Finding the Set of k-additive Dominating Measures Viewed as a Flow Problem

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Abstract. In this paper we deal with the problem of obtaining the set of k-additive measures dominating a fuzzy measure. This problem extends the problem of deriving the set of probabilities dominating a fuzzy measure, an important problem appearing in Decision Making and Game Theory. The solution proposed in the paper follows the line developed by Chateauneuf and Jaffray for dominating probabilities and continued by Miranda et al. for dominating k-additive belief functions. Here, we address the general case transforming the problem into a similar one such that the involved set functions have non-negative Möbius transform; this simplifies the problem and allows a result similar to the one developed for belief functions. Although the set obtained is very large, we show that the conditions cannot be sharpened. On the other hand, we also show that it is possible to define a more restrictive subset, providing a more natural extension of the result for probabilities, such that it is possible to derive any k-additive dominating measure from it.

Keywords: Fuzzy measures · Dominance · k-additivity

1 Introduction

Fuzzy measures, also called capacities, nonadditive measures, are widely used in the representation of uncertainty, decision making and cooperative game theory. A particular class of fuzzy measures which is of interest in this paper can be found in the Theory of Evidence developed by Dempster [4] and Shafer [22]. In this theory, uncertainty is represented by a pair of "lower probability" (or "degree of belief") and "upper probability" (or "degree of plausibility") assigned to every event. These upper and lower probabilities have been well studied [25,26]; they are not additive in general, and are called by Shafer belief and plausibility functions.

The problem of finding the set of probability measures dominating a given fuzzy measure appears in many situations, especially in decision theory and in cooperative game theory. In decision theory, it may happen that the available information is not sufficient to assign an exact probability to events, but it only

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allows an interval of compatible probability values. In this case, we obtain a set of possible probabilities, denoted by $\mathcal{P}$. If we consider $\mu := \inf_{P \in \mathcal{P}} P$, then μ is a fuzzy measure (but not necessarily a belief function [24]), called the "coherent lower probability". As a consequence, for any probability P' dominating μ, it follows that $E_{P'}(f) \geq \mathcal{C}_\mu(f)$, for any function f, where $\mathcal{C}_\mu$ denotes the Choquet integral [3]. In [2], Chateauneuf and Jaffray use this result and the fact that $\mu \leq P$, $\forall P \in \mathcal{P}$ to obtain an easy method to compute $\inf_{P \in \mathcal{P}} E_P(f)$. Note that this method is based on the knowledge of the set of all probability distributions dominating μ. The same can be applied for obtaining an upper bound.

In cooperative game theory, a TU-game is a set function μ vanishing on the empty set (it is not necessarily a fuzzy measure, however). One of the most important problems in this field is to obtain a sharing function for the game, that is, assuming that the grand coalition X is formed and the benefit $\mu(X)$ is obtained, we are looking for a rational and equitable way to divide $\mu(X)$ among all players. Any possible sharing function is called a solution of the game. Among the many concepts of solutions in the literature (see, e.g., [5]), one of the most popular is the *core* of the game [23], which is defined as the set of additive games dominating μ and coinciding with μ on the grand coalition X. The core is a bounded polyhedron, possibly empty, and much research has been devoted to its study (see a survey in [10]).

On the other hand, a natural extension of probabilities or additive measures is the concept of k-additive measure [7,8]. They constitute a mid-term between probability measures (which are too restrictive in many situations) and general fuzzy measures (whose complexity is too high to deal with in practice). Thus, a natural extension of the previous dominance problem is to look for the set of k-additive measures dominating a given fuzzy measure. There are some cases where this could be useful. First, suppose a situation that can be modelled via a k-additive measure (an axiomatic characterization to this situation can be found in [19]), but where our information is not enough to completely determine the measure. Then, we have to work with a set of compatible k-additive measures (let us call it $\mathcal{U}_k$). A second example is the identification of a capacity in a practical situation. It can be proved that the available information may not be sufficient to determine a single solution, but there exists a set of k-additive measures, all equally suitable [16]. Moreover, the set of all these measures is a convex set and consequently, the measure for an event $A \subseteq X$ lies in an interval of possible values (a deeper study about the uniqueness of the solution and the structure of the set of solutions can be found in [18]). As for probabilities, if $\mu = \inf_{\mu_k \in \mathcal{U}_k} \mu_k$, then μ is a fuzzy measure and $\mathcal{C}_{\mu'_k}(f) \geq \mathcal{C}_\mu(f)$, for any k-additive measure μ'_k dominating μ. Therefore, it seems interesting to find the set of all k-additive fuzzy measures dominating μ, thus extending the results in [2].

