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Linear nonlocal diffusion problems in metric measure spaces

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Abstract

The aim of this paper is to provide a comprehensive study of some linear nonlocal diffusion problems in metric measure spaces. These include, for example, open subsets in $\mathbb{R}^N$, graphs, manifolds, multi-structures or some fractal sets. For this, we study regularity, compactness, positiveness and the spectrum of the stationary nonlocal operator. Then we study the solutions of linear evolution nonlocal diffusion problems, with emphasis in similarities and differences with the standard heat equation in smooth domains. In particular prove weak and strong maximum principles and describe the asymptotic behaviour using spectral methods.

1 Introduction

Diffusion is the natural process by which some magnitude (heat or matter, for example) is transported from one part of a system to another as a result of random molecular motions. As such, diffusion has a prominent role in distinct fields such as biology, thermodynamics and even economics.

In smooth media (e.g. an open region in the Euclidean space or a smooth manifold) classical diffusion models include differential operator such as the Laplacian and diffusion problems are usually described in terms of partial differential equations [12]. As the real world is nonsmooth, in the last decade there has been great effort in developing similar

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techniques and structures from the realm of differential equations to analyze diffusion processes in nonsmooth media, including some fractal like sets, see e.g. [5, 26, 20].

There is another approach, however, that allows to describe and model diffusion processes by means of nonlocal models, see e.g. [2], which we apply here in smooth and nonsmooth media. Assume then that (Ω, μ) is a measure space and $u(x,t)$ is the density of some population at the point $x \in \Omega$ at time t. Also assume $J(x,y)$ is a nonnegative function defined in $\Omega \times \Omega$, that represents the density of probability of a member of that population to jump from a location y to x. Hence $\int_\Omega J(y,x)dy = 1$ for all $x \in \Omega$. Then $\int_\Omega J(x,y)u(y,t)dy$ is the rate at which the individuals arrive to location x from all other locations $y \in \Omega$. On the other hand, $-\int_\Omega J(y,x)dy\, u(x,t) = -u(x,t)$ is the rate at which the individuals are leaving from location x to all other locations $y \in \Omega$. Then, the time evolution of the population u in Ω can be written as

$$\begin{cases} u_t(x,t) &= \displaystyle\int_\Omega J(x,y)u(y,t)dy - u(x,t), \quad x \in \Omega, \\ u(x,0) &= u_0(x), \qquad\qquad\qquad\qquad\quad x \in \Omega. \end{cases} \tag{1.1}$$

where u_0 is the initial distribution of the population. This problem and variations of it have been previously used to model diffusion processes, in [2], [8], [13], and [18], for example, with Ω and open set in $\mathbb{R}^N$. However, nonlocal diffusion models like (1.1) can be naturally defined in measure spaces, since we just need to consider the density of probability of jumping from a location x in Ω to a location y in Ω, given by the function $J(x,y)$. This allows us studying diffusion processes in very different type of spaces, like: graphs, (which are used to model complicated structures in chemistry, molecular biology or electronics, or they can also represent basic electric circuits into digital computers), compact manifolds, multi-structures composed by several compact sets with different dimensions, (for example a dumbbell domain), or even some fractal sets as the Sierpinski gasket, [5, 21, 26]. Some of this spaces are introduced in Section 2.

Since it is always convenient to speak about continuity, in this work, we consider problems like (1.1) defined in metric measure spaces, (Ω, μ, d), which are defined as follows. For more information see [24].

Definition 1.1. *A* **metric measure space** (Ω, μ, d) *is a metric space* (Ω, d) *with a* σ*-finite, regular, and complete Borel measure* μ *in* Ω*, and that associates a finite positive measure to the balls of* Ω*.*

In this context, we take $X = L^p(\Omega)$, $1 \le p \le \infty$, or $X = \mathcal{C}_b(\Omega)$ and consider nonlocal diffusion problems of the form

$$\begin{cases} u_t(x,t) &= K_J u(x,t) - h(x)u(x,t), \quad x \in \Omega,\ t > 0, \\ u(x,t_0) &= u_0(x), \quad x \in \Omega, \end{cases} \tag{1.2}$$

where $u_0 \in X$, $h \in L^\infty(\Omega)$ or in $\mathcal{C}_b(\Omega)$, and the nonlocal diffusion operator $K_J u$ is given by

$$K_J u(x,t) = \int_\Omega J(x,y)u(y,t)dy.$$

We will not assume, unless otherwise made explicit, that $\int_\Omega J(x,y)dy = 1$. A particular case which we will pay attention below is when $h(x) = \int_\Omega J(x,y)dy$.

One of our main goals in this paper is to show some similarities and differences between (1.2) and solutions of the classical heat equation. We will show in particular that both models share positivity properties such as the strong maximum principle. However solutions of (1.2) do not smooth in time, except asymptotically as $t \to \infty$.

The paper is organized as follows. In Section 2 we present several metric measure spaces in which all the analysis carried out in this paper holds. Those include open sets of the euclidean space, graphs, compact manifolds, multi-structures (sets composed by several compact sets with different dimensions joint together) or even some fractal sets. In Section 3 we derive a comprehensive study of the linear operator $K_J - hI$. We will discuss in particular continuity and compactness in different function spaces, including the case of convolution-type operators.

We also study the positiveness of the diffusive operator K_J. Under the assumption

$$J(x, y) > 0 \text{ for all } x,\, y \in \Omega, \text{ such that } d(x, y) < R, \tag{1.3}$$

for some $R > 0$ and the geometric condition that Ω is R-connected (see Definition 3.9), we show that for a nonnegative nontrivial function z, the set of points in Ω where $K_J z$ is strictly positive is larger than that of z. This will also allow us to use Kreĭn-Rutman Theorem, (see [22]), to obtain that the spectral radius in $\mathcal{C}_b(\Omega)$ of the operator K_J is a positive simple eigenvalue, with a strictly positive eigenfunction associated. Condition (1.3) is also shown to be somehow optimal.

In the last part of Section 3 we study similar questions for the nonlocal operator $K_J - hI$, with $h \in L^\infty(\Omega)$. In particular, we derive Green's formulas in the spirit of [2] and characterize the spectrum, which is also shown to be independent of the function space.

In Section 4 we analyze the solutions of (1.2), as well as the monotonicity properties of the solutions. In particular we will show that (1.3) implies that (1.2) has a strong maximum principle.

We then show that although solutions of (1.2) do not regularize, because they carry the singularities of the initial data, there is a subtle asymptotic smoothness for large times. In particular the semigroup $S(t)$ of (1.2) is asymptotically smooth as in [16, p. 4].

Finally, using the techniques of Riesz projections and the fact that the spectrum is independent of the space, we are able to describe the asymptotic behavior of the solutions of (1.2).

2 Examples of metric measure spaces

In the following sections we will consider a general measure metric space (Ω, μ, d) as in Definition 1.1. Below we enumerate some examples to which we can apply the theory developed throughout this work.

• A SUBSET OF $\mathbb{R}^N$: Let Ω be a Lebesgue measurable set of $\mathbb{R}^N$ with positive measure. A particular case is the one in which Ω is an open subset of $\mathbb{R}^N$, which can be even $\Omega = \mathbb{R}^N$. We consider the metric measure space (Ω, μ, d) where $\Omega \subseteq \mathbb{R}^N$, μ is the Lebesgue measure on $\mathbb{R}^N$, and d is the Euclidean metric of $\mathbb{R}^N$.

• GRAPHS: We consider a non empty, connected and finite graph in $\mathbb{R}^N$ defined by $G = (V, E)$, where $V \subset \mathbb{R}^N$ is the finite set of *vertices*, and the *edge set* E, consists of a collection of Jordan curves

$$E = \left\{ \pi_j : [0,\, 1] \to \mathbb{R}^N \middle|\, j \in \{1,\, 2,\, 3, ...,\, n\} \right\}$$

where $\pi_j \in \mathcal{C}^1\left([0,1]\right)$ is injective with $\pi_j(0), \pi_j(1) \in V$. We identify the graph with its associated network.

$$G = \bigcup_{j=1}^{n} e_j = \bigcup_{j=1}^{n} \pi_j\left([0,\, 1]\right) \subset \mathbb{R}^N$$

and we assume that any two edges $e_j \neq e_h$ satisfy that the intersection $e_j \cap e_h$ is either empty, one vertex or two vertices.

We define the measure structure of this graph. The edges have associated the one dimensional Lebesgue measure. Hence a set $A \subset e_i$ is **measurable** if and only if $\pi_i^{-1}(A) \subset [0,1]$ is measurable, and for any measurable set $A \subset e_i$, we consider the measure μ_i, defined as

$$\mu_i(A) = \int_{\pi_i^{-1}(A)} \|\pi_i'(t)\| dt.$$

In particular, the length of the edge e_i is defined as the length of the curve π_i,

$$\mu_i(e_i) = \int_0^1 \|\pi_i'(t)\| dt. \tag{2.1}$$

Therefore, a set $A \subset G$ is **measurable** if and only if $A \cap e_i$ is measurable for every $i \in \{1,\, 2,\, 3, ...,\, n\}$, and its measure is given by

$$\mu_G(A) = \sum_{i=1}^{n} \mu_i(A \cap e_i).$$

With this, it turns out that a function $f : G \to \mathbb{R}$ is measurable if and only if $f_{|e_i} : e_i \to \mathbb{R}$ is measurable, if and only if $f \circ \pi_j : [0,1] \to \mathbb{R}$ is measurable.

For $1 \leq p < \infty$, we set $f \in L^p(G) = \prod_{i=1}^{n} L^p(e_i)$, with norm $\|f\|_{L^p(G)} = \sum_{i=1}^{n} \|f\|_{L^p(e_i)} < \infty$, where, $\|f\|_{L^p(e_i)} = \left(\int_0^1 |f(\pi_i(t))|^p \, \|\pi_i'(t)\| dt\right)^{1/p} = \left(\int_0^1 |f(\pi_i(\cdot))|^p \, d\mu_i\right)^{1/p}$. For $p = \infty$, $f \in L^\infty(G) = \prod_{i=1}^{n} L^\infty(e_i)$, with norm $\|f\|_{L^\infty(G)} = \max_{i=1,\ldots,n} \|f\|_{L^\infty(e_i)} < \infty$, where, $\|f\|_{L^\infty(e_i)} = \sup_{t\in[0,1]} |f(\pi_i(t))|$.

Now, we describe the metric associated to the graph. For $v,\, w \in G$ the **geodesic distance** from v to w, $d_g(v, w)$, is the length of the shortest path from v to w. This distance, d_g, defines the metric structure associated to the graph G. Observe that since the graph is connected, there always exists the path from v to w, and since the graph is finite the geodesic metric d_g is equivalent to euclidean metric in $\mathbb{R}^N$. With this, a continuous function $f : G \to \mathbb{R}$ has a norm $\|f\|_{\mathcal{C}(G)} = \max_{i=1,\ldots,n} \|f\|_{\mathcal{C}(e_i)} < \infty$, where $\|f\|_{\mathcal{C}(e_i)} = \sup_{t\in[0,1]} |f(\pi_i(t))|$.

Thus the graph defines a metric measure space (G, μ_G, d_g).

• Compact Manifolds: Let $\mathcal{M} \subset \mathbb{R}^N$ be a compact manifold that we define as follows. Let U be an open bounded set of $\mathbb{R}^d$, with $d \leq N$, and let $\varphi : U \to \mathbb{R}^N$ be an application such that it defines a diffeomorphism from $\overline{U}$ onto its image $\varphi(\overline{U})$. Then we define the compact manifold as $\mathcal{M} = \varphi(\overline{U})$.

A natural measure in $\mathcal{M}$ is the one for which, $A \subset \mathcal{M}$ is measurable if and only if $\varphi^{-1}(A) \subset \mathbb{R}^d$ is measurable. Hence for any measurable set $A \subset \mathcal{M}$, we define the measure μ as,

$$\mu(A) = \int_{\varphi^{-1}(A)} \sqrt{g}\, dx, \tag{2.2}$$

where $g = det(g_{ij})$ and $g_{ij} = \langle \frac{\partial \varphi}{\partial x_i}, \frac{\partial \varphi}{\partial x_j} \rangle$. Since the compact manifold $\mathcal{M} = \varphi(\overline{U}) \subset \mathbb{R}^N$ and $U \subset \mathbb{R}^d$, then the measure (2.2) is equal to the d-Hausdorff measure in $\mathbb{R}^N$ restricted to $\mathcal{M}$, (see [25, p. 48]).

To define a natural metric in $\mathcal{M}$, let $\ell(c)$ be the length of a curve, c, in $\mathbb{R}^N$ defined as in (2.1). Then we define the **geodesic distance** between two points p, q in the manifold $\mathcal{M}$ as

$$d_g(p, q) := \inf\{\ell(c) \mid c : [0, 1] \to \mathcal{M} \text{ smooth curve, } c(0) = p,\, c(1) = q\}.$$

Since $\mathcal{M} \subset \mathbb{R}^N$ is compact, the geodesic metric, d_g, and the euclidean metric of $\mathbb{R}^N$, d, are equivalent.

Thus we have the metric measure space $(\mathcal{M}, \mathcal{H}^d, d)$ where $\mathcal{H}^d$ is the d-dimensional Hausdorff measure in $\mathbb{R}^N$ and d_g is the geodesic metric, which is equivalent to the Euclidean metric of $\mathbb{R}^N$.

• Multi-structures: Now, we consider a multi-structure, composed by several compact sets with different dimensions. For example, we can think in a piece of plane joined to a curve that is joined to a sphere in $\mathbb{R}^N$, or we can think also in a dumbbell domain. Therefore, we are going to define an appropriate measure and metric for these multi-structures.

Consider a collection of metric measure spaces $\{(X_i, \mu_i, d_i)\}_{i\in\{1,\ldots,n\}}$, with its respective measures, μ_i, and metrics, d_i, defined as above. Moreover, we assume the measure spaces $\{(X_i, \mu_i)\}_{i\in\{1,\ldots,n\}}$ satisfy

$$\mu_i(X_i \cap X_j) = \mu_j(X_i \cap X_j) = 0,$$

for $i \neq j$, and $i, j \in \{1, \ldots, n\}$.

Then we define

$$X = \bigcup_{i\in\{1,\ldots,n\}} X_i,$$

and we say that $E \subset X$ is measurable if and only if $E \cap X_i$ is μ_i-measurable for all $i \in \{1, \ldots, n\}$. Moreover we define the measure μ_X as

$$\mu_X(E) = \sum_{i=1}^{n} \mu_i(E \cap X_i).$$

Now let us define the metric that we consider in X. We assume that $X_i \subset \mathbb{R}^N$ is compact for all $i \in \{1, \ldots, n\}$, and the metrics d_i associated to each X_i, are equivalent to the euclidean metric in $\mathbb{R}^N$. Therefore, the metric d that we consider for the multi-structure, is the euclidean metric in $\mathbb{R}^N$.

Thus, we have the metric measure space, (X, μ_X, d), which is called the direct sum of metric measure spaces (X_i, μ_i, d_i), $i \in \{1, \ldots, n\}$.

• Spaces with finite Hausdorff measure and geodesic distance: There exist examples of compact sets $F \subset \mathbb{R}^N$ of Hausdorff dimension $\mathcal{H}_{dim}(F) = s < N$ and finite s-Hausdorff measure, i.e., $\mathcal{H}^s(F) < \infty$, which are pathwise connected, with finite length paths. Some of this sets can be constructed as self-similar affine fractal sets, and such an example is provided by the Sierpinski gasket, see e.g. [21], [9] and [23].

For such sets, we can consider the metric measure space $(F, \mathcal{H}^s, d_g)$ where d_g is the geodesic metric which may not be equivalent to the euclidean metric in $\mathbb{R}^N$.

