\beta}_{ij} \cdot y = \beta_{ij} \cdot y, \quad \text{for } j = i+1 \quad \text{and} \quad i \neq t-1, t, \\ \tilde{\gamma}_{t-1,t} &= \tilde{\gamma}_{t,t+1} = 0, \\ \tilde{\gamma}_{i,j} &= 0, \quad \text{for } j \geq i+2. \end{aligned}$$

Then, $\overline{\tilde{\beta}_{ij}} \cdot \omega = \sum_{i \leq k \leq j-1} \overline{\tilde{\alpha}_{ik}} \cdot \tilde{\alpha}_{k+1,j}$ and $0 = \sum_{i \leq k \leq j-1} (\overline{\tilde{\alpha}_{ik}} \cdot \tilde{\beta}_{k+1,j} - \overline{\tilde{\beta}_{ik}} \cdot \overline{\tilde{\alpha}_{k+1,j}})$. This defines the element

$$\tilde{b} = \left[\sum_{1 \leq k \leq m-1} \overline{\tilde{\gamma}_{1k}} \cdot \tilde{\gamma}_{k+1,m} \right],$$

which is the zero cohomology class because each term of $\sum_{1 \leq k \leq m-1} \overline{\tilde{\gamma}_{1k}} \cdot \tilde{\gamma}_{k+1,m}$ vanishes. \square

Remark 4.6. It should be mentioned that Proposition 4.5 is false without the assumption of the hard Lefschetz property for \mathcal{A} . This was realized after the referee produced a counterexample.

Proof of Theorem 4.4 : We know that Massey products on a manifold M can be computed by using any model for M . Now, let (M, η, ξ, ϕ, g) be a compact Sasakian manifold of dimension $2n + 1$. Put $\mathcal{A} = H_B^*(M)$ the basic cohomology of M . From Theorem 4.3 we know that $(\mathcal{A} \otimes \wedge(x), D)$, with $|x| = 1$, $D(\mathcal{A}) = 0$ and $Dx = [d\eta]_B$, is a model for M .

Since (M, η, ξ, ϕ, g) is a Sasakian manifold, there is the orthogonal decomposition of the tangent bundle TM of M

$$TM = \mathcal{D} \oplus \mathcal{L},$$

where \mathcal{L} is the trivial line subbundle generated by ξ , and the transversal foliation \mathcal{D} is Kähler with respect to $\omega = [d\eta]_B \in H_B^2(M)$. Then, we can apply El Kacimi's basic hard Lefschetz Theorem for transversally Kähler foliations \mathcal{F} on a compact manifold N such that $H_B^{\text{cod}\mathcal{F}}(N) \neq 0$. (See [12] for a more complete statement of the theorem.) This implies that $(\mathcal{A}, 0)$ satisfies the hard Lefschetz property with respect to ω and so, by Proposition 4.5, all higher Massey products of the model $(\mathcal{A} \otimes \wedge(x), D)$ are zero. \square

Proposition 4.7. *In the set-up of Proposition 4.5, all m^{th} order a -Massey products $\langle a; b_1, \dots, b_m \rangle$ of $(\mathcal{A} \otimes \wedge(y), d)$, where $m \geq 3$, are zero.*

Proof. Suppose that the a -Massey product $\langle a; b_1, \dots, b_m \rangle$ is defined, where a and b_i are closed elements in $(\mathcal{A} \otimes \wedge(y), d)$, $\deg a = \text{even}$ and $m \geq 3$. Then, the degree of $\langle a; b_1, \dots, b_m \rangle$ is

$$\deg \langle a; b_1, \dots, b_m \rangle = (m-1) \deg a + \left(\sum_{i=1}^m \deg b_i \right) - m + 1 \leq 2n + 1.$$

This implies $\deg a \leq n$ and there is at most one b_i with $\deg b_i \geq n + 1$.