Another interest in finding the set of dominating k-additive measures can be found in game theory. As we have said above, the core of a game μ may be empty [1]. Considering instead the set of its k-additive dominating games, called the k-additive core, it is shown in [17] that the k-additive core is never empty, as soon as $k \geq 2$.

In this paper we deal with the problem of characterizing the set of all k-additive measures dominating a given fuzzy measure μ. Previous attempts in this direction appear in [9,20]. As it will become apparent below, we have to face many difficulties that do not arise in the case of probabilities, except in very restrictive situations. One of these situations is the case of k-additive belief functions dominating a belief function. We will use the results in this case to derive a general result for any fuzzy measure and any dominating k-additive measure.

The rest of the paper is organized as follows: in the next section, we explain the basic facts and results in order to fix notation and to be self-contained; then, we derive the results for characterizing the set of dominating k-additive measures. We end the paper with concluding remarks and open problems.

2 Basic Results

Consider a finite referential set of n elements, $X = \{1, ..., n\}$. The set of subsets of X is denoted by $\mathcal{P}(X)$ and we denote $\mathcal{P}^*(X) = \mathcal{P}(X)\backslash\{\emptyset\}$; the set of subsets whose cardinality is less or equal than k is denoted by $\mathcal{P}^k(X)$, or $\mathcal{P}^k_*(X)$ if the emptyset is not included. Subsets of X are denoted $A, B, ...$; we will sometimes write $i_1 \cdots i_k$ instead of $\{i_1, \ldots, i_k\}$ in order to avoid a heavy notation; braces are usually omitted for singletons and subsets of two elements.

We define a **fuzzy measure** as a set function $\mu : \mathcal{P}(X) \to [0, 1]$ satisfying the boundary conditions $\mu(\emptyset) = 0, \mu(X) = 1$ and monotonicity ($\mu(A) \leq \mu(B)$ whenever $A \subseteq B$). Fuzzy measures are denoted by μ, μ^* and so on, and the set of all fuzzy measures on X is denoted $\mathcal{FM}(X)$.

Given a set function μ (not necessarily a fuzzy measure), an equivalent representation of μ can be obtained via the **Möbius transform** [21], given by

$$m(A) := \sum_{B \subseteq A} (-1)^{|A \backslash B|} \mu(B), \; \forall A \subseteq X.$$

The Möbius transform is also widely used in the field of Game Theory, where it is known as *dividends* [13]. It is worth noting that $m(A)$ can attain negative values. The set of fuzzy measures μ such that the corresponding Möbius transform satisfies $m(A) \geq 0, \; \forall A \subseteq X$ is known as the set of **belief functions**, denoted $\mathcal{BEL}(X)$. Belief functions come from the Theory of Evidence developed by Dempster [4] and Shafer [22]. Given the Möbius transform, it is possible to recover the original fuzzy measure through the *Zeta transform*:

$$\mu(A) = \sum_{B \subseteq A} m(B).$$

Contrarily to fuzzy measures, for which it holds $0 \leq \mu(A) \leq 1, \; \forall A \subseteq X$, the upper and lower bounds for the Möbius transform are not trivial. These bounds are given in the next result.