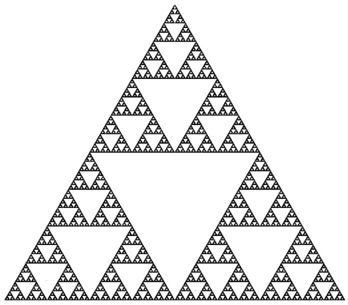

Figure 1: Sierpinski Gasket.

3 The linear nonlocal diffusion operator

Let (Ω, μ, d) be a metric measure space and consider a linear nonlocal diffusion operator of the form

$$K_J u(x) = \int_\Omega J(x,y)u(y)dy,$$

where the function J, defined in Ω as

$$\Omega \ni x \mapsto J(x,\cdot) \geq 0.$$

We will not assume, unless otherwise made explicit, that Ω has a finite measure nor that $\int_\Omega J(x,y)\,dy = 1$.

Hereafter for $1 \leq p \leq \infty$ we will denote by p' its conjugate exponent, that is, satisfying $1 = 1/p + 1/p'$. Notice that the dual space of $L^p(\Omega)$ is given by $(L^p(\Omega))' = L^{p'}(\Omega)$, for $1 \leq p < \infty$, and for $p = \infty$, $(L^\infty(\Omega))' = \mathcal{M}(\Omega)$, where $\mathcal{M}(\Omega)$ is the space of Radon measures, for more information see [14, chap. 7].

3.1 Properties of the operator K_J

We begin with the following result.

Proposition 3.1.

i. Assume $1 \leq p, q \leq \infty$ *and* $J \in L^q(\Omega, L^{p'}(\Omega))$. *Then* $K_J \in \mathcal{L}(L^p(\Omega), L^q(\Omega))$ *and the mapping* $J \longmapsto K_J$ *is linear and continuous, and*

$$\|K_J\|_{\mathcal{L}(L^p(\Omega),L^q(\Omega))} \leq \|J\|_{L^q(\Omega,L^{p'}(\Omega))}. \tag{3.1}$$

ii. *Assume* $1 \le p \le \infty$, $J \in L^\infty(\Omega, L^{p'}(\Omega))$ *and for any measurable set* $D \subset \Omega$ *satisfying* $\mu(D) < \infty$,

$$\lim_{x \to x_0} \int_D J(x,y)dy = \int_D J(x_0,y)dy, \qquad \forall x_0 \in \Omega. \tag{3.2}$$

Then $K_J \in \mathcal{L}(L^p(\Omega), \mathcal{C}_b(\Omega))$ *and the mapping* $J \longmapsto K_J$ *is linear and continuous, and*

$$\|K_J\|_{\mathcal{L}(L^p(\Omega),\mathcal{C}_b(\Omega))} \le \|J\|_{L^\infty(\Omega,\, L^{p'}(\Omega))}. \tag{3.3}$$

In particular, if $J \in \mathcal{C}_b(\Omega, L^{p'}(\Omega))$, *then* $K_J \in \mathcal{L}(L^p(\Omega), \mathcal{C}_b(\Omega))$, *and*

$$\|K_J\|_{\mathcal{L}(L^p(\Omega),\mathcal{C}_b(\Omega))} \le \|J\|_{\mathcal{C}_b(\Omega,\, L^{p'}(\Omega))}.$$

iii. *Assume* $\Omega \subset \mathbb{R}^N$ *is* **open**, $1 \le p, q \le \infty$, *and* $J \in W^{1,q}(\Omega, L^{p'}(\Omega))$. *Then* $K_J \in \mathcal{L}(L^p(\Omega), W^{1,q}(\Omega))$ *and the mapping* $J \longmapsto K_J$ *is linear and continuous, and*

$$\|K_J\|_{\mathcal{L}(L^p(\Omega),W^{1,q}(\Omega))} \le \|J\|_{W^{1,q}(\Omega,L^{p'}(\Omega))}. \tag{3.4}$$

Proof.
i. Thanks to Hölder's inequality, we have for $1 \le q < \infty$ and $1 \le p \le \infty$,

$$\begin{aligned}\|K_J u\|^q_{L^q(\Omega)} &= \int_\Omega \left| \int_\Omega J(x,y)u(y)\,dy \right|^q dx \\ &\le \|u\|^q_{L^p(\Omega)} \int_\Omega \|J(x,\cdot)\|^q_{L^{p'}(\Omega)} dx \quad = \|u\|^q_{L^p(\Omega)} \|J\|^q_{L^q\left(\Omega, L^{p'}(\Omega)\right)}.\end{aligned}$$

For $q = \infty$ and $1 \le p \le \infty$, for each $x \in \Omega$,

$$|K_J u(x)| = \left| \int_\Omega J(x,y)u(y)dy \right| \le \|u\|_{L^p(\Omega)} \|J(x,\cdot)\|_{L^{p'}(\Omega)},$$

and taking supremum in $x \in \Omega$, we obtain the result.
ii. Note that since $J \in L^\infty(\Omega, L^{p'}(\Omega))$, from part *i.* with $q = \infty$, we have that $K_J \in \mathcal{L}(L^p(\Omega), L^\infty(\Omega))$. Also note that the hypothesis (3.2) can also be written as

$$\lim_{x \to x_0} \int_\Omega J(x,y)\chi_D(y)dy = \int_\Omega J(x_0,y)\chi_D(y)dy, \qquad \forall x_0 \in \Omega,$$

where χ_D is the characteristic function of $D \subset \Omega$, with $\mu(D) < \infty$, which means that $K_J(\chi_D)$ is continuous and bounded in Ω. Since $\mu(D) < \infty$, then $\chi_D \in L^p(\Omega)$, for $1 \le p \le \infty$. Moreover, the space

$$V = span\,[\,\chi_D\,;\, D \subset \Omega \ \text{ with } \ \mu(D) < \infty]\,,$$

is dense in $L^p(\Omega)$, for $1 \le p \le \infty$ and $K_J : V \to \mathcal{C}_b(\Omega)$, and then

$$K_J(L^p(\Omega)) = K_J\left(\overline{V}\right) \subset \overline{K_J(V)} \subset \mathcal{C}_b(\Omega)$$

and we get (3.3).

In particular if $J \in C_b\big(\Omega, L^{p'}(\Omega)\big)$, then the hypothesis (3.2) is satisfied.

iii. As a consequence of Fubini's Theorem, and since Ω is open we have that for all $u \in L^p(\Omega)$ and $i = 1, \dots, N$, the weak derivative of $K_J u$ is given by

$$\begin{aligned}
\left\langle \frac{\partial}{\partial x_i} K_J u, \varphi \right\rangle &= -\langle K_J u, \partial_{x_i}\varphi\rangle = -\int_\Omega \int_\Omega J(x,y) u(y) \partial_{x_i}\varphi(x)\, dy\, dx \\
&= -\langle \langle J(\cdot, y), \partial_{x_i}\varphi\rangle, u\rangle = \langle \langle \partial_{x_i} J(\cdot, y), \varphi\rangle, u\rangle \\
&= \int_\Omega \int_\Omega \partial_{x_i} J(x,y) u(y)\, \varphi(x)\, dy\, dx = \langle K_{\frac{\partial J}{\partial x_i}} u, \varphi\rangle.
\end{aligned} \tag{3.5}$$

for all $\varphi \in \mathcal{C}_c^\infty(\Omega)$. Therefore

$$\frac{\partial}{\partial x_i} K_J u = K_{\frac{\partial J}{\partial x_i}} u. \tag{3.6}$$

Since $J \in W^{1,q}(\Omega, L^{p'}(\Omega))$, and from part *i.* and (3.6), we have that

$$\|K_J\|_{\mathcal{L}(L^p(\Omega), L^q(\Omega))} \le \|J\|_{L^q(\Omega, L^{p'}(\Omega))} \tag{3.7}$$

and for $i = 1, \dots, N$,

$$\left\| \frac{\partial}{\partial x_i} K_J \right\|_{\mathcal{L}(L^p(\Omega), L^q(\Omega))} = \left\| K_{\frac{\partial J}{\partial x_i}} \right\|_{\mathcal{L}(L^p(\Omega), L^q(\Omega))} \le \left\| \frac{\partial J}{\partial x_i} \right\|_{L^q(\Omega, L^{p'}(\Omega))}. \tag{3.8}$$

Hence, $K_J \in \mathcal{L}(L^p(\Omega), W^{1,q}(\Omega))$, for all $1 \le p, q \le \infty$ and from (3.7) and (3.8) we have (3.4). □

The following result collects cases in which $K_J \in \mathcal{L}(X, X)$, with $X = L^p(\Omega)$ or $X = \mathcal{C}_b(\Omega)$.

Corollary 3.2.

i. If $J \in L^p(\Omega, L^{p'}(\Omega))$ then $K_J \in \mathcal{L}(L^p(\Omega), L^p(\Omega))$.

ii. If $J \in \mathcal{C}_b(\Omega, L^1(\Omega))$ then $K_J \in \mathcal{L}(\mathcal{C}_b(\Omega), \mathcal{C}_b(\Omega))$.

iii. If $\mu(\Omega) < \infty$ and $J \in L^\infty(\Omega, L^\infty(\Omega))$ then $K_J \in \mathcal{L}(L^p(\Omega), L^p(\Omega))$, for all $1 \le p \le \infty$.

Proof. i. From Proposition 3.1 we have the result.

ii. If $J \in \mathcal{C}_b(\Omega, L^1(\Omega))$ then, thanks to the previous Proposition 3.1, K_J belongs to $\mathcal{L}(L^\infty(\Omega), \mathcal{C}_b(\Omega))$. Moreover, since $\mathcal{C}_b(\Omega) \subset L^\infty(\Omega)$, we have that $K_J \in \mathcal{L}(\mathcal{C}_b(\Omega), \mathcal{C}_b(\Omega))$.

iii. From Proposition 3.1 we have that $K_J \in \mathcal{L}(L^1(\Omega), L^\infty(\Omega))$. Moreover, since $\mu(\Omega) < \infty$,

$$L^p(\Omega) \hookrightarrow L^1(\Omega) \xrightarrow{K_J} L^\infty(\Omega) \hookrightarrow L^p(\Omega).$$

□

The particular case where the nonlocal diffusion term is given by a convolution in $\Omega = \mathbb{R}^N$ with a function $J_0 : \mathbb{R}^N \to \mathbb{R}$, i.e. $J(x,y) = J_0(x-y)$ and $K_J u = J_0 * u$, has been widely considered, e.g. [1, 3, 8, 11] and references therein. Hence, we consider here such type of operators. For this, let $\Omega \subset \mathbb{R}^N$ be a measurable set, (it can be $\Omega = \mathbb{R}^N$, or just a subset $\Omega \subset \mathbb{R}^N$) and consider the kernel

$$J(x,y) = J_0(x-y), \qquad x, y \in \Omega. \tag{3.9}$$

where J_0 is a function in $L^{p'}(\mathbb{R}^N)$, for $1 \leq p \leq \infty$, and the nonlocal operator

$$K_J u(x) = \int_\Omega J_0(x-y)u(y)dy, \quad x \in \Omega.$$

Straight from Proposition 3.1 we get the following.

Corollary 3.3. *For $1 \leq p \leq \infty$, let $\Omega \subseteq \mathbb{R}^N$ be a measurable set, $J_0 \in L^{p'}(\mathbb{R}^N)$ and J defined in (3.9). Then $K_J \in \mathcal{L}(L^p(\Omega), L^\infty(\Omega))$. In particular if $\mu(\Omega) < \infty$, then $K_J \in \mathcal{L}(L^p(\Omega), L^q(\Omega))$, for $1 \leq q \leq \infty$.*

Proof. If $J_0 \in L^{p'}(\mathbb{R}^N)$ then $J \in L^\infty(\Omega, L^{p'}(\Omega))$, since

$$\sup_{x\in\Omega} \|J(x,\cdot)\|_{L^{p'}(\Omega)} = \sup_{x\in\Omega} \|J_0(x-\cdot)\|_{L^{p'}(\Omega)} \leq \|J_0\|_{L^{p'}(\mathbb{R}^N)} < \infty.$$

Thus, thanks to Proposition 3.1, we have that $K_J \in \mathcal{L}(L^p(\Omega), L^\infty(\Omega))$. In particular, if $\mu(\Omega) < \infty$ then $K_J \in \mathcal{L}(L^p(\Omega), L^q(\Omega))$, for all $1 \leq q \leq \infty$. $\square$

On the other hand, if $\mu(\Omega) = \infty$, (as in the case of $\Omega = \mathbb{R}^N$), then K_J is not necessarily in $\mathcal{L}(L^p(\Omega), L^q(\Omega))$, for $q \neq \infty$. In the proposition below we prove the cases which **can not** be obtained as a consequence of Proposition 3.1.

Proposition 3.4. *With the notations above, let $\Omega \subseteq \mathbb{R}^N$ be a measurable set with $\mu(\Omega) = \infty$ and let $1 \leq p \leq \infty$.*

i. If $J_0 \in L^r(\mathbb{R}^N)$ and $\frac{1}{q} = \frac{1}{p} + \frac{1}{r} - 1$ then $K_J \in \mathcal{L}(L^p(\Omega), L^q(\Omega))$, and

$$\|K_J\|_{\mathcal{L}(L^p(\Omega), L^q(\Omega))} \leq \|J_0\|_{L^r(\mathbb{R}^N)}.$$

In particular, if $r = 1$ we can take $p = q$.

*ii. If $\Omega \subset \mathbb{R}^N$ is **open**, $J_0 \in W^{1,r}(\mathbb{R}^N)$ and $\frac{1}{q} = \frac{1}{p} + \frac{1}{r} - 1$ then $K_J \in \mathcal{L}(L^p(\Omega), W^{1,q}(\Omega))$, and*

$$\|K_J\|_{\mathcal{L}(L^p(\Omega), W^{1,q}(\Omega))} \leq \|J_0\|_{W^{1,r}(\mathbb{R}^N)}.$$

Proof. i. If u is defined in Ω, let us denoted by $\hat{u}$ the extension by zero of u to $\mathbb{R}^N$. Thus, we have for $x \in \Omega$

$$K_J u(x) = \int_\Omega J_0(x-y)u(y)dy = \int_{\mathbb{R}^N} J_0(x-y)\hat{u}(y)dy = (J_0 * \hat{u})(x).$$

Now, we define the extension of the operator K_J as

$$\widehat{K}_J u(x) = (J_0 * \hat{u})(x), \quad \text{for } x \in \mathbb{R}^N,$$

then $K_J u(x) = \left(\widehat{K}_J u\right)\Big|_\Omega (x)$, for $x \in \Omega$. Thanks to Young's inequality, see [7, p. 104], we have

$$\|K_J u\|_{L^q(\Omega)} \leq \|\widehat{K}_J u\|_{L^q(\mathbb{R}^N)} \leq \|J_0\|_{L^r(\mathbb{R}^N)} \|\hat{u}\|_{L^p(\mathbb{R}^N)} = \|J_0\|_{L^r(\mathbb{R}^N)} \|u\|_{L^p(\Omega)}.$$

Hence, $\|K_J u\|_{L^q(\Omega)} \le \|J_0\|_{L^r(\mathbb{R}^N)}\|u\|_{L^p(\Omega)}$, for all p, q, r such that $\frac{1}{q} = \frac{1}{p} + \frac{1}{r} - 1$.

ii. Following the same arguments made in Proposition 3.1 in (3.5), we know that for $x \in \Omega$,

$$\frac{\partial}{\partial_{x_i}} K_J u = K_{\frac{\partial J}{\partial x_i}} u = \left(\widehat{K}_{\frac{\partial J}{\partial x_i}} u \right)\Big|_{\Omega}$$

Then, applying part *i.* to $\|K_J u\|_{L^q(\Omega)}$ and $\|K_{\frac{\partial J}{\partial x_i}} u\|_{L^q(\Omega)}$ we have that for p, q, r such that $\frac{1}{q} = \frac{1}{p} + \frac{1}{r} - 1$, $K_J \in \mathcal{L}(L^p(\Omega), W^{1,q}(\Omega))$. Thus, the result. □

Now we prove that under the hypotheses on J in Proposition 3.1, the operator K_J is compact. For this we will use the following result.