By Lemma 2.5, the a -Massey product $\langle a; b_1, \dots, b_m \rangle$ only depends on the cohomology classes $[a]$, $[b_i]$ and not on the particular elements representing these classes.

If all $\deg b_i \leq n$, then the Lefschetz property for (\mathcal{A}, ω) implies that not only $a \in \mathcal{A}$ but also $b_i \in \mathcal{A}$, for all $1 \leq i \leq m$. According to section 2, an element $c \in \langle a; b_1, \dots, b_m \rangle$ is a cohomology class in $H^*(\mathcal{A} \otimes \wedge(y), d)$, which is given by taking $\xi_i = \mu_i + \nu_i \cdot y \in \mathcal{A} \otimes \wedge(y)$, where $1 \leq i \leq m$, such that $d\xi_i = a \cdot b_i$. Then, we have

$$c = \left[\sum_i (-1)^{|\xi_1| + \dots + |\xi_{i-1}|} \xi_1 \cdot \dots \cdot \xi_{i-1} \cdot b_i \cdot \xi_{i+1} \cdot \dots \cdot \xi_m \right].$$

Now, from the elements ξ_i , we choose $\tilde{\xi}_i = \tilde{\mu}_i + \tilde{\nu}_i \cdot y \in \mathcal{A} \otimes \wedge(y)$, $1 \leq i \leq m$, given by

$$\tilde{\xi}_i = \tilde{\nu}_i \cdot y = \nu_i \cdot y.$$

Since $\mu_i \in \mathcal{A}$, we have $d\mu_i = 0$. So $d\tilde{\xi}_i = d(\nu_i \cdot y) = d\xi_i = a \cdot b_i$. This means that the elements $\tilde{\xi}_i$ define the element

$$\tilde{c} = \left[\sum_i (-1)^{|\xi_1| + \dots + |\xi_{i-1}|} \tilde{\xi}_1 \cdot \dots \cdot \tilde{\xi}_{i-1} \cdot b_i \cdot \tilde{\xi}_{i+1} \cdot \dots \cdot \tilde{\xi}_m \right] = 0,$$

because each term of $\sum_i (-1)^{|\xi_1| + \dots + |\xi_{i-1}|} \tilde{\xi}_1 \cdot \dots \cdot \tilde{\xi}_{i-1} \cdot b_i \cdot \tilde{\xi}_{i+1} \cdot \dots \cdot \tilde{\xi}_m$ vanishes.

Suppose that $\langle a; b_1, \dots, b_m \rangle$ is defined but there is one b_t ($1 \leq t \leq m$) with $\deg b_t \geq n + 1$. Thus, $a, b_i \in \mathcal{A}$, for $i \neq t$, and $b_t = \beta_t \cdot y$. So a cohomology class $c \in \langle a; b_1, \dots, b_m \rangle$

is given by taking $\xi_i = \mu_i + \nu_i \cdot y \in \mathcal{A} \otimes \wedge(y)$, where $1 \leq i \leq m$, such that $d\xi_i = a \cdot b_i$. Then,

$$c = \left[\sum_i (-1)^{|\xi_1| + \dots + |\xi_{i-1}|} \xi_1 \cdot \dots \cdot \xi_{i-1} \cdot b_i \cdot \xi_{i+1} \cdot \dots \cdot \xi_m \right].$$

We choose $\tilde{\xi}_i = \tilde{\mu}_i + \tilde{\nu}_i \cdot y \in \mathcal{A} \otimes \wedge(y)$, $1 \leq i \leq m$, as follows

$$\tilde{\xi}_i = \tilde{\nu}_i \cdot y = \nu_i \cdot y, \quad \text{for } i \neq t, \quad \tilde{\xi}_t = 0.$$