Theorem 1. *[12] For any fuzzy measure μ, its Möbius transform satisfies for any $A \subseteq N$, $|A| > 1$:*

$$l_{|A|} := -\binom{|A|-1}{c'_{|A|}} \leqslant m(A) \leqslant \binom{|A|-1}{c_{|A|}} := u_{|A|},$$

with

$$c_{|A|} = 2\left\lfloor \frac{|A|}{4} \right\rfloor, \quad c'_{|A|} = 2\left\lfloor \frac{|A|-1}{4} \right\rfloor + 1,$$

and for $|A| = 1 < n$:

$$0 \leqslant m(A) \leqslant 1,$$

and $m(A) = 1$ if $|A| = n = 1$. These upper and lower bounds are attained by the fuzzy measures μ^*_A, μ_{A*}, respectively:

$$\mu^*_A(B) = \begin{cases} 1, & \text{if } |A| - l_{|A|} \leqslant |B \cap A| \\ 0, & \text{otherwise} \end{cases},$$

$$\mu_{A*}(B) = \begin{cases} 1, & \text{if } |A| - l'_{|A|} \leqslant |B \cap A| \\ 0, & \text{otherwise} \end{cases},$$

for any $B \subseteq N$.

We give in Table 1 the first values of the bounds.

Table 1. Lower and upper bounds for the Möbius transform of a fuzzy measure

$\|A\|$	1	2	3	4	5	6	7	8	9	10	11	12
u.b. of $m(A)$	1	1	1	3	6	10	15	35	70	126	210	462
l.b. of $m(A)$	1(0)	−1	−2	−3	−4	−10	−20	−35	−56	−126	−252	−462

Let us now introduce the concept of k-additivity. A problem appearing in the practical use of fuzzy measures is their complexity. Contrary to the case of probabilities, where just $n-1$ values suffice to completely determine a probability on a set of cardinality n, in order to determine a fuzzy measure $2^n - 2$ values are necessary. As a consequence, complexity grows exponentially with n. In an attempt to reduce this complexity, Grabisch has defined the concept of k-additive measure [7].

A fuzzy measure μ is said to be **k-additive** if its Möbius transform vanishes for any $A \subseteq X$ such that $|A| > k$ and there exists at least one subset A with exactly k elements such that $m(A) \neq 0$.

Thus, it can be seen that probabilities are just 1-additive measures (and also 1-additive belief functions). As a consequence, k-additive measures generalize probability measures and they fill the gap between probability measures and general fuzzy measures. For a k-additive measure, the number of coefficients is reduced to

$$\sum_{i=1}^{k} \binom{n}{i}.$$

More about k-additive measures can be found, e.g., in [8]. We define the set $\mathcal{FM}^k(X)$ (resp. $\mathcal{BEL}^k(X)$) as the set of fuzzy measures (resp. belief functions) μ whose corresponding Möbius transform m satisfies $m(A) = 0$ if $|A| > k$.

Finally, we say that a fuzzy measure μ^* **dominates** μ, and we denote it by $\mu^* \geq \mu$, if

$$\mu^*(A) \geq \mu(A), \forall A \subseteq X.$$

For general set functions, dominance is defined by

$$\mu^*(A) \geq \mu(A), \forall A \subseteq X, \mu^*(X) = \mu(X).$$

Given a fuzzy measure μ, we define the set $\mathcal{FM}^k_{\geq}(\mu)$ (or $\mathcal{BEL}^k_{\geq}(\mu)$ if we restrict to dominating k-additive belief functions) as the set of fuzzy measures (resp. belief functions) in $\mathcal{FM}^k(X)$ (resp. $\mathcal{BEL}^k(X)$) dominating μ.

3 Characterizing the Set of Dominating Fuzzy Measures

Consider a fuzzy measure μ and let us turn to the problem of obtaining the set $\mathcal{FM}^k_{\geq}(\mu)$. In [2], the following result is proved.

Theorem 2. *Let μ be a fuzzy measure on X, m its Möbius transform, and suppose $P \in \mathcal{FM}^1_{\geq}(\mu)$. Then, P can be put under the following form:*

$$P(\{i\}) = \sum_{B \ni i} \lambda(B, i) m(B), \forall i \in X.$$

The function $\lambda : \mathcal{P}_*(X) \times X \to [0, 1]$ is a weight function satisfying:

$$\sum_{i \in B} \lambda(B, i) = 1, \forall \emptyset \neq B \subseteq X.$$

$$\lambda(B, i) = 0 \textit{ whenever } i \notin B.$$

Dempster has shown the same result in [4] and also Shapley in [23], but both of them only for belief functions.