Lemma 3.5. *For $1 \le q < \infty$ and $1 \le p \le \infty$, let (Ω, μ) be a measure space, then any function $H \in L^q(\Omega, L^{p'}(\Omega))$ can be approximated in $L^q(\Omega, L^{p'}(\Omega))$ by functions of separated variables.*

Then we have.

Proposition 3.6.

i. For $1 \le p \le \infty$ and $1 \le q < \infty$, if $J \in L^q(\Omega, L^{p'}(\Omega))$ then $K_J \in \mathcal{L}(L^p(\Omega), L^q(\Omega))$ is compact.

ii. For $1 \le p \le \infty$, if $J \in BUC(\Omega, L^{p'}(\Omega))$, then $K_J \in \mathcal{L}(L^p(\Omega), \mathcal{C}_b(\Omega))$ is compact. In particular, $K_J \in \mathcal{L}(L^p(\Omega), L^\infty(\Omega))$ is compact.

iii. For $1 \le p \le \infty$ and $1 \le q < \infty$, if $\Omega \subset \mathbb{R}^N$ is **open** *and $J \in W^{1,q}(\Omega, L^{p'}(\Omega))$ then $K_J \in \mathcal{L}(L^p(\Omega), W^{1,q}(\Omega))$ is compact.*

Proof. i. Since $J \in L^q(\Omega, L^{p'}(\Omega))$, for $1 \le p \le \infty$ and $1 \le q < \infty$, we know from Lemma 3.5 that there exist $M(n) \in \mathbb{N}$ and $f_j^n \in L^q(\Omega)$, $g_j^n \in L^{p'}(\Omega)$ with $j = 1, ..., M(n)$ such that $J(x, y)$ can be approximated by functions that are a finite linear combination of functions with separated variables defined as, $J^n(x, y) = \sum_{j=1}^{M(n)} f_j^n(x) g_j^n(y)$ and $\|J - J^n\|_{L^q(\Omega, L^{p'}(\Omega))} \to 0$, as n goes to ∞. Then define $K_{J^n}u(x) = \sum_{j=1}^{M(n)} f_j^n(x) \int_\Omega g_j^n(y)u(y)dy$. Thus, since $K_J - K_{J^n} = K_{J-J^n}$, thanks to Proposition 3.1, we have that,

$$\|K_J - K_{J^n}\|_{\mathcal{L}(L^p(\Omega), L^q(\Omega))} \le \|J - J^n\|_{L^q(\Omega, L^{p'}(\Omega))} \to 0, \quad \text{as } n \text{ goes to } \infty.$$

Since K_{J^n} has finite rank, then $K_J \in \mathcal{L}\left(L^p(\Omega), L^q(\Omega)\right)$ is compact, e.g. [7, p.157].

ii. If $J \in BUC(\Omega, L^{p'}(\Omega))$, then hypothesis (3.2) of Proposition 3.1 is satisfied and then $K_J \in \mathcal{L}(L^p(\Omega), \mathcal{C}_b(\Omega))$. Now, we consider $u \in B \subset L^p(\Omega)$, where B is the unit ball in $L^p(\Omega)$. Now, we prove using Ascoli-Arzela Theorem (see [7, p. 111]), that $K_J(B)$ is relatively compact in $\mathcal{C}_b(\Omega)$. Let $x, z \in \Omega$, $u \in B$, thanks to Hölder's inequality, we have,

$$|K_J u(z) - K_J u(x)| = \left| \int_\Omega \Big(J(z, y) - J(x, y) \Big) u(y) dy \right| \le \|J(z, \cdot) - J(x, \cdot)\|_{L^{p'}(\Omega)}.$$

Since $J \in BUC(\Omega, L^{p'}(\Omega))$, then for all $\varepsilon > 0$, there exists $\delta > 0$ such that if $x, z \in \Omega$ satisfy that $d(z,x) < \delta$, then $\|J(z,\cdot) - J(x,\cdot)\|_{L^{p'}(\Omega)} < \varepsilon$. Hence, we have that $K_J(B)$ is equicontinuous.

On the other hand, thanks to Hölder's inequality, for all $x \in \Omega$ and $u \in B$

$$|K_J u(x)| = \left| \int_\Omega J(x,y)u(y)dy \right| \le \|J(x,\cdot)\|_{L^{p'}(\Omega)} < \infty.$$

Thus, we have that $K_J(B)$ is precompact and therefore $K_J \in \mathcal{L}(L^p(\Omega), \mathcal{C}_b(\Omega))$ is compact. Also, the second part of the result is immediate.

iii. Thanks to the argument (3.5) in Proposition 3.1, we have that $\frac{\partial}{\partial x_i} K_J u = K_{\frac{\partial J}{\partial x_i}} u$. Since $J \in W^{1,q}(\Omega, L^{p'}(\Omega))$, we have that $J \in L^q(\Omega, L^{p'}(\Omega))$ and moreover $\frac{\partial J}{\partial x_i} \in L^q(\Omega, L^{p'}(\Omega))$, for all $i = 1, \ldots, N$. Using part *i.* we obtain that $K_{\frac{\partial J}{\partial x_i}} \in \mathcal{L}\left(L^p(\Omega),\, L^q(\Omega)\right)$ is compact. Thus, if B is the unit ball in $L^p(\Omega)$, we have that $K_J(B)$ and $K_{\frac{\partial J}{\partial x_i}}(B)$ are precompact for all $i = 1, \ldots, N$. Now we consider the mapping

$$\begin{array}{rccc} \mathcal{T}: & L^p(\Omega) & \longrightarrow & (L^q(\Omega))^{N+1} \\ & u & \longmapsto & \left(K_J u,\, K_{\frac{\partial J}{\partial x_1}} u, \ldots,\, K_{\frac{\partial J}{\partial x_N}} u \right). \end{array}$$

Thanks to Tíkhonov's Theorem, we know that $\mathcal{T}(B)$ is precompact in $(L^q(\Omega))^{N+1}$. Moreover, we consider the mapping

$$\begin{array}{rccc} \mathcal{S}: & W^{1,q}(\Omega) & \hookrightarrow & (L^q(\Omega))^{N+1} \\ & g & \longmapsto & \left(g, \frac{\partial g}{\partial x_1}, \ldots, \frac{\partial g}{\partial x_N} \right). \end{array}$$

Since $\mathcal{S}$ is an isometry, then we have that $\mathcal{S}^{-1}|_{Im(\mathcal{S})} : Im(\mathcal{S}) \subset (L^q(\Omega))^{N+1} \to W^{1,q}(\Omega)$ is continuous. On the other hand, thanks to the hypotheses on J and Proposition 3.1, we have that $K_J \in \mathcal{L}\left(L^p(\Omega),\, W^{1,q}(\Omega)\right)$. Thus, $Im(\mathcal{T}) \subset Im(\mathcal{S})$.
Hence, the operator $K_J : L^p(\Omega) \to W^{1,q}(\Omega)$, can be written as

$$K_J u = \mathcal{S}^{-1}|_{Im(\mathcal{S})} \circ \mathcal{T} u.$$

Therefore, we have that K_J is the composition of a continuous operator $\mathcal{S}^{-1}|_{Im(\mathcal{S})}$, with a compact operator $\mathcal{T}$. Thus, the result. □

Remark 3.7. *Observe that the assumptions in Proposition 3.6 are the same as in Proposition 3.1 except for the case* $K \in \mathcal{L}(L^p(\Omega), L^\infty(\Omega))$ *where we assume in the former that* $J \in BUC(\Omega, L^{p'}(\Omega))$*, instead of* $J \in L^\infty(\Omega, L^{p'}(\Omega))$ *as in the latter.*

Now we derive several consequences from interpolation. Note that the following result is valid for a general operator K, not necessarily an integral operator.

Proposition 3.8. *Let* (Ω, μ) *be a measure space, with* $\mu(\Omega) < \infty$*. Assume that for* $1 \le p_0 < p_1 < \infty$*,* $K \in \mathcal{L}(L^{p_0}(\Omega), L^{p_0}(\Omega))$ *and* $K \in \mathcal{L}(L^{p_1}(\Omega), L^{p_1}(\Omega))$*. Then* $K \in \mathcal{L}(L^p(\Omega), L^p(\Omega))$*, for all* $p \in [p_0,\, p_1]$*. Additionally, suppose that either:*

i. $K \in \mathcal{L}(L^{p_0}(\Omega), L^{p_0}(\Omega))$ *is compact, or*

ii. $K \in \mathcal{L}(L^{p_1}(\Omega), L^{p_1}(\Omega))$ *is compact .*

Then $K \in \mathcal{L}(L^p(\Omega), L^p(\Omega))$ *is compact for all* $p \in [p_0, p_1]$.

Proof. From Riesz-Thorin Theorem, (see [6, p. 196]), we have $K \in \mathcal{L}(L^p(\Omega), L^p(\Omega))$, for all $p \in [p_0, p_1]$. The proof of the compactness can be found in [10, p. 4]. □

Now we analyze positive preserving properties of nonlocal operators. For this we will need some positivity properties of the kernel J and some connectedness of Ω. To do this, we first introduce the following.

Definition 3.9. *Let* (Ω, μ, d) *be a metric measure space and* $R > 0$. *We say that* Ω *is* R-**connected** *if for any* $x, y \in \Omega$, *there exists a finite* R-*chain connecting* x *and* y. *By this we mean that there exist* $N \in \mathbb{N}$ *and a finite set of points* $\{x_0, \dots, x_N\}$ *in* Ω *such that* $x_0 = x$, $x_N = y$ *and* $d(x_{i-1}, x_i) < R$, *for all* $i = 1, \dots, N$.

Then we have the following.

Lemma 3.10. *If* Ω *is compact and connected then* Ω *is* R-*connected for any* $R > 0$.

Proof. We fix an arbitrary $x_0 \in \Omega$, and we define the increasing sequence of open sets

$$P^1_{R,x_0} = B(x_0, R) \quad \text{and} \quad P^n_{R,x_0} = \bigcup_{x \in P^{n-1}_{R,x_0}} B(x, R) \quad \text{for} \quad n \in \mathbb{N}. \tag{3.10}$$

Observe that P^n_{R,x_0} is the set of points in Ω that for which there exists an R-chain of n steps, connecting with x_0. Then $A = \bigcup_{n=1}^{\infty} P^n_{R,x_0}$ is open. Lets us show that it is also closed. In such a case since Ω is connected we would have $\Omega = A$ which implies that Ω is R–connected, since x_0 is arbitrary. Indeed if $y \in \Omega \setminus A$, then we claim the $B(y, R) \subset \Omega \setminus A$, since otherwise $B(y, R)$ would intersect some P^n_{R,x_0}, which implies that $y \in P^{n+1}_{R,x_0}$ which is absurd. □

With this, we get the following.

Lemma 3.11. *Let* (Ω, μ, d) *be a metric measure space such that* Ω *is* R-*connected. For any fixed* $x_0 \in \Omega$ *consider the sets* P^n_{R,x_0} *as in (3.10).*

Then, for every compact set in $\mathcal{K} \subset \Omega$, *there exists* $n(x_0) \in \mathbb{N}$ *such that* $\mathcal{K} \subset P^n_{R,x_0}$ *for all* $n \geq n(x_0)$.

Furthermore, if Ω *is compact, there exists* $n_0 \in \mathbb{N}$ *such that for any* $y \in \Omega$, $\Omega = P^n_{R,y}$, *for all* $n \geq n_0$.

Proof. Since Ω is R-connected, for any $y \in \Omega$, consider an R-chain connecting x_0 and y, $\{x_0, \dots, x_M\}$ such that $x_M = y$ and $d(x_{i-1}, x_i) < R$, for all $i = 1, \dots, M$. Thus, $x_1 \in B(x_0, R) = P^1_{R,x_0}$, $x_2 \in B(x_1, R) \subset P^2_{R,x_0}$, $B(x_i, R) \subset P^{i+1}_{R,x_0}$, for all $i = 1, \dots, M$. In particular, $y \in P^M_{R,x_0}$ and

$$B(y, R) \subset P^{M+1}_{R,x_0}. \tag{3.11}$$

On the other hand, since $\mathcal{K}$ is compact, $\mathcal{K} \subset \bigcup_{y \in \mathcal{K}} B(y, R)$, there exists $n \in \mathbb{N}$ and $y_i \in K$, such that $\mathcal{K} \subset \bigcup_{i=1}^{n} B(y_i, R)$. From (3.11), for every i there exists M_i such that

$B(y_i,R) \subset P^{M_i+1}_{R,x_0}$. We choose $n(x_0) = \max_{i=1,\dots,n}\{M_i+1\}$, and we obtain that $\mathcal{K} \subset P^{n(x_0)}_{R,x_0}$. Therefore, $\mathcal{K} \subset P^{n}_{R,x_0}$, for all $n \geq n(x_0)$. Thus, the result.

If Ω is compact, from the previous result we know that for a fixed $x_0 \in \Omega$, there exists $N = N(x_0)$ such that $\Omega = P^N_{R,x_0}$. Therefore, any two points in Ω are connected by an R-chain of N steps to x_0. Thus any two points in Ω are connected to each other by and R-chain of $2N$ steps. In other words $\Omega = P^n_{R,y}$, for all $n \geq 2N$ for all $y \in \Omega$. □

Now we define the *essential support* of a nonnegative measurable function.

Definition 3.12. *Let z be a measurable nonnegative function $z : \Omega \to \mathbb{R}$. We define the* **essential support** *of z as:*

$$P(z) = \big\{x \in \Omega \,:\, \forall \delta > 0,\ \mu\big(\{y \in \Omega :\ z(y) > 0\} \cap B(x,\delta)\big) > 0\big\},$$

where $B(x,\delta)$ is the ball centered in x, with radius δ.

It is not difficult to check that $z \geq 0$ not identically zero iff $P(z) \neq \emptyset$ which is equivalent to $\mu\,(P(z)) > 0$.

Let us introduce the following definitions.

Definition 3.13. *Let z be a measurable nonnegative function $z : \Omega \to \mathbb{R}$. Then we denote*

$$P^0(z) = P(z),$$

the essential support of z, and for any $R > 0$, we define the increasing sequence of open sets

$$P^1_R(z) = \bigcup_{x\in P^0(z)} B(x,R), \qquad P^2_R(z) = \bigcup_{x\in P^1_R(z)} B(x,R), \qquad \dots \qquad P^n_R(z) = \bigcup_{x\in P^{n-1}_R(z)} B(x,R),$$

for all $n \in \mathbb{N}$.

Now, we prove the main result.