Hence, $\tilde{\xi}_i$ satisfy $d\tilde{\xi}_i = a \cdot b_i$. For $i \neq t$, we have

$$d\xi_i = d\nu_i \cdot y = a \cdot b_i,$$

and for $i = t$ we have $d\tilde{\xi}_t = 0$, and

$$d\xi_t = a \cdot b_t = a \cdot \beta_t \cdot y.$$

As the image of d is contained in \mathcal{A} , it follows that $a \cdot \beta_t = 0$ and hence $a \cdot b_t = 0 = d\tilde{\xi}_t$. Therefore, the elements $\tilde{\xi}_i$ define the cohomology class $\tilde{c} \in \langle a; b_1, \dots, b_m \rangle$ given by $\tilde{c} = \left[\sum_i (-1)^{|\xi_1| + \dots + |\xi_{i-1}|} \tilde{\xi}_1 \cdot \dots \cdot \tilde{\xi}_{i-1} \cdot b_i \cdot \tilde{\xi}_{i+1} \cdot \dots \cdot \tilde{\xi}_m \right] = 0$. \square

Proposition 4.8. *All a -Massey products of a compact Sasakian manifold are zero.*

Proof. Let (M, η, ξ, ϕ, g) be a compact Sasakian manifold. As we noticed in the proof of Theorem 4.4, the differential graded commutative algebra $(\mathcal{A} = H_B^*(M), 0)$, where $\omega = [d\eta]_B \in H_B^2(M)$, satisfies the hard Lefschetz property with respect to ω . Then, Proposition 4.7 implies that all a Massey products of the model $(\mathcal{A} \otimes \wedge(x), D)$ for M are zero. \square

Theorem 4.9. *Let M be a simply connected compact symplectic manifold of dimension $2k$ with an integral symplectic form ω . Assume that the quadruple Massey product in $H^*(M)$ is non-zero. There exists a sphere bundle*

$$S^{2m+1} \longrightarrow E \longrightarrow M,$$

for $m+1 > k$, such that the total space E is K -contact, but E does not admit any Sasakian structure.

Proof. Let

$$S^1 \longrightarrow P \longrightarrow M$$

be the principal S^1 -bundle corresponding to $[\omega] \in H^2(M, \mathbb{Z})$. Choose a unitary representation of S^1 in \mathbb{C}^{m+1} whose all weights are positive. Consider the S^{2m+1} -bundle

$$S^{2m+1} \longrightarrow E := P \times_{S^1} S^{2m+1} \longrightarrow M$$

associated to the principal S^1 -bundle P for this unitary representation. By Theorem 4.2 applied to S^{2m+1} , we obtain a fiberwise K -contact structure on the total space E (see also [19]). Clearly, there is an algebraic model of E of the form

$$(\mathcal{A}_{PL}(M), d) \longrightarrow (\mathcal{A}_{PL} \otimes \wedge(z), D) \longrightarrow (\wedge(z), 0),$$

where $(\wedge(z), 0)$ is the minimal model of S^{2m+1} with one odd generator z of degree $2m+1$. Note that by the degree reasons, $D(z) = 0$. Indeed, $D(z)$ represents a cohomology class in $H^{2m+2}(M)$ (it is the Euler class of the corresponding vector bundle). Since

$2m + 2 > 2k = \dim M$, this class has to be zero. But this means that $\mathcal{A}_{PL}(E)$ must be weakly equivalent $\mathcal{A}_{PL}(M \times S^{2m+1})$ (see [13, p. 202, Example 4]). It now follows that

$$(\mathcal{A}_{PL}(M) \otimes \bigwedge(z), D) \simeq (\mathcal{A}_{PL}(M), d) \otimes (\bigwedge(z), 0),$$

and the latter is a model of E . Assume now that E is Sasakian. By Theorem 4.4, all higher order Massey products in $H^*(E)$ must be zero. But this contradicts the assumption in the theorem because $(\mathcal{A}_{PL}(M), d) \otimes (\bigwedge(z), 0)$ is a model of E . Therefore, E is not Sasakian. \square

Remark 4.10. Theorem 4.9 can also be proved using the differential algebra $(\Omega^*(M), d)$ of the de Rham forms instead of $\mathcal{A}_{PL}(M)$. We prefer the latter because it enables us to use citation of [13] directly.