If we restrict our attention to the case of the set of k-additive belief functions dominating a belief function, the following result appears in [20].

Theorem 3. *Let $\mu, m, \mu^* : \mathcal{P}(X) \to \mathbb{R}$, where μ is a fuzzy measure, m its Möbius inverse, and $\mu^* \in \mathcal{BEL}^k_{\geq}(\mu)$. Then, necessarily the Möbius transform m^* of μ^* can be put under the following form:*

$$m^*(A) = \sum_{B | A \cap B \neq \emptyset} \lambda(B, A) m(B), \forall A \in \mathcal{P}^k_*(X),$$

where function $\lambda : \mathcal{P}_*(X) \times \mathcal{P}^k_*(X) \rightarrow [0,1]$ *is such that*

$$\sum_{A|B\cap A\neq\emptyset} \lambda(B,A) = 1, \forall B \in \mathcal{P}_*(X). \tag{1}$$

$$\lambda(B,A) = 0, \ if \ A\cap B = \emptyset. \tag{2}$$

We have to keep in mind that Eqs. 1 and 2 lead to a non-empty intersection condition; from a mathematical point of view, another possibility (with better properties) of generalizing Theorem 2 could be a more restrictive inclusion condition, i.e. satisfying $\lambda(B,A) = 0$ whenever $A \not\subseteq B$. However, this condition fails to obtain all dominating k-additive belief functions, as it is shown in [20]. When dealing with general fuzzy measures, it happens that we have to permit functions λ attaining negative values [9]. Thus, we obtain a very wide set of functions, many of them failing to satisfy monotonicity or dominance.

In this paper, we are going to apply the result for belief functions to obtain a more handy result for the general case.

Theorem 4. *Let* $\mu, \mu^* : \mathcal{P}(X) \rightarrow \mathbb{R}$, *where* $\mu \in \mathcal{FM}^k(X)$ *and* $\mu^* \in \mathcal{FM}^k_{\geq}(\mu)$, *for* $k = 1,...,n$, *and let us denote by* m *and* m^* *their respective Möbius transforms. Let us define:*

$$m_{aux}(A) = m(A) - l_{|A|}, \ m^*_{aux}(A) = m^*(A) - l_{|A|},$$

where l_i *denotes the lower bound for the Möbius transform of subsets of cardinality* $i, i = 1,...,k$. *Then, necessarily* m^*_{aux} *can be put under the following form:*

$$m^*_{aux}(A) = \sum_{B|A\cap B\neq\emptyset} \lambda(B,A) m_{aux}(B), \forall A \in \mathcal{P}^k_*(X),$$

where function $\lambda : \mathcal{P}^k_*(X) \times \mathcal{P}^k_*(X) \rightarrow [0,1]$ *is such that*

$$\sum_{A|A\cap B\neq\emptyset} \lambda(B,A) = 1, \forall B \in \mathcal{P}^k_*(X). \tag{3}$$

$$\lambda(B,A) = 0, \ if \ A\cap B = \emptyset. \tag{4}$$

Indeed, in this result, function λ is a sharing function of $m_{aux}(B)$ among any subset A such that $A \cap B \neq \emptyset$. Thus, this problem can be turned into a transshipment problem in a flow network. Figure 1 shows the corresponding flow network for $k = 2$.

The proof of the result is based on Gale's Theorem for a transshipment network [6], where subset A offers $m(A) - l_{|A|}$ to be shared among subsets intersecting with A. However, the underlying idea of the result relies on the result for k-additive dominating belief functions. For belief functions, Theorem 3 is an extension of Theorem 2; on the other hand, this result cannot be applied for general $k-$additive dominating measures, as shown in [9]. The idea then is to transform μ and μ^* into other set functions μ_{aux} and μ^*_{aux} resp., having