Proposition 3.14. *Let (Ω,μ,d) be a metric measure space, and let $J \geq 0$ satisfy that*

$$J(x,y) > 0 \text{ for all } x,\ y \in \Omega, \text{ such that } d(x,y) < R, \tag{3.12}$$

for some $R > 0$. If $z \geq 0$ is a nontrivial measurable function defined in Ω then,

$$P(K^n_J(z)) \supset P^n_R(z), \quad \text{for all } \ n \in \mathbb{N}.$$

In particular, if Ω is R-connected, then for any compact set $\mathcal{K} \subset \Omega$,

$$\exists\, n_0(z) \in \mathbb{N}, \quad \text{such that } \ P(K^n_J(z)) \supset \mathcal{K}, \text{ for all } \ n \geq n_0(z).$$

If Ω is compact and connected, then $\exists\, n_0 \in \mathbb{N}$, such that, for all $z \geq 0$ measurable and not identically zero

$$P(K^n_J(z)) = \Omega, \text{ for all } \ n \geq n_0.$$

Proof. First of all we prove that $P\left(K_J z\right) \supset P^1_R(z)$. Since $z \geq 0$, not identically zero, then $\mu\left(P(z)\right) > 0$ and then

$$K_J z(x) = \int_\Omega J(x,y)z(y)dy \geq \int_{P(z)} J(x,y)z(y)dy.$$

From (3.12) we have that

$$K_J z(x) > 0 \text{ for all } x \in \bigcup_{y\in P(z)} B(y,R) = P^1_R(z). \tag{3.13}$$

Since $P^1_R(z)$ is an open set in Ω, we have that, if $x \in P^1_R(z)$, then

$$\mu\left(B(x,\delta) \cap P^1_R(z)\right) > 0 \quad \text{for all } 0 < \delta. \tag{3.14}$$

Thus, thanks to (3.13) and (3.14), we have that

$$P\left(K_J z\right) \supset P^1_R(z). \tag{3.15}$$

Applying K_J to $K_J z$, following the previous arguments and thanks to (3.15), we obtain

$$P\left(K^2_J(z)\right) \supset P^1_R(K_J z) = \bigcup_{x\in P(K_J z)} B(x,R) \supset \bigcup_{x\in P^1_R(z)} B(x,R) = P^2_R(z).$$

Therefore, iterating this process, we finally obtain that

$$P(K^n_J(z)) \supset P^n_R(z),\ \forall n \in \mathbb{N}. \tag{3.16}$$

Now consider $\mathcal{K} \subset \Omega$ a compact set in Ω, and taking $x_0 \in P(z)$, then thanks to Lemma 3.11 there exists $n_0(z) \in \mathbb{N}$, such that $\mathcal{K} \subset P^n_R(z)$ for all $n \geq n_0(z)$, then thanks to (3.16), $\mathcal{K} \subset P(K^n_J(z))$ for all $n \geq n_0(z)$.

Now, if Ω is compact and connected, thanks to Lemma 3.10, Ω is R- connected. From Lemma 3.11 there exists $n_0 \in \mathbb{N}$ such that for any $y \in \Omega$, $\Omega = P^n_{R,y}$, for all $n \geq n_0$. Hence, from (3.16), for any $z \geq 0$ not identically zero, taking $y \in P(z)$, $P(K^n_J(z)) \supset P^n_{R,y} = \Omega$, $\forall\, n \geq n_0$. □

Remark 3.15. *Notice that the hypothesis* (3.12) *is somehow an optimal condition, as the following counterexample shows.*

Let $\Omega = [0,1] \subset \mathbb{R}$ *and take* $x_0 = 1/2$, *and* $0 < R < 1/2$ *such that* $(1/2 - R, 1/2 + R) \subset [0,1]$. *We consider a function* J *satisfying that* $J \geq 0$ *defined as*

$$J(x,y) = \begin{cases} 1, & (x,\,y) \in [0,1]^2 \setminus \left(\frac{1}{2} - R, \frac{1}{2} + R\right)^2, \textit{ with } d(x,y) < R, \\ 0 & \textit{for the rest of } (x,\,y). \end{cases} \tag{3.17}$$

Now, we consider a function $z_0 : \Omega \to \mathbb{R}$, $z_0 \geq 0$, *such that* $P(z_0) \subset [1/2, 1]$. *Since* $z_0(y) = 0$ *in* $[0, 1/2]$, *we have that* $K_J z_0(x) = \int_\Omega J(x,y)z_0(y)dy = \int_{1/2}^1 J(x,y)z_0(y)dy$. *Moreover, from* (3.17), *we have that for* $\tilde{x} \in [0, 1/2)$, $J(\tilde{x}, y) = 0$ *for all* $y \in [1/2, 1]$, *(see Figure 2).*

Hence $K_J z_0(\tilde{x}) = 0$ *in* $[0, 1/2)$, *and therefore* $P(K_J z_0) \subset [1/2, 1]$. *Iterating this argument, we obtain that*

$$P(K^n_J(z_0)) \subset [1/2, 1] \quad \textit{for all } n \in \mathbb{N}.$$

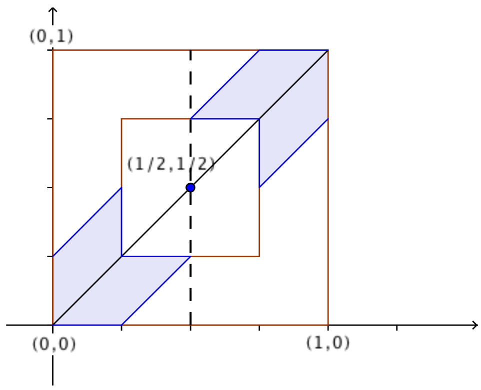

Figure 2: The shadowed area are the points (x, y) where J is strictly positive, $R = 1/4$.

Now we describe the adjoint operator associated to K_J, and we prove that if $J \in L^2(\Omega\times\Omega)$ and $J(x, y) = J(y, x)$ then the operator K_J is selfadjoint in $L^2(\Omega)$.

Proposition 3.16. *For* $1 \le p < \infty$, $1 \le q < \infty$. *Let* (Ω, μ) *be a measure space. We assume that the mapping*

$$x \mapsto J(x, \cdot) \ \textit{ satisfies that } \ J \in L^q(\Omega, L^{p'}(\Omega)),$$

and the mapping

$$y \mapsto J^*(y, \cdot) := J(\cdot, y) \ \textit{ satisfies that } \ J^* \in L^{p'}(\Omega, L^q(\Omega)).$$

Then the adjoint operator associated to $K_J \in \mathcal{L}(L^p(\Omega), L^q(\Omega))$, *is*

$$K_J^* : L^{q'}(\Omega) \to L^{p'}(\Omega), \quad \textit{with } K_J^* = K_{J^*}.$$

In particular, if J *satisfies that*

$$J(x, y) = J(y, x),$$

and $J \in L^2(\Omega \times \Omega)$, *the operator* K_J *is selfadjoint in* $L^2(\Omega)$.

Proof. We consider $u \in L^p(\Omega)$ and $v \in L^{q'}(\Omega)$. Thanks to Fubini's Theorem and the hypotheses on J

$$\langle K_J u, v\rangle_{L^q(\Omega), L^{q'}(\Omega)} = \int_\Omega\int_\Omega J(x, y)u(y)dy\, v(x)dx = \int_\Omega\int_\Omega J(x, y)v(x)dx\, u(y)dy,$$

and $\displaystyle\int_\Omega\int_\Omega J(x, y)v(x)dx\, u(y)dy = \langle u, K_J^* v\rangle_{L^p(\Omega), L^{p'}(\Omega)}$, with

$$K_J^* v(y) = \int_\Omega J(x, y)v(x)dx = \int_\Omega J^*(y, x)v(x)dx = K_{J^*}v(y).$$

The symmetric case in $L^2(\Omega)$ is now obvious. □

We will now prove that under certain hypotheses on K_J the spectrum $\sigma_X(K_J)$ is independent of X, with $X = L^p(\Omega)$, $1 \le p \le \infty$ or $X = \mathcal{C}_b(\Omega)$. We also characterize the spectrum of K_J when K_J is selfadjoint in $L^2(\Omega)$, and prove that under the same hypothesis on the positivity of J in Proposition 3.14, the spectral radius of K_J in $\mathcal{C}_b(\Omega)$ is a simple eigenvalue that has a strictly positive associated eigenfunction.

The proposition below is for a general compact operator K, not only for the integral operator K_J (see Propositions 3.6 to check compactness for operators with kernel, K_J).

Proposition 3.17. *Let (Ω, μ, d) be a metric measure space with $\mu(\Omega) < \infty$.*

i. Assume $K \in \mathcal{L}(L^{p_0}(\Omega), L^{p_1}(\Omega))$ for some $1 \le p_0 < p_1 < \infty$ and $K \in \mathcal{L}(L^{p_0}(\Omega), L^{p_0}(\Omega))$ is compact. Then $K \in \mathcal{L}(L^p(\Omega), L^p(\Omega))$, for all $p \in [p_0, p_1]$, and $\sigma_{L^p}(K)$ is independent of p.

ii. Assume $K \in \mathcal{L}(L^{p_0}(\Omega), L^{p_1}(\Omega))$ is compact for some $1 \le p_0 < p_1 \le \infty$. Then $K \in \mathcal{L}(L^p(\Omega), L^p(\Omega))$, for all $p \in [p_0, p_1]$, and $\sigma_{L^p}(K)$ is independent of p.

iii. Assume $K \in \mathcal{L}(L^{p_0}(\Omega), \mathcal{C}_b(\Omega))$ for some $1 \le p_0 \le \infty$ is compact and $X = \mathcal{C}_b(\Omega)$ or $X = L^r(\Omega)$ with $r \in [p_0, \infty]$. Then $K \in \mathcal{L}(X, X)$, and $\sigma_X(K)$ is independent of X.

Proof.

i. Thanks to Proposition 3.8, we have that $K \in \mathcal{L}(L^p(\Omega), L^p(\Omega))$ is compact for all $p \in [p_0, p_1]$. Thus the spectrum of K is composed by zero and a discrete set of eigenvalues of finite multiplicity, (see [7, chap. 6]). Let us prove now that the eigenvalues of the spectrum $\sigma_{L^p(\Omega)}(K)$ are independent of p.

Now if $\lambda \in \sigma_{L^{p_1}}(K)$ is an eigenvalue, consider an associated eigenfunction $\Phi \in L^{p_1}(\Omega)$. Since $\mu(\Omega) < \infty$ we have that $\Phi \in L^p(\Omega)$ for all $p \in [p_0, p_1]$. Hence, $\lambda \in \sigma_{L^p}(K)$ for all $p \in [p_0, p_1]$.

On the other hand, if $\lambda \in \sigma_{L^p(\Omega)}(K)$ is an eigenvalue, with $p \in [p_0, p_1)$, then any associated eigenfunction $\Phi \in L^p(\Omega)$ satisfies that

$$K\Phi = \lambda\Phi. \tag{3.18}$$

Since $L^p(\Omega) \hookrightarrow L^{p_0}(\Omega)$ and $K : L^{p_0}(\Omega) \to L^{p_1}(\Omega)$, then $K\Phi \in L^{p_1}(\Omega)$. From (3.18), we obtain that $\Phi \in L^{p_1}(\Omega)$. Hence, $\lambda \in \sigma_{L^{p_1}(\Omega)}(K)$.

ii. We know that $K \in \mathcal{L}(L^{p_0}(\Omega), L^{p_1}(\Omega))$ is compact, and we have that

$$L^{p_1}(\Omega) \hookrightarrow L^{p_0}(\Omega) \xrightarrow{K} L^{p_1}(\Omega) \hookrightarrow L^{p_0}(\Omega).$$

Therefore $K \in \mathcal{L}(L^{p_1}(\Omega), L^{p_1}(\Omega))$ is compact, and the hypotheses of Proposition 3.8 are satisfied. Therefore $K \in \mathcal{L}(L^p(\Omega), L^p(\Omega))$ is compact for all $p \in [p_0, p_1]$. From part *i.*, we have the result.

iii. We know that $K \in \mathcal{L}(L^{p_0}(\Omega), \mathcal{C}_b(\Omega))$ is compact. Since $\mu(\Omega) < \infty$, we have that for any $r \in [p_0, \infty]$

$$\mathcal{C}_b(\Omega) \hookrightarrow L^r(\Omega) \hookrightarrow L^{p_0}(\Omega) \xrightarrow{K} \mathcal{C}_b(\Omega) \hookrightarrow L^r(\Omega) \hookrightarrow L^{p_0}(\Omega)$$

Therefore, $K \in \mathcal{L}(X, X)$ is compact for $X = \mathcal{C}_b(\Omega)$ or $X = L^r(\Omega)$ with $r \in [p_0, \infty]$. Hence, following the arguments in *i.* we have that $\sigma_X(K)$ is independent of X. □

The following Proposition gives more details about the spectrum of K_J.

Proposition 3.18. *Let (Ω, μ, d) be a metric measure space with $\mu(\Omega) < \infty$. We assume $K_J \in \mathcal{L}(L^{p_0}(\Omega), \mathcal{C}_b(\Omega))$ is compact for some $p_0 \leq 2$. Let $X = L^p(\Omega)$, with $p \in [p_0, \infty]$, or $X = \mathcal{C}_b(\Omega)$, and assume J satisfies that*

$$J(x, y) = J(y, x).$$

Then $K_J \in \mathcal{L}(X, X)$ and $\sigma_X(K_J) \setminus \{0\}$ is a real sequence of eigenvalues of finite multiplicity, independent of X, that converges to 0.

Moreover, if we consider

$$m = \inf_{\substack{u \in L^2(\Omega) \\ \|u\|_{L^2(\Omega)}=1}} \langle K_J u, u \rangle_{L^2(\Omega)} \quad and \quad M = \sup_{\substack{u \in L^2(\Omega) \\ \|u\|_{L^2(\Omega)}=1}} \langle K_J u, u \rangle_{L^2(\Omega)}, \tag{3.19}$$

then $\sigma_X(K_J) \subset [m, M] \subset \mathbb{R}$, $m \in \sigma_X(K_J)$ and $M \in \sigma_X(K_J)$. In particular, $L^2(\Omega)$ admits an orthonormal basis consisting of eigenfunctions of K_J.

Proof. Thanks to Proposition 3.16, K_J is selfadjoint in $L^2(\Omega)$, then $\sigma_{L^2}(K_J) \setminus \{0\}$ is a real sequence of eigenvalues of finite multiplicity that converges to 0, (see [7, chap.6]). Furthermore, from Proposition 3.17 we have that $\sigma_X(K_J)$ is independent of X. Thus, the result.

On the other hand, we have that $\sigma_X(K_J) \subset [m, M] \subset \mathbb{R}$, with $m \in \sigma_X(K_J)$ and $M \in \sigma_X(K_J)$, where m and M are given by (3.19), and thanks to the Spectral Theorem (see [7, chap.6]), we know that $L^2(\Omega)$ admits an orthonormal basis consisting of eigenfunctions of K_J. □

The following Corollary states that under the hypotheses of Proposition 3.14, any non-negative eigenfunction associated to the operator K_J is in fact strictly positive as well as its associated eigenvalue.

Corollary 3.19. *Let J satisfy the hypotheses of Proposition 3.14 and assume Ω is R-connected. If $\Phi \geq 0$, is an eigenfunction associated to an eigenvalue λ of the operator K_J, then $\Phi > 0$, and the eigenvalue, λ, is also strictly positive.*

Proof. Thanks to Proposition 3.14, we know that, for every function $\Phi \geq 0$, not identically zero defined in Ω, it happens that $P(K_J^n(\Phi)) \supset P_R^n(\Phi)$, $\forall n \in \mathbb{N}$.

On the other hand, since Φ is an eigenfunction associated to an eigenvalue λ of the operator K_J, we have that $K_J^n(\Phi) = \lambda^n \Phi$, $\forall n \in \mathbb{N}$. Moreover, from Proposition 3.14, we know that for any compact set $\mathcal{K} \subset \Omega$, there exists $n_0 \in \mathbb{N}$ such that $P(K_J^n(\Phi)) \supset \mathcal{K}$ for all $n \geq n_0$. Thus, $K_J^n(\Phi) = \lambda^n \Phi$ is strictly positive in $\mathcal{K}$ for all $n \geq n_0$. Therefore Φ must be strictly positive in any compact set $\mathcal{K}$ of Ω. Hence, $\lambda > 0$ and Φ must be strictly positive in Ω. □

Now, let us give some results about the spectral radius of the operator K

$$r(K) = \sup |\sigma(K)|.$$

For this, we will use Kreĭn-Rutman Theorem, [22]. We will work in the space $\mathcal{C}_b(\Omega)$, with Ω compact, and we consider the positive cone $C = \{f \in \mathcal{C}_b(\Omega);\ f \geq 0\}$, with $\mathrm{Int}(C) = \{f \in$

$\mathcal{C}_b(\Omega);\ f(x) > 0,\ \forall x \in \Omega\}$. Thus, in the proposition below, we prove that the spectral radius of the operator K is a simple eigenvalue that has an associated eigenfunction that is strictly positive.