Examples 4.11.

- (1) It is proved in [2] that there exist simply connected compact symplectic manifolds with non-zero quadruple Massey products. Therefore, any such manifold M is a base of some sphere bundle which is K -contact but non-Sasakian.
- (2) There exists an 8-dimensional simply connected compact symplectic manifold M with a non-zero triple a -Massey product. It was constructed in [16]. One can easily modify Theorem 4.9, assuming that the base M is symplectic, simply connected and possesses a non-zero triple a -Massey product. In this way one obtains a 17-dimensional K -contact non-Sasakian simply connected compact manifold. More such manifolds can be constructed using results of [7], since the property of having non-zero a -Massey products is inherited by symplectic blow-ups and symplectic resolutions.

5. K -CONTACT AND SASAKIAN GROUPS

Following Boyer and Galicki [5] we will call a group Γ to be K -contact if it can be realized as the fundamental group of a compact K -contact manifold, and we will call Γ to be a *Sasakian group* if there exists a compact Sasakian manifold M with $\pi_1(M) \cong \Gamma$. In [5] the authors pose a problem of realizing finitely presented groups as fundamental groups of K -contact or Sasakian manifolds. Since Sasakian manifolds constitute an odd-dimensional counterpart of the class of Kähler manifolds, it is natural to expect that not all finitely presented groups are Sasakian.

On the other hand, we will show in this section that any finitely presented group is K -contact.

Using the analogy between Kähler and Sasakian manifolds, we ask the following question: *Can lattices in semisimple Lie groups be Sasakian?*

We show that the above question is related to the *orbifold fundamental groups* of Kähler orbifolds. Some restrictions of Sasakian groups were found in [8]. In particular, it was proved that some lattices in semisimple Lie groups cannot be fundamental groups of regular Sasakian manifolds. Here we strengthen this result showing that such groups cannot even be Sasakian.

Proposition 5.1. *Let X be any compact connected symplectic manifold. There exists a Boothby-Wang fibration corresponding to some integral multiple of the given symplectic*

form

$$S^1 \longrightarrow P \longrightarrow X$$

such that the total space M of the associated fiber bundle

$$S^3 \longrightarrow M := P \times^{S^1} S^3 \longrightarrow X$$

admits a K -contact structure, and also $\pi_1(M) \cong \pi_1(X)$.

Proof. We know that Boothby-Wang fibrations are fat with the moment map having nonzero values. The image of the moment map consists of fat vectors (see Theorem 4.2). Moreover, S^3 is K -contact, and the Hopf S^1 -action preserves the K -contact structure; this is straightforward, but one can also use a general description of K -contact manifolds given in [5, Chapter 7] or in [3]. By Lerman's Theorem 4.1, any Boothby-Wang fibration yields an associated bundle whose total space $M = P \times^{S^1} S^3$ admits a K -contact structure as well. Clearly, we have $\pi_1(M) \cong \pi_1(X)$. \square

Theorem 5.2. *Any finitely presented group is K -contact.*

Proof. A well-known result of Gompf [17] shows that any finitely presentable group Γ can be realized as the fundamental group of some closed symplectic manifold of dimension ≥ 4 . Therefore, the theorem follows using Proposition 5.1. \square

We now recall a theorem of Margulis from [27].