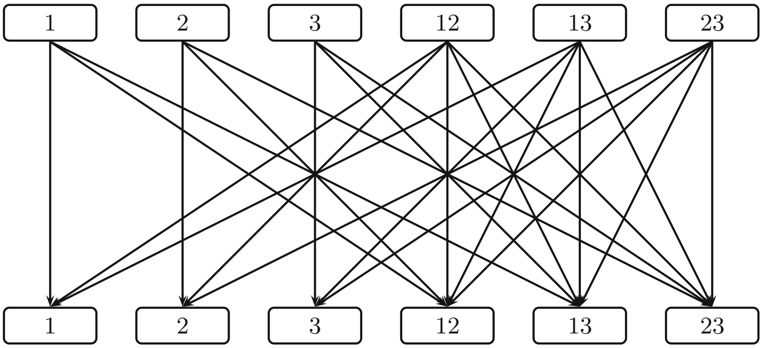

Fig. 1. Example of flow network for $k = 2$

properties similar to belief functions. More concretely, the corresponding Möbius transform is non-negative (but these set functions are not normalized!). In this sense, we can add other constraints instead of $l_{|A|}$, or even the same constant regardless the cardinality, but these are the more accurate [12].

Remark that the condition allowing positive weights for non-empty intersections in Eqs. 3 and 4 cannot be turned into an inclusion condition, as the next example shows.

Example 1. Consider $|X| = 3$ and the 2-additive fuzzy measure μ whose Möbius transform m, and whose corresponding m_{aux} are given in next table

A	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$
μ	0.4	0.3	0.4	0.4	1	0.7
m	0.4	0.3	0.4	-0.3	0.2	0
m_{aux}	0.4	0.3	0.4	0.7	1.2	1

Now, consider μ^* the 2-additive measure, with m^*, m^*_{aux} given by

A	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$
μ^*	0.4	0.3	0.5	0.5	1	0.7
m^*	0.4	0.3	0.5	-0.2	0.1	-0.1
m^*_{aux}	0.4	0.3	0.5	0.8	1.1	0.9

Then, $m^*_{aux}(12) > m_{aux}(12)$, while the only subset containing $\{1,2\}$ is $\{1,2\}$ itself.

The previous result can be extended when we are looking for k'-additive measures dominating k-additive measures when $k \neq k'$. For this, it suffices to notice that $\mathcal{FM}^{k'}(X) \subset \mathcal{FM}^{k}(X)$ if $k' < k$. Consequently, any measure in

$\mathcal{FM}^{k'}_{\geq}(\mu)$ can be derived from the previous theorem considering k. Similarly, if $k' > k$, any k'-additive measure dominating μ can be derived from the previous theorem just taking into account that $\mu \in \mathcal{FM}^{k'}(X)$.

Corollary 1. *Let $\mu, \mu^* : \mathcal{P}(X) \to \mathbb{R}$, where $\mu \in \mathcal{FM}^k(X)$ and $\mu^* \in \mathcal{FM}^{k'}_{\geq}(\mu)$, for $k, k' = 1, ..., n$, and let us denote by m and m^* their respective Möbius transforms. Assume $k \geq k'$ and let us define:*

$$m_{aux}(A) = m(A) - l_{|A|},\ m^*_{aux}(A) = m^*(A) - l_{|A|},$$

where l_i denotes the lower bound for the Möbius transform of subsets of cardinality $i, i = 1, ..., k$. Then, necessarily m^*_{aux} can be put under the following form:

$$m^*_{aux}(A) = \sum_{B|A\cap B\neq\emptyset} \lambda(B, A) m_{aux}(B),\ \forall A \in \mathcal{P}^{k'}_*(X),$$

where function $\lambda : \mathcal{P}^k_*(X) \times \mathcal{P}^{k'}_*(X) \to [0, 1]$ is such that

$$\sum_{A|A\cap B\neq\emptyset} \lambda(B, A) = 1,\ \forall B \in \mathcal{P}^k_*(X).$$

$$\lambda(B, A) = 0,\ \textit{if}\ A \cap B = \emptyset.$$

Corollary 2. *Let $\mu, \mu^* : \mathcal{P}(X) \to \mathbb{R}$, where $\mu \in \mathcal{FM}^k(X)$ and $\mu^* \in \mathcal{FM}^{k'}_{\geq}(\mu)$, for $k, k' = 1, ..., n$, and let us denote by m and m^* their respective Möbius transforms. Assume $k \leq k'$ and let us define:*