Proposition 3.20. *Let (Ω, μ, d) be a metric measure space, with Ω compact and connected. We assume that J satisfies*

$$J(x, y) = J(y, x)$$

and

$$J(x, y) > 0,\ \forall x,\ y \in \Omega \ \text{ such that } \ d(x, y) < R, \ \text{for some } R > 0,$$

and $K_J \in \mathcal{L}(L^p(\Omega), \mathcal{C}_b(\Omega))$, for some $1 \leq p \leq \infty$, is compact, (see Proposition 3.6 ii.).

Then $K_J \in \mathcal{L}(\mathcal{C}_b(\Omega), \mathcal{C}_b(\Omega))$ is compact, the spectral radius $r_{\mathcal{C}_b(\Omega)}(K_J)$ is a positive simple eigenvalue, and its associated eigenfunction can be taken strictly positive.

Proof. Since Ω is compact and connected then from Proposition 3.14 we obtain that, there exists $n_0 \in \mathbb{N}$ such that, for any nontrivial nonnegative $u \in \mathcal{C}_b(\Omega)$, $\Omega = P_R^n(u)$, for all $n \geq n_0$, (see Definition 3.13), and $\forall n \in \mathbb{N}$, $P(K^n u) \supset P_R^n(u)$. Therefore $\Omega = P_R^n(u) \subset P(K^n u)$ for all $n \geq n_0$, i.e., for any nonnegative $u \in \mathcal{C}_b(\Omega)$, $K_J^n u > 0$ in Ω for all $n \geq n_0$. Hence, K_J is strongly positive in $\mathcal{C}_b(\Omega)$. Moreover $K_J : \mathcal{C}_b(\Omega) \hookrightarrow L^p(\Omega) \longrightarrow \mathcal{C}_b(\Omega)$ is compact. Hence, thanks to Kreĭn-Rutman Theorem, (see [22]), the spectral radius $r_{\mathcal{C}_b(\Omega)}(K_J)$ is a positive simple eigenvalue with an eigenfunction Φ associated to it that is strictly positive. $\square$

A similar result was proved by Bates and Zhao [4], for $\Omega \subset \mathbb{R}^N$ open, but with the stronger assumption $J(x, y) > 0$ for all $x,\ y \in \Omega$.

3.2 Properties of $K - hI$

Let (Ω, μ, d) be a metric measure space. We will always assume below that

- If $X = L^p(\Omega)$, with $1 \leq p \leq \infty$, we assume $h \in L^\infty(\Omega)$.
- If $X = \mathcal{C}_b(\Omega)$, we assume $h \in \mathcal{C}_b(\Omega)$.

The following result collects some properties of multiplication operators. Note that below we denote $R(h)$ the range of the function $h : \Omega \to \mathbb{R}$ and $\overline{R(h)}$, its closure.

Proposition 3.21. *Let h be as above and consider the multiplication operator hI, that maps*

$$u(x) \mapsto h(x)u(x).$$

Then the resolvent set and spectrum of the multiplication operator are independent of X and are given by

$$\rho_X(hI) = \mathbb{C} \setminus \overline{R(h)}, \quad \sigma_X(hI) = \overline{R(h)},$$

respectively. Moreover, for $X = L^p(\Omega)$, the eigenvalues associated to hI have infinite multiplicity and satisfy

$$EV(hI) = \{\alpha \ ; \ \mu\left(\{x \in \Omega \ ; \ h(x) = \alpha\}\right) > 0\}.$$

On the other hand, for $X = \mathcal{C}_b(\Omega)$, the eigenvalues have infinite multiplicity and satisfy

$$EV(hI) \supset \{\alpha; \ \{x \in \Omega \ ; \ h(x) = \alpha\} \ \text{has nonempty interior}\}$$

Proof. If $X = L^p(\Omega)$ consider $f \in L^p(\Omega)$ and $u \in L^p(\Omega)$, such that $h(x)u(x) - \lambda u(x) = f(x)$, that is,

$$u(x) = \frac{f(x)}{h(x) - \lambda} = \frac{1}{h(x) - \lambda} f(x).$$

Then we have that $\lambda \in \rho_{L^p(\Omega)}(hI)$ if and only if $(hI - \lambda I)^{-1} \in \mathcal{L}(L^p(\Omega))$, if and only if $\frac{1}{h - \lambda} \in L^\infty(\Omega)$, and this happens if and only if $\lambda \notin \overline{\mathrm{R}(h)}$. Thus, $\rho_{L^p(\Omega)}(h) = \mathbb{C} \setminus \overline{\mathrm{R}(h)}$.

If λ is an eigenvalue, then for some $\Phi \in L^p(\Omega)$ with $\Phi \not\equiv 0$, we have

$$h(x)\Phi(x) = \lambda\Phi(x)$$

and this only happens if there exists a set $A \subset \Omega$, with $\mu(A) > 0$, such that $h(x) = \lambda$ for all $x \in A \subset \Omega$. Hence, we have that $\mathrm{Ker}(hI - \lambda I) = L^p(A)$. Thus, the result.

If $X = \mathcal{C}_b(\Omega)$ the arguments run as above. Just note that if $\{\lambda;\ \{x \in \Omega\ ;\ h(x) = \lambda\}$ has nonempty interior, A, then $\mathrm{Ker}(hI - \lambda I) = \{\Phi \in \mathcal{C}_b(\Omega) : \Phi(x) = 0,\ \forall x \in \Omega \setminus A\}$. □

Now we consider the particular case of the function

$$h_0(x) = \int_\Omega J(x, y)dy.$$

for which we assume $J \in L^\infty(\Omega, L^1(\Omega))$ and hence $h_0 \in L^\infty(\Omega)$.

Then we have

Proposition 3.22. (Green's formulas) *Let $\mu(\Omega) < \infty$. Assume $J \in L^\infty(\Omega, L^1(\Omega)) \cap L^p(\Omega, L^{p'}(\Omega))$, for some $1 \le p < \infty$, and*

$$J(x, y) = J(y, x).$$

Then for $u \in L^p(\Omega)$ and $v \in L^{p'}(\Omega)$,

$$\langle K_J u - h_0 I u, v\rangle_{L^p, L^{p'}} = -\frac{1}{2}\int_\Omega\int_\Omega J(x, y)(u(y) - u(x))(v(y) - v(x))dy\, dx. \tag{3.20}$$

In particular, if $p = 2$ we have that for $u \in L^2(\Omega)$

$$\langle K_J u - h_0\, I u,\, u\rangle_{L^2, L^2} = -\frac{1}{2}\int_\Omega\int_\Omega J(x, y)(u(y) - u(x))^2 dy\, dx.$$

Proof. Observe that

$$\begin{aligned} I_1 &= \int_\Omega\int_\Omega J(x, y)(u(y) - u(x))(v(y) - v(x))dy\, dx \\ &= \int_\Omega\int_\Omega J(x, y)(u(y) - u(x))v(y)dydx - \int_\Omega\int_\Omega J(x, y)(u(y) - u(x))v(x)dydx. \end{aligned}$$

Relabeling variables in the first term above, we obtain

$$I_1 = \int_\Omega\int_\Omega J(y, x)(u(x) - u(y))v(x)dx\, dy - \int_\Omega\int_\Omega J(x, y)(u(y) - u(x))v(x)dy\, dx.$$

Now, since $J(x,y) = J(y,x)$,

$$I_1 = \int_\Omega \int_\Omega J(x,y)(u(x) - u(y))v(x)dxdy - \int_\Omega \int_\Omega J(x,y)(u(y) - u(x))v(x)dydx.$$

Thanks to Fubini's Theorem, we have that

$$I_1 \quad = -2 \int_\Omega \int_\Omega J(x,y)(u(y) - u(x))v(x)dy\, dx. \tag{3.21}$$

On the other hand, thanks to the hypothesis on J, $h_0 \in L^\infty(\Omega)$ and from Propositions 3.1 we have that $K_J - h_0 I \in \mathcal{L}(L^p(\Omega))$, for all $1 \le p \le \infty$. Hence, if $u \in L^p(\Omega)$ and $v \in L^{p'}(\Omega)$

$$\begin{aligned} \langle K_J u - h_0 I u, v\rangle_{L^p, L^{p'}} &= \int_\Omega \left(\int_\Omega J(x,y)u(y)dy - \int_\Omega J(x,y)dy\, u(x) \right) v(x)\, dx \\ &= \int_\Omega \int_\Omega J(x,y)(u(y) - u(x))v(x)dy\, dx. \end{aligned} \tag{3.22}$$

Hence, from (3.21) and (3.22), we obtain (3.20). The second part of the proposition is an immediate consequence of (3.20). □

Now we describe the spectrum of $K - hI \in \mathcal{L}(X, X)$, where $X = L^p(\Omega)$, with $1 \le p \le \infty$, or $X = \mathcal{C}_b(\Omega)$, and we prove that, under certain conditions on the operator K, it is independent of X. Moreover, we give conditions on J and h under which the spectrum of $K_J - hI$ is nonpositive. For this, we start introducing some definitions used in the following theorems.

Definition 3.23. *If T is a linear operator in a Banach space Y, a* **normal point** *of T is any complex number which is in the resolvent set, or is an isolated eigenvalue of T of finite multiplicity. Any other complex number is in the* **essential spectrum** *of T.*

To describe the spectrum of $K - hI$, we use the following theorem that can be found in [17, p. 136].

Theorem 3.24. *Suppose Y is a Banach space, $T : D(T) \subset Y \to Y$ is a closed linear operator, $S : D(S) \subset Y \to Y$ is linear with $D(S) \supset D(T)$ and $S(\lambda_0 - T)^{-1}$ is compact for some $\lambda_0 \in \rho(T)$. Let U be an open connected set in $\mathbb{C}$ consisting entirely of normal points of T, which are points of the resolvent of T, or isolated eigenvalues of T of finite multiplicity. Then either U consists entirely of normal points of $T + S$, or entirely of eigenvalues of $T + S$.*

Remark 3.25. *If $S : Y \to Y$ is compact, Theorem 3.24 implies that the perturbation S can not change the essential spectrum of T.*

The next theorem describes the spectrum of the operator $K - hI$ in X. Recall that if $X = L^p(\Omega)$, with $1 \le p \le \infty$, we assume $h \in L^\infty(\Omega)$, while if $X = \mathcal{C}_b(\Omega)$, we assume $h \in \mathcal{C}_b(\Omega)$.

Theorem 3.26. *If $K \in \mathcal{L}(X, X)$ is compact, (see Proposition 3.6), then*

$$\sigma(K - hI) = \overline{R(-h)} \cup \{\mu_n\}_{n=1}^M, \quad \textit{with} \;\; M \in \mathbb{N} \cup \{\infty\}.$$

If $M = \infty$, then $\{\mu_n\}_{n=1}^\infty$ is a sequence of eigenvalues of $K - hI$ with finite multiplicity, that accumulates in $\overline{R(-h)}$.

Proof. With the notations of Theorem 3.24, we consider the operators

$$S = K \text{ and } T = -hI.$$

First of all, we prove that $\mathbb{C} \setminus \overline{R(-h)} \subset \rho(K - hI)$. We choose the set U in Theorem 3.24 as

$$U = \rho(-hI) = \rho(T) = \mathbb{C} \setminus \overline{R(-h)}$$

which is an open, connected set. Since $U = \rho(T)$, every $\lambda \in U$ is a normal point of T.

On the other hand, if $\lambda_0 \in \rho(T)$, then $(T - \lambda_0)^{-1} \in \mathcal{L}(X, X)$, and $S = K$ is compact. Then, we have that $S(\lambda_0 - T)^{-1} \in \mathcal{L}(X, X)$ is compact. Thus, all the hypotheses of Theorem 3.24 are satisfied. Now, thanks to Theorem 3.24, we have that $U = \mathbb{C} \setminus \overline{R(-h)}$ consists entirely of eigenvalues of $T + S = K - hI$ or U consists entirely of normal points of $T + S = K - hI$.

If $U = \mathbb{C} \setminus \overline{R(-h)}$ consists entirely of eigenvalues of $T + S = K - hI$, we arrive to contradiction, because the spectrum of $K - hI$ is bounded. So $U = \mathbb{C} \setminus \overline{R(-h)}$ has to consist entirely of normal points of $T + S$. Then, they are points of the resolvent or isolated eigenvalues of $T + S = K - hI$. Since any set of isolated points in $\mathbb{C}$ is a finite set, or a numerable set, we have that the isolated eigenvalues are

$$\{\mu_n\}_{n=1}^{M}, \quad \text{with } M \in \mathbb{N} \text{ or } M = \infty.$$

Moreover, since the spectrum of $K - hI$ is bounded, if $M = \infty$ then $\{\mu_n\}_{n=1}^{\infty}$ is a sequence of eigenvalues of $K - hI$ with finite multiplicity, that accumulates in $\overline{R(-h)}$.

Now we prove that $\overline{R(-h)} \subset \sigma(K - hI)$. We argue by contradiction. Suppose that there exists a $\tilde{\lambda} \in \overline{R(-h)}$ that belongs to $\rho(K - hI)$. Since the resolvent set is open, there exists a ball $B_\varepsilon(\tilde{\lambda})$ centered in $\tilde{\lambda}$, that is into the resolvent of $K - hI$. Then $U = B_\varepsilon(\tilde{\lambda})$ is an open connected set that consists of normal points of $K - hI$. With the notation of Theorem 3.24, we consider the operators

$$T = K - hI \quad \text{and} \quad S = -K$$

and the open, connected set

$$U = B_\varepsilon(\tilde{\lambda}).$$

Arguing like in the previous case, if $\lambda_0 \in \rho(T)$, we have that $S(\lambda_0 - T)^{-1}$ is compact, thus the hypotheses of Theorem 3.24 are satisfied. Hence $U = B_\varepsilon(\tilde{\lambda})$ consists entirely of eigenvalues of $T + S = -hI$ or $U = B_\varepsilon(\tilde{\lambda})$ consists entirely of normal points of $T + S = -hI$.

If $U = B_\varepsilon(\tilde{\lambda})$ consists entirely of eigenvalues of $T + S = -hI$, we would arrive to contradiction, because the eigenvalues of $-hI$ are only inside $\overline{R(-h)}$, and the ball $B_\varepsilon(\tilde{\lambda})$ is not inside $\overline{R(-h)}$. So $U = B_\varepsilon(\tilde{\lambda})$ has to consist of normal points of $T + S = -hI$, so they are points of the resolvent of $-hI$ or isolated eigenvalues of finite multiplicity of $-hI$. Since $\rho(-hI) = \mathbb{C} \setminus \overline{R(-h)}$, and $\tilde{\lambda} \in \overline{R(-h)}$, we have that $\tilde{\lambda}$ has to be an isolated eigenvalue of $-hI$, with finite multiplicity. But from Proposition 3.21, we know that the eigenvalues of $-hI$ have infinity multiplicity. Thus, we arrive to contradiction. Hence, we have proved that $\overline{R(-h)} \subset \sigma(K - hI)$. With this, we have finished the proof of the theorem. □

In the following proposition we give conditions that guarantee that the spectrum of $K - hI$ is independent of $X = L^p(\Omega)$ with $1 \le p \le \infty$, or $X = \mathcal{C}_b(\Omega)$.