Theorem 5.3 ([27]). *Let \mathcal{G} be a connected semisimple Lie group of rank at least two and with no co-compact factors. Let $\Gamma \subset \mathcal{G}$ be an irreducible arithmetic lattice in \mathcal{G} , and let Σ be a normal subgroup of Γ . Then either $\Sigma \subset Z(\mathcal{G})$ (the center of \mathcal{G}), or Γ/Σ is finite.*

Proposition 5.4. *Let Γ be an irreducible arithmetic lattice in a semisimple real Lie group \mathcal{G} of rank at least two with no co-compact factors and with trivial center. If Γ is Sasakian, then it must be isomorphic to the group $\pi_1^{orb}(M)$ of some Kähler orbifold. Moreover, Γ cannot be a co-compact arithmetic lattice in $SO(1, n)$, $n > 2$, or $F_{4(20)}$, or a simple real non-Hermitian Lie group of noncompact type with real rank at least 20.*

Proof. Assume that Γ is a Sasakian group. Let $\Gamma \cong \pi_1(X)$, where X is a compact quasi-regular Sasakian manifold. We know that X is a rational circle bundle over a compact Kähler orbifold M . Let

$$S^1 \hookrightarrow X \xrightarrow{\pi} M$$

be this orbi-bundle. By [5, Theorem 4.3.18], for any orbi-bundle obtained by an action of a Lie group G

$$G \longrightarrow X \longrightarrow M,$$

there is a long homotopy exact sequence

$$\cdots \longrightarrow \pi_i(G) \longrightarrow \pi_i^{orb}(X) \longrightarrow \pi_i^{orb}(M) \longrightarrow \pi_{i-1}(G) \longrightarrow \cdots$$

Since in our case $G = S^1$, we have the exact sequence

$$\cdots \longrightarrow \mathbb{Z} \longrightarrow \Gamma \longrightarrow \Gamma' \longrightarrow \{1\},$$

where $\Gamma' = \pi_1^{orb}(M)$. Now by Theorem 5.3, we have $\Gamma \cong \pi_1^{orb}(M)$, because the image of \mathbb{Z} must be in the center of \mathcal{G} , which is trivial by the assumption on \mathcal{G} . Since there is also a surjection $\pi_1^{orb}(M) \rightarrow \pi_1(M)$, we get a surjection

$$h : \Gamma \rightarrow \pi_1(M).$$

Consider the locally symmetric Riemannian space $B = \Gamma \backslash \mathcal{G}/K$, where K is a maximal compact subgroup of \mathcal{G} . Then we have $\Gamma \cong \pi_1(B)$. Since the sectional curvature of B is non-positive, we have the following:

- (1) there is a harmonic map $f : X \rightarrow B$ such that $f_* : \pi_1(X) \rightarrow \pi_1(B)$ is an isomorphism (Eells-Sampson theorem [11]).
- (2) $f_*(\xi) = 0$ for the Reeb vector field ξ on X (see [30]).

The above statement (2) shows that f is constant on orbits of the Reeb vector field. The fibers of p are circles. Therefore, f factors through M , meaning, there exists $g : M \rightarrow B$ such that $g \circ p = f$. Therefore, on the level of fundamental groups one obtains $g_* \circ p_* = f_*$, which shows that there is also a surjection

$$g_* : \pi_1(M) \rightarrow \pi_1(B) = \Gamma.$$

Hence there is a sequence of two surjections

$$\Gamma \xrightarrow{h} \pi_1(M) \xrightarrow{g_*} \Gamma.$$

Recall that Γ is an arithmetic lattice in a semisimple Lie group with trivial center. It means that $g_* \circ h$ cannot have non-trivial kernel (again, by Theorem 5.3). Hence, h cannot have a non-trivial kernel as well, and, therefore, it must be an isomorphism. So $\pi_1(M) \cong \Gamma$.

It is known that the fundamental group of the topological space underlying a Kähler orbifold is Kähler. Indeed, a resolution of the singularity of the underlying space of a Kähler orbifold is a compact Kähler manifold. On the other hand, by [24, p. 203, Theorem (7.5.2)], this resolution of singularity gives an isomorphism of fundamental groups because the singularities are quotient of a smooth variety by the action of a finite group. Therefore, the fundamental group of the topological space underlying a Kähler orbifold is Kähler.

Thus, Γ must be Kähler. But the latter is impossible, by a result of Carlson and Hernández [6]. \square

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