$$m_{aux}(A) = m(A) - l_{|A|},\ m^*_{aux}(A) = m^*(A) - l_{|A|},$$

where l_i denotes the lower bound for the Möbius transform of subsets of cardinality $i, i = 1, ..., k'$. Then, necessarily m^*_{aux} can be put under the following form:

$$m^*_{aux}(A) = \sum_{B|A\cap B\neq\emptyset} \lambda(B, A) m_{aux}(B),\ \forall A \in \mathcal{P}^{k'}_*(X),$$

where function $\lambda : \mathcal{P}^k_*(X) \times \mathcal{P}^{k'}_*(X) \to [0, 1]$ is such that

$$\sum_{A|A\cap B\neq\emptyset} \lambda(B, A) = 1,\ \forall B \in \mathcal{P}^k_*(X).$$

$$\lambda(B, A) = 0,\ \textit{if}\ A \cap B = \emptyset.$$

As we have seen in Example 1, a non-empty intersection condition is needed. However, it is possible to obtain all dominating k-additive dominating measures from set functions that can be derived via an inclusion condition. This is stated in next result.

Theorem 5. *Let* $\mu, m, \mu^*, m^* : \mathcal{P}(X) \to \mathbb{R}$*, where* $\mu \in \mathcal{FM}^k(X)$*,* $\mu^* \in \mathcal{FM}^k_{\geq}(\mu)$ *and* m, m^* *their corresponding Möbius inverses. Let us define*

$$m_{aux}(A) = m(A) - l_{|A|},\ m^*_{aux}(A) = m^*(A) - l_{|A|},$$

where l_i *denotes the lower bound for the Möbius transform of subsets of cardinality* $i, i = 1, ..., k$*. Then, there exists a set function (not necessarily a fuzzy measure)* μ' *dominating* μ *whose Möbius transform* m' *is such that the corresponding* m'_{aux} *can be written as*

$$m'_{aux}(B) = \sum_{A|B\subseteq A} \lambda'(A, B) m_{aux}(A),\ \forall B \in \mathcal{P}^k_*(X),$$

where $\lambda' : \mathcal{P}^k_*(X) \times \mathcal{P}^k_*(X) \to [0, 1]$ *is such that*

$$\sum_{B|B\subseteq A} \lambda'(A, B) = 1,\ \forall A \in \mathcal{P}^k_*(X). \tag{5}$$

$$\lambda'(A, B) = 0 \ if\ B \not\subseteq A, \tag{6}$$

and m^*_{aux} *can be derived from* m'_{aux} *through*

$$m^*_{aux}(C) = \sum_{B|B\subseteq C} \lambda^*(B, C) m'_{aux}(B),\ \forall C \in \mathcal{P}^k_*(X),$$

where $\lambda^* : \mathcal{P}^k_*(X) \times \mathcal{P}^k_*(X) \to [0, 1]$ *is such that*

$$\sum_{C|B\subseteq C} \lambda^*(B, C) = 1,\ \forall B \in \mathcal{P}^k_*(X).$$

$$\lambda^*(B, C) = 0 \ if\ B \not\subseteq C.$$

This result is explained in Fig. 2 for $|X| = 3$ and $k = 2$.

It is worthnoting the differences with a similar result appearing in [20]; in that result, applying for dominating k-additive belief functions, any set function obtained using Eqs. 5 and 6 is a k-additive dominating belief function. However, in this more general situation, we cannot ensure monotonicity, as next example shows.