Proposition 3.27. *Let $\mu(\Omega) < \infty$.*

i. Assume, for some $1 \le p_0 < p_1 < \infty$, $K \in \mathcal{L}(L^{p_0}(\Omega), L^{p_1}(\Omega))$, $K \in \mathcal{L}(L^{p_0}(\Omega), L^{p_0}(\Omega))$ is compact and $h \in L^\infty(\Omega)$. Then $K - hI \in \mathcal{L}(L^p(\Omega), L^p(\Omega))$, $\forall p \in [p_0, p_1]$, and $\sigma_{L^p}(K - hI)$ is independent of p .

ii. Assume, for some $1 \le p_0 < p_1 \le \infty$, $K \in \mathcal{L}(L^{p_0}(\Omega), L^{p_1}(\Omega))$ is compact and $h \in L^\infty(\Omega)$. Then $K - hI \in \mathcal{L}(L^p(\Omega), L^p(\Omega))$, $\forall p \in [p_0, p_1]$, and $\sigma_{L^p}(K - hI)$ is independent of p.

iii. Assume, for some $1 \le p_0 \le \infty$, $K \in \mathcal{L}(L^{p_0}(\Omega), \mathcal{C}_b(\Omega))$ is compact and $X = \mathcal{C}_b(\Omega)$ or $X = L^r(\Omega)$ with $r \in [p_0, \infty]$, and $h \in \mathcal{C}_b(\Omega)$. Then $K - hI \in \mathcal{L}(X, X)$ and $\sigma_X(K - hI)$ is independent of X.

Proof. Following the same arguments in Proposition 3.17, we have that in any of the cases *i.*, *ii.*, or *iii.*, $K \in \mathcal{L}(X, X)$ is compact, where $X = L^p(\Omega)$ with $p_0 \le p \le p_1$ for the cases *i.* and *ii.*, and $X = L^p(\Omega)$ with $p_0 \le p \le \infty$, or $X = \mathcal{C}_b(\Omega)$ for the case *iii.*. Then, from Theorem 3.26 we have that

$$\sigma_X(K - hI) = \overline{R(-h)} \cup \{\mu_n\}_{n=1}^M, \quad \text{with } M \in \mathbb{N} \text{ with } or M = \infty,$$

where $\{\mu_n\}_n$ are eigenvalues of $K - hI$, with finite multiplicity $\forall n \in \{1, \dots, M\}$.

Since $\overline{R(-h)}$ is independent of X, we just have to prove that the eigenvalues $\lambda \in \sigma_X(K - hI)$ satisfying that $\lambda \notin \overline{R(-h)}$ are independent of X. Let $\lambda \in \sigma_X(K - hI)$ be an eigenvalue such that $\lambda \notin \overline{R(-h)}$. We denote by Φ an eigenfunction associated to $\lambda \in \sigma_X(K - hI)$, then

$$K\Phi(x) - h(x)\Phi(x) = \lambda\Phi(x) \tag{3.23}$$

Since $\lambda \notin \overline{R(-h)}$, then from (3.23) we obtain

$$\Phi(x) \;=\; \frac{1}{h(x) + \lambda} K\Phi(x) \tag{3.24}$$

and $\frac{1}{h(\cdot)+\lambda} \in L^\infty(\Omega)$.

For the cases *i.* and *ii.*, thanks to the hypotheses on K, we have

$$\frac{1}{h(\cdot) + \lambda} K \in \mathcal{L}(L^{p_0}(\Omega), L^{p_1}(\Omega)). \tag{3.25}$$

Now assume $\lambda \in \sigma_{L^{p_1}}(K - hI)$ is an eigenvalue with associated eigenfunction $\Phi \in L^{p_1}(\Omega)$. Since $\mu(\Omega) < \infty$ we have that $L^{p_1}(\Omega) \hookrightarrow L^p(\Omega)$ for all $p \le p_0$, then $\Phi \in L^p(\Omega)$ and $\lambda \in \sigma_{L^p}(K - hI)$ for all $p \in [p_0, p_1]$.

On the other hand, if $\lambda \in \sigma_{L^p(\Omega)}(K - hI)$ is an eigenvalue, with an associated eigenfunction $\Phi \in L^p(\Omega)$, with $p \in [p_0, p_1)$, then by $L^p(\Omega) \hookrightarrow L^{p_0}(\Omega)$ and (3.25), we have that $\frac{1}{h(\cdot)+\lambda} K\Phi \in L^{p_1}(\Omega)$. Hence, from (3.24), we obtain that $\Phi \in L^{p_1}(\Omega)$. Therefore, $\Phi \in L^p(\Omega)$ for $p \in [p_0, p_1]$, and the spectrum is independent of the space in cases *i.* and *ii.*.

The case *iii.* is analogous to the previous result, using that $h \in \mathcal{C}_b(\Omega)$ and $\lambda \notin \overline{R(-h)}$, then

$$\frac{1}{h(\cdot) + \lambda} K\Phi \in \mathcal{L}(L^{p_0}(\Omega), \mathcal{C}_b(\Omega)).$$

Thus, the result. □

The following results give conditions for the spectrum of $K_J - hI$ to be nonpositive.

Corollary 3.28. *Let $\mu(\Omega) < \infty$. Assume for some $1 \le p_0 \le 2$, $K_J \in \mathcal{L}(L^{p_0}(\Omega), X)$ is compact with $X = L^p(\Omega)$, with $p_0 \le p \le \infty$, or $X = \mathcal{C}_b(\Omega)$ and J satisfies that*

$$J(x,y) = J(y,x).$$

Then

i. If $h \equiv c$, with $c \in \mathbb{R}$ such that $c > r(K_J)$, where $r(K_J)$ is the spectral radius of K_J then $\sigma_X(K_J - hI)$ is real and nonpositive.

ii. If $J \in L^\infty(\Omega, L^1(\Omega))$ and $h(x) = h_0(x) = \displaystyle\int_\Omega J(x,y)dy \in L^\infty(\Omega)$, satisfies that $h_0(x) \ge \alpha > 0$ for all $x \in \Omega$, then $\sigma_X(K_J - h_0 I)$ is nonpositive and 0 is an isolated eigenvalue with finite multiplicity. Moreover if J satisfies that

$$J(x,y) > 0, \ \forall x,\, y \in \Omega \text{ such that } d(x,y) < R$$

and Ω is R–connected, then $\{0\}$ is a simple eigenvalue with only constant eigenfunctions.

iii. If $h \in L^\infty(\Omega)$ satisfies that $h \ge h_0$ in Ω, then $\sigma_X(K_J - hI)$ is nonpositive.

Proof. Under the hypotheses and thanks to the previous Proposition 3.27, we have that $\sigma_X(K - hI)$ is independent of X. Hence the rest of the results will be proved in $L^2(\Omega)$.

i. From Proposition 3.16 we have that K_J is selfadjoint in $L^2(\Omega)$, and so is $K_J - hI$. Thus $\sigma_{L^2(\Omega)}(K_J)$ is composed by real values that are less or equal to $r(K_J)$, (see [7, p.165]).

On the other hand, $\sigma_{L^2(\Omega)}(K_J - hI) = \sigma_{L^2(\Omega)}(K_J) - c$ and $c > r(K_J)$, then we have that $\sigma_{L^2(\Omega)}(K_J - hI)$ is real and nonpositive.

ii. Under the hypotheses we have that $K \in \mathcal{L}(X, X)$ is compact, then thanks to Theorem 3.26, we know that

$$\sigma_X(K - h_0 I) = \overline{R(-h_0)} \cup \{\mu_n\}_{n=1}^M, \quad \text{with } M \in \mathbb{N} \text{ or } M = \infty.$$

Then, thanks to Proposition 3.22,

$$\langle (K_J - h_0)u, u\rangle_{L^2, L^2} = -\frac{1}{2}\int_\Omega \int_\Omega J(x,y)(u(x) - u(y))^2 dy\, dx \le 0, \tag{3.26}$$

From this we get

$$\sigma_{L^2(\Omega)}(K_J - h_0) \le \sup_{\substack{u \in L^2(\Omega) \\ \|u\|_{L^2(\Omega)} = 1}} \langle (K_J - h_0)u, u\rangle_{L^2(\Omega), L^2(\Omega)} \le 0.$$

Observe that clearly constant function satisfy $(K - h_0)u = 0$ and since $0 \notin \overline{R(-h_0)}$, then 0 is an isolated eigenvalue with finite multiplicity. Let us prove below that $\{0\}$ is a simple eigenvalue. We consider φ an eigenfunction associated to $\{0\}$. Thanks to Proposition 3.22 in $L^2(\Omega)$ we have

$$0 = \langle (K - h_0\, I)\varphi,\, \varphi\rangle_{L^2(\Omega), L^2(\Omega)} = -\frac{1}{2}\int_\Omega \int_\Omega J(x,y)(\varphi(y) - \varphi(x))^2 dy\, dx.$$

Since $J(x,y) > 0$, $\forall x,\, y \in \Omega$ such that $d(x,y) < R$, then for all $x \in \Omega$, $\varphi(x) = \varphi(y)$ for any $y \in B(x,R)$. Thus, since Ω is R–connected, φ is a constant function in Ω. Therefore, $\{0\}$ is simple.

iii. If $h \geq h_0$ from (3.26), we have

$$\langle (K_J - hI)u, u\rangle_{L^2(\Omega),L^2(\Omega)} = \langle (K_J - h_0)u, u\rangle_{L^2(\Omega),L^2(\Omega)} + \langle (h_0 - h)u, u\rangle_{L^2(\Omega),L^2(\Omega)} \leq 0.$$

□

4 The linear evolution equation

Let (Ω, μ, d) be a metric measure space. Let $X = L^p(\Omega)$, with $1 \leq p \leq \infty$ or $X = \mathcal{C}_b(\Omega)$. The problem we are going to work with in this section, is the following

$$\begin{cases} u_t(x,t) &= (K_J - hI)u(x,t) = Lu(x,t), \quad x \in \Omega,\, t > 0 \\ u(x,0) &= u_0(x), \qquad\qquad\qquad\qquad\quad x \in \Omega, \end{cases} \tag{4.1}$$

with $K_J u(x) = \int_\Omega J(x,y)u(y)dy$, $J \geq 0$, $u_0 \in X$, $K_J \in \mathcal{L}(X,X)$ and if $X = L^p(\Omega)$, with $1 \leq p \leq \infty$, we assume $h \in L^\infty(\Omega)$ while if $X = \mathcal{C}_b(\Omega)$, we assume $h \in \mathcal{C}_b(\Omega)$.

First, since $K_J \in \mathcal{L}(X,X)$ then the problem (4.1) has a unique strong solution $u \in \mathcal{C}^\infty(\mathbb{R}, X)$, given by

$$u(t) = e^{Lt}u_0.$$

The mapping

$$\mathbb{R} \ni t \;\mapsto\; u(t) = e^{Lt}u_0 \in X$$

is analytic. Moreover the mapping $(t, u_0) \;\mapsto\; e^{Lt}u_0$ is continuous.

We denote the group associated to the operator $L = K_J - hI$ with $S_{K,h}$, to remark the dependence on K_J and h. Hence the solution of (4.1) is

$$u(t, u_0) = S_{K,h}(t)u_0 = e^{Lt}u_0.$$

4.1 Maximum principles

First of all, let us consider the problem (4.1), with $h \equiv 0$,

$$\begin{cases} \dfrac{du}{dt} = & K_J u, \\ u(0) = & u_0 \geq 0. \end{cases} \tag{4.2}$$

Then the solution to (4.2) can be written as

$$u(x,t) = e^{K_J t}u_0(x) = \left(\sum_{k=0}^{\infty} \frac{t^k K_J^k}{k!} \right) u_0(x).$$

Since J is nonnegative, we have that $K_J^k u_0$ is nonnegative for any u_0 nonnegative, $\forall k \in \mathbb{N}$. Then we have that the solution $u(x,t)$ is nonnegative. In fact, for any $m \geq 0$

$$u(x,t) \geq u_0(x) \geq 0, \quad u(x,t) \geq u_0(x) + tK_J u_0(x) \geq 0, \quad \text{and} \quad u(x,t) \geq \left(\sum_{k=0}^{m} \frac{t^k K_J^k}{k!} \right) u_0(x) \geq 0.$$

Now, for $h \not\equiv 0$, let u be the solution to (4.1). We take the function

$$v(t) = e^{h(\cdot)t}u(t), \quad \text{for } t \geq 0.$$

This function v satisfies that

$$v_t(x,t) = e^{h(x)t}K_J u(x,t), \quad \text{and} \quad v(x,0) = u_0(x),$$

hence, integrating in time we get

$$u(x,t) = e^{-h(x)t}u_0(x) + \int_0^t e^{-h(x)(t-s)}K_J u(x,s)ds. \tag{4.3}$$

Let $X = L^p(\Omega)$, with $1 \leq p \leq \infty$, or $X = \mathcal{C}_b(\Omega)$. For every $\omega_0 \in X$ and $T > 0$, we consider the mapping $\mathcal{F}_{\omega_0} : \mathcal{C}([0,T];X) \to \mathcal{C}([0,T];X)$ defined as

$$\mathcal{F}_{\omega_0}(\omega)(x,t) = e^{-h(x)t}\omega_0(x) + \int_0^t e^{-h(x)(t-s)}K_J(\omega)(x,s)ds.$$

Then we have the following immediate result.

Lemma 4.1. *If $\omega_0,\ z_0 \in X$, and $\omega,\ z \in X_T = \mathcal{C}([0,T];X)$, then there exist two constants C_1 and C_2 depending on h and T, such that*

$$|||\mathcal{F}_{\omega_0}(\omega) - \mathcal{F}_{z_0}(z)||| \leq C_1(T)\|\omega_0 - z_0\|_X + C_2(T)|||\omega - z|||, \tag{4.4}$$

where $C_1(T) = e^{\|h_-\|_{L^\infty(\Omega)}T}$, $C_2(T) = CTe^{\|h_-\|_{L^\infty(\Omega)}T}$, $C_2 : [0,\infty) \to \mathbb{R}$ is increasing and continuous, and $C_2(T) \to 0$, as $T \to 0$.

With this we can prove the following.

Proposition 4.2. (Weak maximum principle) *For every nonnegative $u_0 \in X$, the solution to the problem (4.1) is nonnegative for all $t \geq 0$.*

Proof. Thanks to (4.3), we know that the solution to (4.1) can be written as

$$u(x,t) = e^{-h(x)t}u_0(x,t) + \int_0^t e^{-h(x)(t-s)}K_J u(x,s)ds = \mathcal{F}_{u_0}u(x,t).$$

We choose T small enough such that $C_2(T)$ in Lemma 4.1 satisfies that $C_2(T) < 1$. Hence, by (4.4) we have that $\mathcal{F}_{u_0}(\cdot)$ is a contraction in $X_T = \mathcal{C}\big([0,T];X\big)$. We consider the sequence of Picard iterations,

$$u_{n+1}(x,t) = \mathcal{F}_{u_0}(u_n)(x,t) \ \ \forall n \geq 1,\ x \in \Omega,\ 0 \leq t \leq T.$$

Then the sequence u_n converges to u in X_T. We take $u_1(x,t) = u_0(x) \geq 0$, then for $t \geq 0$

$$u_2(x,t) = \mathcal{F}_{u_0}(u_1)(x,t) = e^{-h(x)t}u_0(x) + \int_0^t e^{-h(x)(t-s)}K_J(u_0)(x)ds$$

is nonnegative, because K_J is a positive operator. Thus $u_2(x,t) \geq 0$ for all $t \geq 0$. Repeating this argument for all u_n, we get that $u_n(x,t)$ is nonnegative for every $n \geq 1$, for $t \geq 0$. As u_n converges to u in X_T, we have that u is nonnegative.

Since $T > 0$ does not depend on the initial data, if we consider again the same problem with initial data $u(\cdot, T)$, then the solution $u(\cdot, t)$ is nonnegative for all $t \in [T, 2T]$. Since (4.1) has a unique solution then we have proved that the solution of (4.1), $u(x,t) \geq 0$ for all $t \in [0, 2T]$. Repeating this argument, we have that the solution of (4.1) is nonnegative $\forall t \geq 0$. □

Now we show that with the assumptions in Proposition 3.14 we have in fact the strong maximum principle.

Theorem 4.3. (Strong maximum principle) *Assume $K_J \in \mathcal{L}(X, X)$, and $J \geq 0$ satisfies*

$$J(x,y) > 0 \ \text{for all} \ x,\, y \in \Omega, \ \ \text{such that}\ d(x,y) < R,$$

for some $R > 0$, and Ω is R-connected.

Then for every nontrivial $u_0 \geq 0$ in X, the solution $u(t)$ of (4.1) is strictly positive, for all $t > 0$.

Proof. Thanks to Proposition 4.2, we know that $u \geq 0$ in Ω, for all $t \geq 0$. We take

$$v(t) = e^{h(\cdot)t} u(t),$$

then recalling the definition of the essential support in Definition 3.12, we have $P(u(t)) = P(v(t))$, for all $t \geq 0$. From the argument above (4.3), we know that v satisfies

$$v_t(t) = e^{h(\cdot)t} K_J(u(t)) \geq 0, \quad \forall t \geq 0. \tag{4.5}$$

Integrating (4.5) in $[s,\, t]$, we obtain

$$v(t) = v(s) + \int_s^t v_t(r) dr \geq v(s), \quad \text{for any} \ \ t \geq s \geq 0. \tag{4.6}$$

Then $P(v(t)) \supset P(v(s))$, $\forall t \geq s$. Moreover, since $v(t) = e^{h(\cdot)t} u(t)$ and thanks to (4.6), we obtain

$$u(t) \geq e^{-h(\cdot)(t-s)} u(s).$$

This implies that $P(u(t)) \supset P(u(s))$, $\forall t \geq s$. As a consequence of (4.6), we have that for any subset $D \subset \Omega$,

$$v|_D(t) = v|_D(s) + \int_s^t \left(e^{h(\cdot)r} K_J(u(r)) \right)\Big|_D dr. \tag{4.7}$$

Since $P(v(t)) \supset P(v(s))$ for all $t \geq s$, and from (4.7), we have that

$$P(u(t)) \cap D = P(v(t)) \cap D \supset P(K_J u(r)) \cap D, \quad \text{for all} \ \ r \in [s, t]. \tag{4.8}$$

Moreover, applying Proposition 3.14 to $u(s)$, we have

$$P(K_J u(r)) \supset P(K_J(u(s))) \supset P_R^1(u(s)) = \bigcup_{x \in P(u(s))} B(x, R) \ \text{for all} \ \ r \in [s,\, t]. \tag{4.9}$$

Hence, if we consider the set $D = P_R^1(u(s))$. From (4.8) and (4.9), we have that

$$P(u(t)) \supset P_R^1(u(s)), \quad \text{for all} \ \ t > s. \tag{4.10}$$

Hence the essential support of the solution at time t, contains the balls of radius R centered at the points in the support of the solution at time $s < t$.

We fix $t > 0$, and let $\mathcal{C} \subset \Omega$ be a compact set, then Proposition 3.14 implies that exists $n_0 \in \mathbb{N}$, such that $\mathcal{C} \subset P^n(u_0)$ for all $n \geq n_0$. We consider the sequence of times

$$t = t_n,\ t_{n-1} = t(n-1)/n,\ ..., t_j = t\,j/n,\ \dots,\ t_1 = t/n,\ t_0 = 0.$$

Therefore, thanks to (4.10), we have that the essential supports at time t, contains the balls of radius R centered at the points in the essential support at time t_{n-1}, $P^1_R(u(t_{n-1}))$, which contains the balls of radius R centered at the points in the essential support at time t_{n-2}, then $P^2_R(u(t_{n-2}))$. Hence repeating this argument, we have

$$P(u(t)) = P(u(t_n)) \supset P^1_R(u(t_{n-1})) \supset P^2_R(u(t_{n-2})) \supset \ldots \supset P^n_R(u_0) \supset \mathcal{C}.$$

Thus, we have proved that $u(t)$ is strictly positive for every compact set in Ω, $\forall t > 0$. Therefore, $u(t)$ is strictly positive in Ω, for all $t > 0$. □

Corollary 4.4. *Under the assumptions of Theorem 4.3, if $u_0 \geq 0$, not identically zero, with $P(u_0) \neq \Omega$, then the solution to* (4.1) *has to be sign changing in Ω, $\forall t < 0$.*

Proof. We argue by contradiction. Let us assume first that there exists $t_0 < 0$ such that $u(\cdot, t_0) \equiv 0$. We take $u(\cdot, t_0)$ as initial data, then solving forward in time $u(\cdot, t) \equiv 0$, for all $t \geq t_0$. Hence, we arrive to contradiction, and $u(t_0)$ is not identically zero.

Secondly, let us assume that there exists $t_0 < 0$ such that $u(\cdot, t_0) \leq 0$, not identically zero. We take $-u(\cdot, t_0) \geq 0$ as the initial data, then thanks to Theorem 4.3, the solution to (4.1), satisfies that $u(x, 0) < 0$, $\forall x \in \Omega$. Thus, we arrive to contradiction.

Now, we assume that there exists $t_0 < 0$ such that $u(x, t_0) \geq 0$. Let $u(\cdot, t_0) \geq 0$ be the initial data, then thanks to Theorem 4.3, the solution to (4.1), satisfies that $u(x, 0) > 0$, $\forall x \in \Omega$. Thus, we arrive to contradiction.

Therefore, the solution has to be sign changing for all negative times. □

4.2 Asymptotic regularizing effects

In general, the group associated to (4.1) has no regularizing effects. However, we will prove that there exists a part of the group, that we call $S_2(t)$ that is compact, so it somehow regularizes. Moreover, there exists another part of the group that we call $S_1(t)$ which does not regularize, i.e., it carries the singularities of the initial data, but it decays to zero exponentially as t goes to ∞, if $h \geq 0$. Thus, we will have a regularizing effect when t goes to ∞, that is, asymptotic smoothness according to [16, p. 4].

Theorem 4.5. *Let $\mu(\Omega) < \infty$. For $1 \leq p \leq q \leq \infty$, let $X = L^q(\Omega)$ or $\mathcal{C}_b(\Omega)$. If $K_J \in \mathcal{L}(L^p(\Omega), X)$ is compact, (see Proposition 3.6), and h satisfies*

$$h(x) \geq \alpha > 0 \ \text{ for all } \ x \in \Omega,$$

and $u_0 \in L^p(\Omega)$, then the group associated to the problem (4.1)*, satisfies that*

$$u(t) = S_{K,h}(t)u_0 = S_1(t)u_0 + S_2(t)u_0$$

with

i. $S_1(t) \in \mathcal{L}(L^p(\Omega))$ $\forall t > 0$, *and* $\|S_1(t)\|_{\mathcal{L}(L^p(\Omega),L^p(\Omega))} \to 0$ *exponentially, as* t *goes to* ∞.

ii. $S_2(t) \in \mathcal{L}(L^p(\Omega), X)$ *is compact,* $\forall t > 0$.

Therefore $S_{K,h}(t)$ *is asymptotically smooth.*

Proof. We write the solution associated to (4.1), as in (4.3), then we have that

$$u(x,t) = S_{K,h}(t)u_0(x) = e^{-h(x)t}u_0(x) + \int_0^t e^{-h(x)(t-s)}K_J u(x,s)ds, \ \forall x \in \Omega$$

and we define $S_1(t)u_0 = e^{-h(\cdot)t}u_0$, $S_2(t)u_0 = \int_0^t e^{-h(\cdot)(t-s)}K_J u(s)ds$.
i. Since $u_0 \in L^p(\Omega)$ and $h \in L^\infty(\Omega)$ with $h \geq \alpha > 0$, then $S_1(t)u_0 = e^{-h(\cdot)t}u_0 \in L^p(\Omega)$ and

$$\|S_1(t)\, u_0\|_{L^p(\Omega)} = \|e^{-h(\cdot)t}u_0(\cdot)\|_{L^p(\Omega)} \leq e^{-\alpha t}\|u_0\|_{L^p(\Omega)}.$$

ii. Fix $t > 0$, as $h \in L^\infty(\Omega)$, $S_{K,h}(s) \in \mathcal{L}(L^p(\Omega))$ $\forall s \in [0,t]$, and $K_J \in \mathcal{L}(L^p(\Omega), X)$, then

$$\|S_2(t)(u_0)\|_X \leq e^{-\alpha t}\int_0^t \|K_J(S_{K,h}(s)u_0)\|_X ds \leq e^{-\alpha t} t \max_{0\leq s\leq t} \|K_J(S_{K,h}(s)u_0)\|_X < \infty.$$

Let us see now that $S_2(t) \in \mathcal{L}(L^p(\Omega), X)$ is compact $\forall t > 0$. Fix $t > 0$ and consider a bounded set $\mathcal{B}$ of initial data. We denote $S_2(t)u_0 = \int_0^t F_{u_0}(s)ds$, with

$$F_{u_0}(s) = e^{-h(\cdot)(t-s)}K_J(S_{K,h}(s)u_0).$$

Assume we have proved that $F_{u_0}(s) \in \mathscr{C}$, where $\mathscr{C}$ is a compact set in X, for all $s \in [0,t]$ and for all $u_0 \in \mathcal{B}$. Then we have that $\frac{1}{t}S_2(t)(u_0) \in \overline{co}(\mathscr{C})$, $\forall u_0 \in \mathcal{B}$, and thanks to Mazur's Theorem, we obtain that $\frac{1}{t}S_2(t)(\mathcal{B})$ is in a compact set of X. Therefore $S_2(t)$ is compact. Now, we have to prove that $F_{u_0}(s) = e^{-h(\cdot)(t-s)}K_J(S_{K,h}(s)u_0)$ belongs to a compact set, for all $(s,\, u_0) \in [0,t] \times \mathcal{B}$.

First of all, we check that $K_J(S_{K,h}(s)u_0)$ belongs to a compact set $\mathcal{W}$ in X, for all $(s,\, u_0) \in [0,t] \times \mathcal{B}$. Since K_J is compact, we just have to prove that the set

$$B = \{S_{K,h}(s)u_0 : \, (s,\, u_0) \in [0,t] \times \mathcal{B}\}$$

is bounded. In fact, since $K_J - hI \in \mathcal{L}(L^p(\Omega), L^p(\Omega))$, then for some $\delta > 0$

$$\|S_{K,h}(s)u_0\|_{L^p(\Omega)} = \|u(\cdot,s)\|_{L^p(\Omega)} \leq Ce^{\delta s}\|u_0\|_{L^p(\Omega)} \leq Ce^{\delta t}\|u_0\|_{L^p(\Omega)},$$

for all $(s,\, u_0) \in [0,t] \times \mathcal{B}$. Then, since $\mathcal{B}$ is bounded, we obtain that B is bounded in $L^p(\Omega)$.

Finally, we just need to prove that $F_{u_0}(s)$ is in a compact set for all $(s,\, u_0) \in [0,t] \times \mathcal{B}$. Since the mapping

$$\begin{array}{rrcl} M: & [0,\, t] \times X & \longrightarrow & X \\ & (s, f) & \longmapsto & e^{-h\,(t-s)} f \end{array}$$

is continuous, then M sends the compact set $[0,\, t] \times \mathcal{W}$ into a compact set $\mathscr{C}$. Thus, $F_{u_0}(s)$ belongs to a compact set, $\mathscr{C}$, $\forall (s,\, u_0) \in [0,t] \times \mathcal{B}$. □

4.3 The Riesz projection and asymptotic behavior

In this section we study the asymptotic behavior of the solution of the problem (4.1) by using the Riesz projection, which is given in terms of the spectrum of the operator. Since the spectrum of the operator $L = K_J - hI$ has been proved in Proposition 3.27 to be independent of $X = L^p(\Omega)$, with $1 \le p \le \infty$ or $X = \mathcal{C}_b(\Omega)$, then the asymptotic behavior of the solution of (4.1) will be characterized with the Riesz projection that can be explicitly computed in $L^2(\Omega)$.

We now briefly recall the construction of the Riesz projection, for more details see [15, chap. 1] and Section III.6.4 in [19]. Consider an operator $L \in \mathcal{L}(X, X)$, where X is a Banach space and consider the linear problem

$$\begin{cases} u_t(x,t) &= Lu(x,t) \\ u(x,0) &= u_0(x), \text{ with } u_0 \in X. \end{cases} \tag{4.11}$$

Since L is a bounded operator, then $Re(\sigma(L)) \le \delta$, and the norm of the semigroup satisfies that

$$||e^{Lt}||_{\mathcal{L}(X)} \le C_0 e^{(\delta+\varepsilon)t}, \qquad t \ge 0. \tag{4.12}$$

Then, given an isolated part σ_1 of $\sigma(L)$ we define the **Riesz projection** of L corresponding to the isolated part σ_1, Q_{σ_1}, as the bounded linear operator on X given by

$$Q_{\sigma_1} = \frac{1}{2\pi i}\int_\Gamma (\lambda I - L)^{-1} d\lambda,$$

where Γ consists of a finite number of rectifiable Jordan curves, oriented in the positive sense around σ_1, separating σ_1 from $\sigma_2 = \sigma(L) \setminus \sigma_1$. This means that σ_1 belongs to the inner region of Γ, and σ_2 belongs to the outer region of Γ. The operator Q_{σ_1} is independent of the path Γ described as above.

Assume the spectrum of L is the disjoint union of two non-empty closed subsets σ_1 and σ_2. To this decomposition of the spectrum corresponds a direct sum decomposition of the space, $X = U \oplus V$, such that U and V are $L-$ invariant subspaces of X, the spectrum of the restriction $L|U$ is equal to σ_1 and that of $L|V$ to σ_2. If we assume that

$$\delta_2 < \mathrm{Re}(\sigma_1) \le \delta_1, \ \mathrm{Re}(\sigma_2) \le \delta_2\,, \ \text{ with } \ \delta_2 < \delta_1,$$

then we have that the solution to (4.11), can be written as

$$u(t) = Q_{\sigma_1}(u)(t) + Q_{\sigma_2}(u)(t).$$

On the other hand, the solution of (4.11) is equal to $u(t) = e^{Lt}u_0$. Thus, thanks to (4.12) and since $\mathrm{Re}(\sigma_2) \le \delta_2$ we obtain that for $t > 0$

$$\|Q_{\sigma_2}(u(t))\|_X = \|\left(Q_{\sigma_2} \circ e^{L\,t}\right) u_0\|_X = \|e^{L_2 t}\, Q_{\sigma_2}(u_0)\|_X \le C_2\, e^{(\delta_2+\varepsilon)t}\|Q_{\sigma_2}(u_0)\|_X,$$

where, $L_2 = L|\mathrm{Im}\, Q_{\sigma_2}$. With this, we get the following, see [15, chap. 1].