Example 2. Consider $|X| = 3$ and let μ be the $\{0, 1\}$-fuzzy measure such that $\mu(A) = 1$ if and only if $\{1, 2\} \subseteq A$. Then, the corresponding m is given by $m(1, 2) = 1$ and $m(A) = 0$ otherwise. Then, $\mu \in \mathcal{FM}^2(X)$ and $m_{aux}(1, 2) = 2, m_{aux}(i, j) = 1$ for any other pair and $m_{aux}(i) = 0$ for any singleton. Now, if we define

$$\lambda(A, B) = \begin{cases} 1 \text{ if } A = B \\ 0 \text{ otherwise} \end{cases},\ \text{ if } A \neq \{1, 2\},\ \lambda(\{1, 2\}, B) = \begin{cases} 1 \text{ if } B = \{1\} \\ 0 \text{ otherwise} \end{cases}$$

Then, $m'_{aux}(1) = 2, m'_{aux}(1, 2) = 0, m'_{aux}(2) = 0$, whence it follows $m'(1) = 2, m'(1, 2) = -1, m'(2) = 0$ and thus, $\mu'(1) = 2 > \mu'(1, 2) = 1$, violating monotonicity.

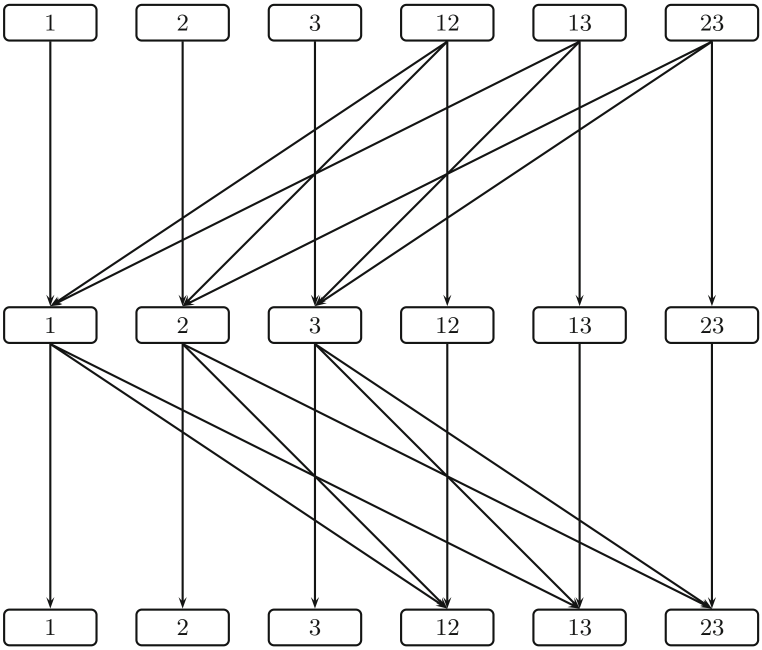

Fig. 2. Example of flow network for $k = 2$.

4 Conclusions and Open Problems

In this paper we have dealt with the problem of obtaining the set of k-additive dominating measures of a k-additive measure. For this, we have used a previous result valid for the special case of belief functions. The result follows the same philosophy of other results derived by Chateauneuf and Jaffray [2] and Miranda et al. [18]; along this line, we have obtained a superset of the set $\mathcal{FM}^k_{\geq}(\mu)$. A natural question is whether $\mathcal{FM}^k_{\geq}(\mu)$ is strictly contained into that set.

We have proved that in general, the non-empty intersection condition cannot be removed, but it seems interesting to search for special cases for which non-empty intersection can be turned into an inclusion condition because this condition seems easier to handle.

However, we feel that the most interesting open problem is to apply these results in a procedure for obtaining the set of vertices of $\mathcal{FM}^k_{\geq}(\mu)$. As it can be easily found, $\mathcal{FM}^k_{\geq}(\mu)$ is a polytope and thus, it is completely determined by its vertices. There are several results concerning the vertices of the core and the set of dominating probabilities, i.e. $\mathcal{FM}^1_{\geq}(\mu)$ [14,23]. For $k \geq 2$, several results have been obtained in [11]. The problem is particularly difficult for $2 < k < n$ because the set of vertices of $\mathcal{FM}^k(X)$ is not the set of $\{0,1\}$-valued measures in $\mathcal{FM}^k(X)$ [15] and the general form of these vertices is not known. Even for a seemingly simple situation, the set $\mathcal{FM}^2_{\geq}(\mu)$, up to our knowledge, the set of vertices is not known for any μ. The results in the paper could shed light on these problems, as they provide bounds for these sets.

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