Theorem 4.6. *Consider $L \in \mathcal{L}(X)$ and let $\sigma(L)$ be a disjoint union of two closed subsets σ_1 and σ_2, with $\delta_2 < Re(\sigma_1) \leq \delta_1$, $Re(\sigma_2) \leq \delta_2$, with $\delta_2 < \delta_1$. Then the solution of* (4.11) *satisfies*

$$\lim_{t\to\infty} \|e^{-\mu t}\big(u(t) - Q_{\sigma_1}(u)(t)\big)\|_X = 0, \qquad \forall \mu > \delta_2.$$

The assumptions of the following proposition, are tailored for the case $L = K_J - hI$ in (4.1) and allows to compute the Riesz projection in terms of the Hilbert projection.

Proposition 4.7. *For $1 \leq p_0 < p_1 \leq \infty$, with $2 \in [p_0, p_1]$, let $X = L^p(\Omega)$, with $p \in [p_0, p_1]$, or $X = \mathcal{C}_b(\Omega)$. We assume $L \in \mathcal{L}(X, X)$ is selfadjoint in $L^2(\Omega)$, the spectrum of L, $\sigma_X(L)$, is independent of X, and the largest eigenvalue associated to L, λ_1 is simple and isolated, with associated eigenfunction $\Phi_1 \in L^p(\Omega) \cap L^{p'}(\Omega)$, for $p \in [p_0, p_1]$, if $X = L^p(\Omega)$, or $\Phi_1 \in \mathcal{C}_b(\Omega) \cap L^1(\Omega)$, if $X = \mathcal{C}_b(\Omega)$, and $\|\Phi_1\|_{L^2(\Omega)} = 1$.*

If $\sigma_1 = \{\lambda_1\}$, and Γ is the curve around only λ_1, then for $u \in X$, the Riesz projection associated to σ_1 is given by

$$Q_{\sigma_1}(u) = \Big(\int_\Omega u\,\Phi_1\Big), \Phi_1. \tag{4.13}$$

Proof. First, working in $L^2(\Omega)$, it is well known that the Riesz projection coincides with the Hilbert projection, that is (4.13) holds for all $u \in L^2(\Omega)$; see from Sections III.6.4 and III.6.5 in [19].

Now in $X = L^p(\Omega)$ for $p \in [p_0, p_1]$ or $X = \mathcal{C}_b(\Omega)$, since the spectrum, $\sigma_X(L)$, is independent of X, we have that the projection $P(u) = \langle u, \Phi_1\rangle \Phi_1$ is well defined for $u \in X$ because by hypothesis, $\Phi_1 \in L^{p'}(\Omega) \cap L^p(\Omega)$ for all $p \in [p_0, p_1]$, if $X = L^p(\Omega)$, or $\Phi_1 \in \mathcal{C}_b(\Omega) \cap L^1(\Omega)$, if $X = \mathcal{C}_b(\Omega)$. In fact $P \in \mathcal{L}(X, X)$. On the other hand, since the set

$$V = span\,[\,\chi_D\,;\, D \subset \Omega \ \text{ with } \ \mu(D) < \infty] \subset L^2(\Omega),$$

where χ_D is the characteristic function of $D \subset \Omega$, is dense in $L^p(\Omega)$, and $Q_{\sigma_1} \equiv P$ in V, they coincide in $X = L^p(\Omega)$. Finally for $X = \mathcal{C}_b(\Omega)$, we use that $L^2(\Omega) \cap \mathcal{C}_b(\Omega)$ is dense in $\mathcal{C}_b(\Omega)$ and again $Q_{\sigma_1} \equiv P$ in X. □

4.4 Asymptotic behaviour of the solution of the nonlocal diffusion problem

Let (Ω, μ, d) be a metric measure space with Ω compact. In this section be apply the results of the previous section about the asymptotic behavior of the solution for the problem

$$\begin{cases} u_t(x,t) &= (K_J - hI)u(x,t), \quad x \in \Omega,\ t > 0, \\ u(x,0) &= u_0(x), \qquad\qquad\qquad \text{with } u_0 \in X. \end{cases}$$

We study two problems to which we apply the results of the previous sections. In particular we consider the cases where h constant or $h = h_0 = \displaystyle\int_\Omega J(\cdot, y)dy$, with $J \in L^\infty(\Omega, L^1(\Omega))$.

Case h constant. For $h = a \in \mathbb{R}$ constant we have the problem

$$\begin{cases} u_t(x,t) &= (K_J - aI)u(x,t), \\ u(x,0) &= u_0(x) \in L^p(\Omega). \end{cases} \tag{4.14}$$

Then we have.

Proposition 4.8. *Let Ω be compact and connected. Let $X = L^p(\Omega)$, with $1 \le p \le \infty$, or $X = \mathcal{C}_b(\Omega)$. Let $K_J \in \mathcal{L}(L^1(\Omega), \mathcal{C}_b(\Omega))$ be compact, (see Proposition 3.6) and assume $J(x,y) = J(y,x)$ with*

$$J(x,y) > 0,\ \forall x,\ y \in \Omega \text{ such that } d(x,y) < R, \quad \text{for some } \ R > 0.$$

Then the solution u of (4.14) *satisfies that*

$$\lim_{t\to\infty} \|e^{-\lambda_1 t} u(t) - C^* \Phi_1\|_X = 0,$$

where $C^* = \displaystyle\int_\Omega u_0 \Phi_1$, and Φ_1 is an eigenfunction associated to λ_1, normalized in $L^2(\Omega)$.

Proof. From Proposition 3.17, we have that $\sigma_X(K_J)$ is independent of X. Moreover, since $J(x,y) = J(y,x)$, then from Proposition 3.18, we know that $\sigma(K_J) \setminus \{0\}$ is a real sequence of eigenvalues $\{\mu_n\}_{n\in\mathbb{N}}$ of finite multiplicity that converges to 0. Furthermore, the hypotheses of Proposition 3.20 are satisfied, then the largest eigenvalue, $\lambda_1 = r(K_J)$, is an isolated simple eigenvalue, and the eigenfunction $\Phi_1 \in \mathcal{C}_b(\Omega)$ associated to it, can be taken positive. Since the spectrum does not depend on X, we have that, $\Phi_1 \in X$, in particular $\Phi_1 \in L^p(\Omega) \cap L^{p'}(\Omega)$, and $\Phi_1 \in \mathcal{C}_b(\Omega) \cap L^1(\Omega)$. Then the spectrum of $K_J - aI$ is $\{\lambda_n = \mu_n - a\}_{n\in\mathbb{N}}$ and Φ_1 is a positive eigenfunction associated to λ_1.

Thus, for $u_0 \in X$ thanks to Theorem 4.6, the solution of (4.14) satisfies

$$\lim_{t\to\infty} \|e^{-\lambda_1 t}\big(u(t) - Q_{\sigma_1}(u)(t)\big)\|_X = 0.$$

and by Proposition 4.7 we have $Q_{\sigma_1} = P$. Thus, since $u(x,t) = e^{(K_J - aI)t} u_0(x)$, we have that

$$Q_{\sigma_1}(u)(t) = Q_{\sigma_1}(e^{(K_J-aI)t}u_0) = e^{(K_J-aI)t} Q_{\sigma_1}(u_0) = C^* e^{(K_J-aI)t}\Phi_1 = C^* e^{\lambda_1 t}\Phi_1.$$

where $C^* = \int_\Omega u_0 \Phi_1$, and we get the result. □

Case $h = h_0 \in L^\infty(\Omega)$. Assume $J \in L^\infty(\Omega, L^1(\Omega))$ and consider the problem

$$\begin{cases} u_t(x,t) & = (K_J - h_0 I) u(x,t) \\ u(x,0) & = u_0(x), \text{ with } u_0 \in L^p(\Omega) \end{cases} \tag{4.15}$$

In the following proposition, we prove that the solution of (4.15) goes exponentially in norm X to the mean value in Ω of the initial data.

Proposition 4.9. *Let $\mu(\Omega) < \infty$, let $X = L^p(\Omega)$, with $1 \le p \le \infty$ or $X = \mathcal{C}_b(\Omega)$. We assume $K_J \in \mathcal{L}(L^1(\Omega), \mathcal{C}_b(\Omega))$ is compact, (see Proposition 3.6) and J satisfies $J \in L^\infty(\Omega, L^1(\Omega))$, $J(x,y) = J(y,x)$ and*

$$J(x,y) > 0,\ \forall x,\ y \in \Omega \text{ such that } d(x,y) < R, \text{ for some } R > 0.$$

We also assume that $h_0(x) > \alpha > 0$, for all $x \in \Omega$.

Then the solution u of (4.15) *satisfies that*

$$\lim_{t\to\infty} \left\| e^{\beta t}\left(u(t) - \frac{1}{\mu(\Omega)} \int_\Omega u_0 \right) \right\|_X = 0,$$

for some $\beta > 0$.

Proof. Since $K_J \in \mathcal{L}(L^1(\Omega), \mathcal{C}_b(\Omega))$ is compact, then $K_J \in \mathcal{L}(X, X)$ is compact. Thanks to Theorem 3.26, we know that

$$\sigma_X(K_J - h_0 I) = \overline{R(-h_0)} \cup \{\mu_n\}_{n=1}^{M}, \quad \text{with } M \in \mathbb{N} \text{ or } M = \infty.$$

If $M = \infty$, then $\{\mu_n\}_{n=1}^{\infty}$ is a sequence of eigenvalues of $K_J - h_0 I$ with finite multiplicity, that has accumulation points in $R(-h)$. Moreover, from Proposition 3.27, $\sigma_X(K_J - h_0 I)$ is independent of X. Also, from Corollary 3.28, we have that $\sigma_X(K_J - h_0 I) \leq 0$, and 0 is an isolated simple eigenvalue with only constant eigenfunctions.

Moreover, since $J(x, y) = J(y, x)$ and thanks to Proposition 3.16, $K_J - h_0 I$ is selfadjoint in $L^2(\Omega)$, thus, $\{\mu_n\} \subset \mathbb{R}$. Hence, we consider $\sigma_1 = \{0\}$ an isolated part of $\sigma(K_J - h_0 I)$, with associated eigenfunction $\Phi_1 = 1/\mu(\Omega)^{1/2}$, and $\sigma_2 = \sigma(K_J - h_0 I) \setminus \{0\}$. Then thanks to Theorem 4.6,

$$\lim_{t \to \infty} \left\| e^{\beta t} \left(u(t) - Q_{\sigma_1}(u)(t) \right) \right\|_X = 0,$$

for some $\beta > 0$ and by Proposition 4.7, and since $Q_{\sigma_1} = P$. Since $u(x, t) = e^{(K_J - h_0 I)t} u_0(x)$,

$$\begin{aligned} Q_{\sigma_1}(u)(t) &= Q_{\sigma_1}(e^{(K_J - h_0 I)t} u_0) = e^{(K_J - h_0 I)t} Q_{\sigma_1}(u_0) \\ &= \left(\int_\Omega u_0 \Phi_1 \right) e^{(K_J - h_0 I)t} \Phi_1 = (\int_\Omega u_0 \Phi_1) \Phi_1 = \frac{1}{\mu(\Omega)} \int_\Omega u_0. \end{aligned}$$

□

Remark 4.10. *Propositions 4.8 and 4.9 where proven in [8], in the case where Ω is an open set in $\mathbb{R}^N$ and for $X = L^2(\Omega)$ or $X = \mathcal{C}(\overline{\Omega})$.*

References

[1] Andreu, F., Mazón, J. M., Rossi, J. D., and Toledo, J. The Neumann problem for nonlocal nonlinear diffusion equations. *J. Evol. Equ. 8*, 1 (2008), 189–215.

[2] Andreu-Vaillo, F., Mazón, J. M., Rossi, J. D., and Toledo-Melero, J. J. *Nonlocal diffusion problems*, vol. 165 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2010.

[3] Bates, P. W., and Chmaj, A. An integrodifferential model for phase transitions: stationary solutions in higher space dimensions. *J. Statist. Phys. 95*, 5-6 (1999), 1119–1139.

[4] Bates, P. W., and Zhao, G. Existence, uniqueness and stability of the stationary solution to a nonlocal evolution equation arising in population dispersal. *J. Math. Anal. Appl. 332*, 1 (2007), 428–440.

[5] ben Avraham, D., and Havlin, S. *Diffusion and reactions in fractals and disordered systems.* Cambridge University Press, Cambridge, 2000.

[6] Bennett, C., and Sharpley, R. *Interpolation of operators*, vol. 129 of *Pure and Applied Mathematics*. Academic Press Inc., Boston, MA, 1988.

[7] Brezis, H. *Functional analysis, Sobolev spaces and partial differential equations*. Universitext. Springer, New York, 2011.

[8] Chasseigne, E., Chaves, M., and Rossi, J. D. Asymptotic behavior for nonlocal diffusion equations. *J. Math. Pures Appl. (9) 86*, 3 (2006), 271–291.

[9] Christensen, E., Ivan, C., and Lapidus, M. L. Dirac operators and spectral triples for some fractal sets built on curves. *Adv. Math. 217*, 1 (2008), 42–78.

[10] Cobos, F. Mini-workshop: Compactness Problems in Interpolation Theory and Function Spaces. *Oberwolfach Rep. 1*, 3 (2004), 2103–2132. Abstracts from the mini-workshop held August 15–21, 2004, Organized by Fernando Cobos and Thomas Kühn, Oberwolfach Reports. Vol. 1, no. 3.

[11] Cortazar, C., Elgueta, M., Rossi, J. D., and Wolanski, N. How to approximate the heat equation with Neumann boundary conditions by nonlocal diffusion problems. *Arch. Ration. Mech. Anal. 187*, 1 (2008), 137–156.

[12] Crank, J. *The mathematics of diffusion*. Oxford, at the Clarendon Press, 1956.

[13] Fife, P. Some nonclassical trends in parabolic and parabolic-like evolutions. In *Trends in nonlinear analysis*. Springer, Berlin, 2003, pp. 153–191.

[14] Folland, G. B. *Real analysis*, second ed. Pure and Applied Mathematics (New York). John Wiley & Sons Inc., New York, 1999. Modern techniques and their applications, A Wiley-Interscience Publication.

[15] Gohberg, I., Goldberg, S., and Kaashoek, M. A. *Classes of linear operators. Vol. I*, vol. 49 of *Operator Theory: Advances and Applications*. Birkhäuser Verlag, Basel, 1990.

[16] Hale, J. K. *Asymptotic behavior of dissipative systems*, vol. 25 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1988.

[17] Henry, D. *Geometric theory of semilinear parabolic equations*, vol. 840 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1981.

[18] Hutson, V., Martinez, S., Mischaikow, K., and Vickers, G. T. The evolution of dispersal. *J. Math. Biol. 47*, 6 (2003), 483–517.

[19] Kato, T. *Perturbation theory for linear operators*, second ed. Springer-Verlag, Berlin, 1976. Grundlehren der Mathematischen Wissenschaften, Band 132.

[20] Kigami, J. *Analysis on fractals*, vol. 143 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2001.

[21] Kigami, J. Measurable Riemannian geometry on the Sierpinski gasket: the Kusuoka measure and the Gaussian heat kernel estimate. *Math. Ann. 340*, 4 (2008), 781–804.

[22] Kreĭn, M. G., and Rutman, M. A. Linear operators leaving invariant a cone in a Banach space. *Amer. Math. Soc. Translation 1950*, 26 (1950), 128.

[23] Lapidus, M. L., and Sarhad, J. J. Dirac operators and geodesic metric on the harmonic sierpinski gasket and other fractal sets. Arxive, December 2012.

[24] Rudin, W. *Real and Complex Analysis.* McGraw-Hill, Inc., 1966.

[25] Simon, L. *Lectures on geometric measure theory*, vol. 3 of *Proceedings of the Centre for Mathematical Analysis, Australian National University.* Australian National University Centre for Mathematical Analysis, Canberra, 1983.

[26] Strichartz, R. S. *Differential equations on fractals.* Princeton University Press, Princeton, NJ, 2006. A tutorial.